普通高等教育“十二五”规划教材

A Concise Course in University Physics

Second Edition Volume 1

大学物理简明教程（英文版）

第二版

上册

王安安 伏云昌 主编

科学出版社

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A Concise Course in University Physics

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王安安 伏云昌 主编
王安安 樊则宾 编写

科学出版社
北京

内 容 简 介

本书是按照《理工科类大学物理课程教学基本要求(2010 年版)》修订的，涵盖了所有 A 类的内容，选择了部分 B 类的内容，如非惯性系、质心、气体的范德瓦尔斯方程、玻尔兹曼分布、几何光学、固体能带论和激光简介等．为方便上、下两学期的教学安排，本次改版将原三册改编为上、下两册．全书共 19 章：上册为力学和电磁学；下册为热学、振动与波动、光学和近代物理．本书配有双语课件光盘．

本书可供理工科非物理专业 112～128 学时双语教学使用，也可供在某一部分内容进行双语教学试点选用，还可供对英文物理感兴趣的广大读者自学或作参考书用．

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第二版前言

根据《理工科类大学物理课程教学基本要求(2010年版)》对教学过程的基本要求第5点"双语教学——在保证教学效果的前提下,有条件的学校可开展物理课程的双语教学,以提高学生查阅外文资料和科技外语交流的能力",为满足坚持双语教学的师生对英文物理教材的需要,我们对 *A Concise Course in University Physics*(《大学物理简明教程(英文版)》,1998～2000年出版)进行了部分修订. 按2010年版教学基本要求,第二版涵盖了所有A类的内容,保留原有的并新选了部分B类内容,如增加了非惯性系和惯性力、质心和质心运动定理、热力学第零定律、范德瓦尔斯方程及几何光学等内容. 为方便目前工科物理两学期教学内容的安排,将第一版的三册改编为上、下两册,供非物理专业112～128学时双语物理教学之用. 第二版保留了第一版的主要特色,力求系统完整、理论叙述准确、文字简明易懂,以达到教师好用、学生易学的目的. 因篇幅所限,第二版省去了各章总结和三个阅读材料.

全书19章编写分工如下:

上册:力学(第1～5章)由王安安编写;电磁学(第6～9章)主要由樊则宾编写.

下册:热学(第10,11章)由王安安编写;振动与波动部分第12、14章及第15章中的几何光学由伏云昌编写,第15章中的波动光学由陈劲波编写,第13章由王安安和陈劲波编写;近代物理(第16～19章)由伏云昌编写. 王安安和伏云昌负责全书的统稿、修改和定稿工作. 为方便本书读者进行双语多媒体课堂教学,伏云昌编制了配套教学课件.

北京大学陆果教授对第二版全书进行了认真的审定,并提出了宝贵的意见和建议,谨此致以诚挚谢意! 编者也对昆明理工大学校领导多年来对我们工作的支持表示衷心的感谢!

双语教学贵在坚持,我们编写及使用英文物理教材的初衷未改,通过双语教学提高学生的综合素质始终是我们的目标,编者愿与致力于这项事业的同行共同努力,持之以恒,为物理教学改革尽绵薄之力!

因时间仓促、水平有限,书中难免有不当之处,恳请同行与读者提出宝贵意见.

编　者

2011年12月于昆明

第一版前言摘要

我们正处在一个高新技术飞速发展、科技信息量激增、知识更新加快、国际交流日益广泛的时代. 我国的进一步改革开放，社会主义市场经济的建立都要求高校毕业生有更强的适应能力，在人才市场上，有效强的外语应用能力、交流型、综合型的毕业生供不应求. 在这种形势下，我们的高等教育正向着重视素质教育的方向转变，而素质与能力是密切相关的，素质的培养要以一定的知识和能力为基础，其中包括独立获取知识的能力. 毋庸置疑，直接用外语为工具获取知识、进行交流的能力是人才素质的一个重要方面.

然而，由于历史原因、文化背景、经济基础、外语教育模式和各类师资外语水平等诸多因素的影响，我们在应用外语进行教学方面的基础性工作十分薄弱. 在普通高校本科生教育中，教材和教学过程基本上只使用中文这一单一语种，实际上已经制约了学生应用外语（主要是英语）获取知识能力的发展. 为了改变这种现状，跟上时代的步伐，试用英文教材，使用英语进行教学的改革便应运而生了.

本教材的编写是编者主持的“试用英文物理教材”教改试点工作的继续，也是编者大学物理教学经验的总结. 从国外引进的教材，虽有诸多优点，但在系统上与我国的大学物理教学基本要求不完全对应，为了满足师生对英文物理教材的需要，编写一套根据我国工科物理教学基本要求，顺应工科物理教学改革的形势，反映编者在多年物理教学实践中总结出来的教学方法与经验，与我们的学生在一、二年级的英文水平相适应的英文“简明物理学教程”的计划就提到日程上来了，这就是我组织编写这套教材的初衷. 这套英文物理教材是 1996 年经国家教委批准列入正式出版计划的. 本教材可供普通高等工科院校本科生物理课 130～140 学时使用.

……

全书 21 章，具体编写分工如下：

第一册：力学（第 1～5 章），分子运动论和热力学基础（第 6、7 章）由王安安编写.

第二册：电学及稳恒电流的磁场（第 8～10 章）由吴光敏编写，磁介质和电磁感应（第 11、12 章）由樊则宾编写，麦克斯韦方程组（第 13 章）由王安安编写.

第三册：机械振动（第 14 章）由伏云昌编写，机械波（第 15 章）由王安安和陈劲波编写. 电磁振荡与电磁波（第 16 章）由吴光敏编写，波动光学部分（第 17 章）中干涉与衍射由陈劲波编写，光的偏振由樊则宾编写. 近代物理部分（第 18～21 章）由伏云昌编写.

第一册绝大部分插图由刘富华用计算机绘制，其余插图由李俊昌教授绘制，封面也由李俊昌设计，谨此致以诚挚谢意.

编写大学英文物理教材是一种大胆的尝试，由于编者水平有限，错误疏漏之处在所难免，希望同行和读者批评指正. 我们相信这本教材的出版将对物理教学的现代化和物理教学与国际接轨作出有益的贡献.

编　者

1997 年 5 月于昆明

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Part One Mechanics

Chapter 1

Kinematics

Kinematics is the study of the geometry of motion: it deals with the mathematical description of motion in terms of position, velocity, and acceleration. Kinematics serves as a prelude to dynamics which studies force as the cause of changes in motion.

1.1 Frame of Reference and Particle

1.1.1 Frame of reference and coordinate system

The world we live in is made of matter, from the largest bodies, such as the Earth, one of the eight major planets in solar system of which the Sun is in the center, the galaxies in which the sun is in one of the spiral arms, and the entire universe, to the smallest particles, such as molecules, atoms and subatomic particles: electrons and nucleus composed of protons and neutrons. Each proton and each neutron is made of two kinds of quarks called up quark and down quark. Although the objects above differ in size by a factor of more than 10^{40}, they have a universality, being in endless motion, and from this point of view, we say that the motion is absolute.

In the remainder of this chapter we shall discuss the position, speed, and acceleration of various objects. To do this scientifically, the first two questions we must answer are: "What position with respect to?" and "What velocity with respect to?". If we choose different objects as the reference to describe the motion of a given body, the indications will be different. For example, if you stand on the ground in a train station and let a ball drop freely from your hand, the motion of the ball seems to be along straight line by you, but along a parabola of trajectory projected horizontally by the observer seating in a moving coach passing through the station. From this point of view, we say that motion is a relative concept and it must always be referred to a particular body that serves as a reference chosen by the observer. Since different observers may use different frames of reference, it is important to know how observations made by different observers are related. For example, when we discuss motions on the surface of the earth, this is the most cases in our course, and then it is convenient to take the earth's surface as our frame of reference. For the motion of the earth or other planets, a particular set of stars, for instance, sun is a good choice, whereas for the motion of the electrons in an atom, the nucleus of the atom is preferred. You are free to choose the frame of reference, but in all cases it is necessary to specify what reference frame is being used and you must always be aware of your choice and be careful to make all your measurements with respect to it.

In physics a frame of reference is usually pictured in terms of a coordinate system, consisting of three mutually perpendicular axes, called the x, y and z axis, relative to which position in space, velocity, acceleration and orbit can be specified. These three axes intersect at the origin O of the coordinate system. In Fig. 1-1, let us consider two observers, one of them on the sun and the other on the earth. Both observers are studying the same motion of an artificial satellite of the earth. To the observer on the earth using frame $x'y'z'$, the satellite appears to describe an almost circular path around the earth. To the solar observer using frame xyz, the satellite's orbit appears as a wavy line.

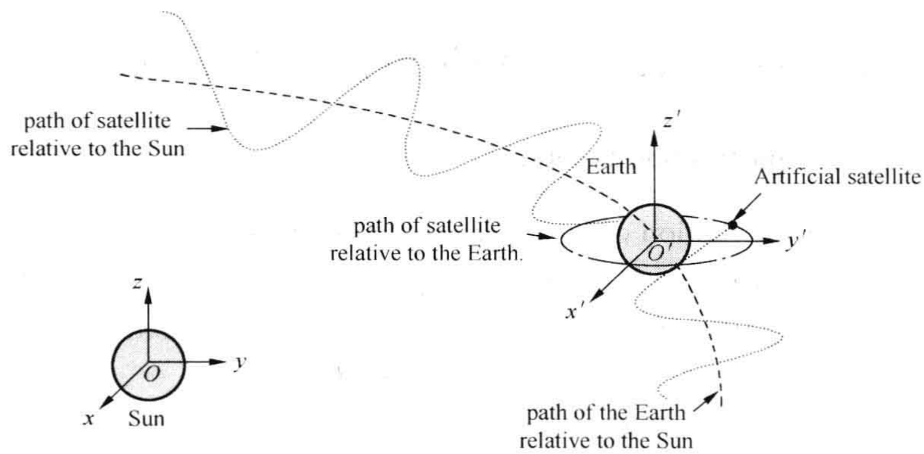


Fig. 1-1 The orbits of a satellite relative to the Earth and to the Sun

1.1.2 Particles

The moving objects that we might examine are among countless possibilities. We shall restrict our concentration on a simple case—translational motion of a particle first, which is defined as the change of position of the particle as a function of time. In the case of an ideal particle—a body with mass, but with no size and no shape, therefore position as a function of time gives a complete description. We can represent an object as a particle (that is as a mass point) if every small part of the object moves in exactly the same way. The concept of particle is an ideal model, the motion of objects are usually more complicated. In some circumstances we are not interested in the size, orientation, and internal structure of a body, and then we can treat the body as a particle, concentrate on its translational motion and ignore all the other motions. For example, we can describe the motion of a ship sailing down a river or a car traveling on a street as a particle motion—for most purposes it is sufficient to know the position of the center of the ship or the car as a function of time.

You must be aware that an object can be treated as a particle in one situation but not in another. The earth behaves pretty much like a particle if we are interested only in its orbital motion around the sun. If we study the rotation of the earth revolving on its own axis, however, the earth is not a particle at all.

It is a very useful method in physics to simplify an object as an ideal model which helps us to solve the major problem in a subject. You will use more ideal models in the other parts of this course.

1.1.3 Time interval and time

It is necessary to distinguish two concepts, time interval and time. When we say time in physics, we mean a give instant. For example, some scheduled flight takes off at 8:00 am from Beijing, lands at 11:00 am on Kunming, 8 o'clock is an instant and so is 11 o'clock. The 3 hours that the whole flying lasts is a time interval. The position of a moving particle is corresponding to a given instant labeled with t while the distance it passed is corresponding to a given time interval labeled with Δt.

1.2 Displacement, Velocity, and Acceleration

1.2.1 Position vector and Position function

When we describe the motion of a particle, the first question is: "Where is it?". In three dimensional world, we need a vector to answer this question. We locate a particle by a vector $\boldsymbol{r}$, extending from the origin of the coordinate system to the particle's position as in Fig. 1-2. Thus,

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{1-1}$$

in which, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors and x, y and z are the components of the vector $\boldsymbol{r}$. The components can be positive, negative or zero.

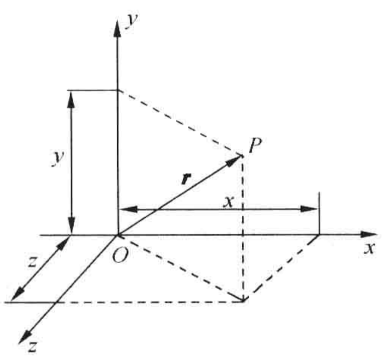


Fig. 1-2 Position vector

We shall define position, displacement, velocity and acceleration for the general case of three dimensions. To simplify the figures, we shall illustrate them in two dimensions in the rest of this chapter.

Mechanical motion is defined as the process of changes in position with time. In principle, the position vector can be correlated with the time by means of a vector function

$$\boldsymbol{r} = \boldsymbol{r}(t) \tag{1-2a}$$

Its three components are written by the following scalar functions

$$x = x(t), \quad y = y(t), \quad z = z(t) \tag{1-2b}$$

Eq. (1-2a) or Eq. (1-2b) is defined as the position function that determines the location of a particle at any given time. Combining Eq. (1-1) and Eq. (1-2b), we have

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} \tag{1-3}$$

which is equivalent with Eq. (1-2a) and Eq. (1-2b).

The path equation can be obtained by eliminating t from Eq. (1-2b)

$$f(x, y, z) = 0$$

If the path of a particle is a straight line, the motion is called as a rectilinear motion; if the path is a curve, the motion is called as a curvilinear motion.

1.2.2 Displacement

Displacement is the change in position during a given time interval. In Fig. 1-3, at time t, the particle is at point A, given by position vector $\boldsymbol{r} = \overrightarrow{OA}$. At a later time t_1, the particle will be at B with $\boldsymbol{r}_1=\overrightarrow{OB}$. Although the particle has moved along the arc $\widehat{AB}=\Delta s$, the displacement is the vector given by

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

or

$$\Delta\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r} \tag{1-4}$$

note that the displacement indicates the change in position not in the path length over the same time interval. Displacement is a vector, its magnitude $|\Delta\boldsymbol{r}|$ is the length of the chord AB; the path is a scalar Δs, the length of the arc AB. In most cases, $|\Delta\boldsymbol{r}|\neq\Delta s$ (Fig. 1-3), only in the limiting case of $\Delta t\to 0$, $|\Delta\boldsymbol{r}|$ can be regarded equal to Δs. For example, a man walks from point A along the rim of a circle of radius R for half a round, his displacement is $2R$, but path is πR. A particle moves back and forth in x axis for one period, its displacement is zero, but path equals to $2A$ (A is the amplitude). You should also be aware of the difference between $|\Delta\boldsymbol{r}|$ and Δr.

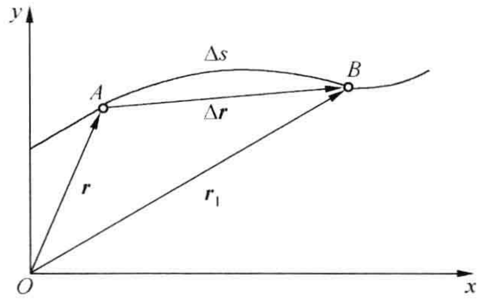


Fig. 1-3 Displacement $\Delta\boldsymbol{r}$ during time interval Δt

1.2.3 Velocity

The second question to describe the motion of a particle is: "How fast is the change of positions?". If $\Delta\boldsymbol{r}$ is the displacement that occurs during the time interval Δt, the average velocity for this interval is defined as

$$\bar{\boldsymbol{v}}=\frac{\Delta\boldsymbol{r}}{\Delta t}$$

The direction of average velocity points in the same direction of displacement (Fig. 1-3); the magnitude of it equals $|\Delta\boldsymbol{r}|/\Delta t$. Obviously, average velocity is related to the specified time interval Δt, and it takes into account only the net displacement in the time interval Δt, ignores the details of the motion, and gives no credits for back and forth motion or the length of the path.

To describe the motion of a particle at a given time t or at a given point, we must make Δt very small. The instantaneous velocity at time t is obtained by evaluating $\Delta\boldsymbol{r}/\Delta t$ in the limit that Δt approaches zero

$$\boldsymbol{v}=\lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{r}}{\Delta t}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \tag{1-5}$$

Thus, the instantaneous velocity is defined as the time derivation of the position vector.

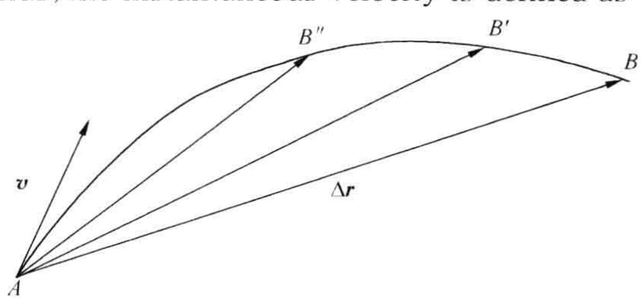


Fig. 1-4 Velocity is tangent to the path at A

Direction of instantaneous velocity:

To determine the direction of instantaneous velocity $\boldsymbol{v}$ at point A, let us see Fig. 1-4. When Δt approaches 0, point B approaches point A, as indicated by B', B'', … with the vector $\overrightarrow{AB}$ changing continuously in both magnitude and direction, in the limit when B

is very close to A, $\overrightarrow{AB}=\Delta\boldsymbol{r}$ coincides in direction with the tangent at A, therefore, the instantaneous velocity is a vector tangent to the path, and points to the advance direction.

Magnitude of instantaneous velocity :

Substituting $\boldsymbol{r}$ from Eq. (1-3) into Eq. (1-5) gives

$$\boldsymbol{v}=\frac{\mathrm{d}}{\mathrm{d}t}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})=\frac{\mathrm{d}x}{\mathrm{d}t}\boldsymbol{i}+\frac{\mathrm{d}y}{\mathrm{d}t}\boldsymbol{j}+\frac{\mathrm{d}z}{\mathrm{d}t}\boldsymbol{k} \tag{1-6}$$

or

$$\boldsymbol{v}=v_x\boldsymbol{i}+v_y\boldsymbol{j}+v_z\boldsymbol{k} \tag{1-7}$$

As we see, the three components of the velocity vector are given by

$$v_x=\frac{\mathrm{d}x}{\mathrm{d}t},\quad v_y=\frac{\mathrm{d}y}{\mathrm{d}t},\quad v_z=\frac{\mathrm{d}z}{\mathrm{d}t} \tag{1-8}$$

And the magnitude of the velocity is

$$v=\sqrt{v_x^2+v_y^2+v_z^2} \tag{1-9}$$

Fore the case of the motion in a plane, angle θ formed between $\boldsymbol{v}$ and $+x$ direction in determined by $\tan\theta=v_y/v_x$ as shown in Fig. 1-5, usually used to indicate the direction of velocity.

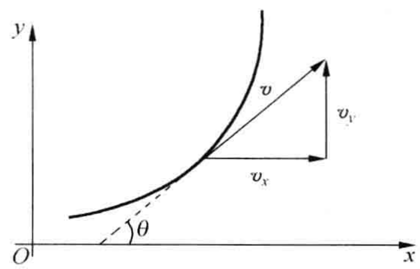


Fig. 1-5 Velocity in two dimensions

Velocity and speed:

On the other hand, the magnitude of velocity vector can be written as

$$v=|\boldsymbol{v}|=\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|=\lim_{\Delta t\to 0}\frac{|\Delta\boldsymbol{r}|}{\Delta t} \tag{1-10}$$

Let Δs represent the path length over Δt, which is given by the length of the arc AB (Fig. 1-3), and the closer B is to A, the closer magnitude of $\Delta\boldsymbol{r}$ is to Δs, that is

$$\lim_{\Delta t\to 0}\frac{|\Delta\boldsymbol{r}|}{\Delta s}=1$$

Therefore

$$v=\lim_{\Delta t\to 0}\frac{|\Delta\boldsymbol{r}|}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=\frac{\mathrm{d}s}{\mathrm{d}t} \tag{1-11}$$

Where $\Delta s/\Delta t$, the path length divided by the time taken, is called the average speed, so $\mathrm{d}s/\mathrm{d}t$ is the instantaneous speed. Note that speed is a scalar, and Eq. (1-11) means that the magnitude of instantaneous velocity equals instantaneous speed, which can be briefly called as velocity and speed.

The unit of speed is m/s, that is, meter per second in SI system.

Example 1-1 The position of a particle moving in x-y plane is given by $x=R+R\cos\omega t$, $y=R\sin\omega t$, here $R=1\mathrm{m}$, $\omega=(\pi/4)\mathrm{s}^{-1}$. Calculate:

(1) the path function $f(x, y)=0$;

(2) velocity at any time;

(3) position vector at $t=0$ and $t=6\mathrm{s}$, the displacement $\Delta\boldsymbol{r}$ and path length Δs during this time interval.

Solution (1) Rearrange the position function as

$$x - R = R\cos\omega t, \quad y = R\sin\omega t$$

then we have

$$(x - R)^2 + y^2 = R^2$$

This is the path function of a circle with radius R and the position of center locates at $(R, 0)$ as Fig. 1-6 shows.

(2) From Eq. (1-8)

$$v_x = \frac{dx}{dt} = -R\omega\sin\omega t$$

$$v_y = \frac{dy}{dt} = R\omega\cos\omega t$$

$$\boldsymbol{v} = -R\omega\sin\omega t\boldsymbol{i} + R\omega\cos\omega t\boldsymbol{j}$$

$$v = \sqrt{v_x^2 + v_y^2} = R\omega = \frac{\pi}{4}\text{m/s}$$

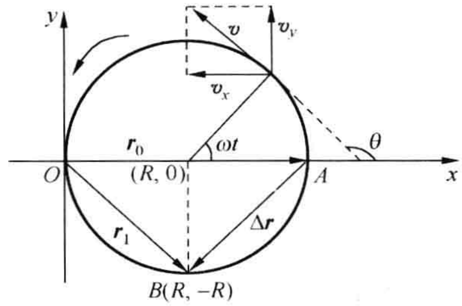


Fig. 1-6 For Example 1-1

which means that the motion is a circular motion with constant speed. The angle θ between $\boldsymbol{v}$ and $+x$ direction is given by

$$\tan\theta = v_y / v_x = -\cot\omega t$$

By inspection of the signs of v_x and v_y at the particular time, you can determine which quadrant the angle is in.

(3) When $t = 0$, we have

$$\boldsymbol{r}_0 = 2R\boldsymbol{i}$$

represented by $\overrightarrow{OA}$, and at $t = 6$ s

$$\boldsymbol{r}_1 = \left(R + R\cos\frac{3}{2}\pi\right)\boldsymbol{i} + R\sin\frac{3}{2}\pi\boldsymbol{j} = R\boldsymbol{i} - R\boldsymbol{j}$$

represented by $\overrightarrow{OB}$, the displacement during $\Delta t = 6$s is

$$\Delta\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_0 = -R\boldsymbol{i} - R\boldsymbol{j}$$

represented by $\overrightarrow{AB}$ in Fig. 1-6.

$$|\Delta\boldsymbol{r}| = \sqrt{(-R)^2 + (-R)^2} = \sqrt{2}R = 1.41\text{ m}$$

While the path length during the same Δt is

$$\Delta s = \text{arc }\widehat{AOB} = \frac{3}{2}\pi R = 4.71\text{ m}$$

1.2.4 Acceleration

The path of a particle moving in two or three dimensions is a curve in general, its velocity changes in both of magnitude and direction. The magnitude of the velocity changes when the particle speeds up or slows down. The direction of the velocity changes because the velocity is tangent to the path and the path bend continuously. Fig. 1-7 indicates the velocity $\boldsymbol{v}$ at time t, and $\boldsymbol{v}_1$ at t_1, corresponding to the position A and B, respectively. The change in velocity during the time interval $\Delta t = t_1 - t$, is represented by $\Delta\boldsymbol{v}$ in the vector triangle in which $\boldsymbol{v} + \Delta\boldsymbol{v} = \boldsymbol{v}_1$, then $\Delta\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}$. To describe the average rate of change in velocity for the time interval Δt, the average acceleration is defined by

$$\bar{\boldsymbol{a}} = \frac{\Delta \boldsymbol{v}}{\Delta t}$$

Using the same method as in definition of the velocity, the instantaneous acceleration at time t, refereed simply as acceleration is given by

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{1-12}$$

which is the time derivation of velocity vector.

Direction of acceleration:

Acceleration vector has the same direction as the limit direction of the change in velocity when $\Delta t \to 0$, which is always pointing toward the concavity of the curve, and because $\Delta \boldsymbol{v}$ is always in the direction in which the curve bends, as shown in Fig. 1-7. Suppose that the direction of acceleration is at an angle of α to the velocity, $\alpha < 90°$, $\alpha > 90°$, and $\alpha = 90°$ corresponding to the cases of $|\boldsymbol{v}_1| > |\boldsymbol{v}|$, $|\boldsymbol{v}_1| < |\boldsymbol{v}|$, and $|\boldsymbol{v}_1| = |\boldsymbol{v}|$, respectively. It is important to be aware that there is an acceleration whenever the velocity changes in either magnitude or direction.

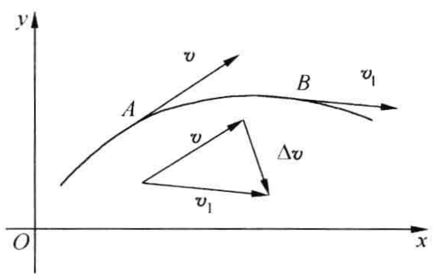


Fig. 1-7 Acceleration in curvilinear motion

Magnitude of acceleration:

From Eq. (1-5), we can also write Eq. (1-12) in the form

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \tag{1-13}$$

Substituting Eq. (1-3) into Eq. (1-13) gives

$$\boldsymbol{a} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\boldsymbol{i} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\boldsymbol{j} + \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\boldsymbol{k}$$

or

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \tag{1-14}$$

The three components of acceleration are given by

$$a_x = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \quad a_y = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \quad a_z = \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} \tag{1-15}$$

And the magnitude of the acceleration is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1-16}$$

The unit of acceleration is $\mathrm{m/s^2}$ in SI system.

In the case of a motion in x-y plane, suppose φ is the angle formed by $\boldsymbol{a}$ and $+x$ direction, thus

$$\tan\varphi = \frac{a_y}{a_x} \tag{1-17}$$

Example 1-2 Suppose the position function is the same as in Example 1-1. Find the acceleration at any time.

Solution From the result of Example 1-1, we have

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = -R\omega^2\cos\omega t$$

$$a_y = \frac{dv_y}{dt} = -R\omega^2 \sin \omega t$$

$$a = \sqrt{a_x^2 + a_y^2} = R\omega^2 = \frac{v^2}{R} \text{ m/s}^2 = 0.62 \text{ m/s}^2$$

which means that the magnitude of $\boldsymbol{a}$ is a constant, the direction of it can be represented by angle α between $\boldsymbol{a}$ and $+x$ direction, and

$$\tan\alpha = \frac{a_y}{a_x} = \tan \omega t$$

For example, if $t = 3$ s, $\tan \alpha = \tan \frac{3}{4}\pi = -1$, because $a_x > 0$, $a_y < 0$, so that $\alpha = -\pi/4$, in the forth quadrant, as shown in Fig. 1-8. On the other hand, we can rewrite a_x and a_y as

$$a_x = -(x-R)\omega^2, \quad a_y = -y\omega^2$$

that is

$$\boldsymbol{a} = -\omega^2[(x-R)\boldsymbol{i} + y\boldsymbol{j}]$$

Note that, there is a vector

$$\boldsymbol{R} = (x-R)\boldsymbol{i} + y\boldsymbol{j}$$

which is pointing from the center of the circle to the position of particle in Fig. 1-8, therefore

$$\boldsymbol{a} = -\omega^2\boldsymbol{R}$$

which means the acceleration is always pointing toward the center of the circle. So it is called as centripetal acceleration.

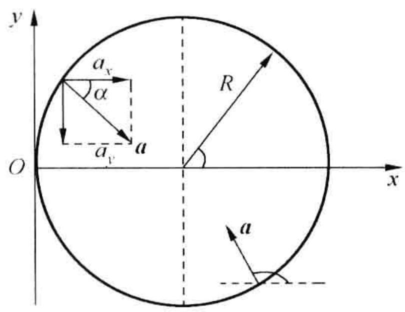


Fig. 1-8 For Example 1-2

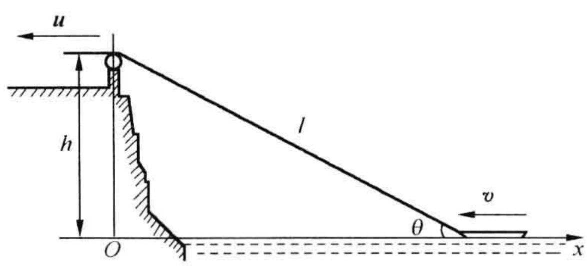


Fig. 1-9 For Example 1-3

Example 1-3 A seaman standing on a cliff, pulls a boat by a pulley, as shown in Fig. 1-9. Suppose that the height of the cliff is h, the rate of the rope pulled is u. Find:

(1) the velocity of the boat;

(2) the acceleration of the boat.

Solution Because the motion of the boat is in one dimension, set x axis pointing right, choose the origin at the foot point of the pulley, and let l representing the variable length of the rope at any time. So that, position vector of the boat is $\boldsymbol{r} = x\boldsymbol{i}$, note that $x = \sqrt{l^2 - h^2}$ in which x and l are changing with time, take time derivation of x, we have

$$\frac{dx}{dt} = \frac{l\frac{dl}{dt}}{\sqrt{l^2 - h^2}} = -\frac{lu}{x}$$

here $\frac{dl}{dt}=-u$ is the rate of rope shortened, then the speed of the boat is

$$v=\frac{dx}{dt}=-\frac{lu}{x}=-u\frac{\sqrt{x^2+h^2}}{x}$$

which is the velocity as a function of coordinate x.

The negative sign indicates the velocity is in the $-x$ direction. The acceleration of the boat is then

$$a=\frac{dv}{dt}=-\frac{u^2h^2}{x^3}$$

which is the acceleration as a function of coordinate. The results indicate that $\boldsymbol{a}$ and $\boldsymbol{v}$ are in the same direction of $-x$ in Fig. 1-9, so that, the magnitude of $\boldsymbol{v}$ becomes larger and larger with the value of x decreased, in the other words, the boat is moving with speed increased.

1.3 Two Types of Problems in Kinematics and Sample Problems

The problems solved by means of vector derivation:

From Example 1-1 and Example 1-2, we can make conclusion that if we know the position vector of the moving particle as a function of time, $\boldsymbol{r}=\boldsymbol{r}(t)$, according to Eq. (1-8) and Eq. (1-15), by taking derivation of position function with respect to time, we can get the velocity and the acceleration.

The problems solved by means of vector integration:

In the opposite cases, if we know the acceleration $\boldsymbol{a}=\boldsymbol{a}(t)$, or velocity $\boldsymbol{v}=\boldsymbol{v}(t)$ as a function of time, we can rewrite Eq. (1-12) and Eq. (1-5) as

$$d\boldsymbol{v}=\boldsymbol{a}(t)dt,\quad d\boldsymbol{r}=\boldsymbol{v}(t)dt$$

In order to find $\boldsymbol{v}$and $\boldsymbol{r}$ as the functions of the time t, we must know the velocity and position at some particular time, for instant, when $t=t_0$, $\boldsymbol{v}=\boldsymbol{v}_0$, $\boldsymbol{r}=\boldsymbol{r}_0$, which are called as the initial conditions, then by integrating above equations in both sides

$$\int_{v_0}^{v} d\boldsymbol{v}=\int_{t_0}^{t}\boldsymbol{a}(t)dt \tag{1-18}$$

$$\int_{r_0}^{r} d\boldsymbol{r}=\int_{t_0}^{t}\boldsymbol{v}(t)dt \tag{1-19}$$

We can obtain velocity and position vector as a function of time. It is important to have your attention that Eq. (1-18) and Eq. (1-19) are vector integrations composed of three scalar integrations, but in practical problems the integration usually can be simplified.

Sometimes the acceleration is known as the function of coordinate, for example, $a=a(x)$ is known, separate the variables and integrate, that is

$$a=a(x)=\frac{v}{v}\frac{dv}{dt}=\frac{vdv}{dx}$$

then

$$a(x)dx=vdv$$

The function $v = v(x)$ can be obtained. Problem 1-7, 1-10 and Example 1-5 etc. give several different cases to solve kinematics problems by means of separating the variables and integrating as well.

1.3.1 Motion with constant acceleration

If the acceleration vector in Eq. (1-18) remains constant, and when $t = t_0$, $\boldsymbol{v} = \boldsymbol{v}_0$ and $\boldsymbol{r} = \boldsymbol{r}_0$ are known, the result of the integration is given by

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{a}(t - t_0) \tag{1-20}$$

Substitute Eq. (1-20) into Eq. (1-19), and make integration, the position function

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0(t - t_0) + \frac{1}{2}\boldsymbol{a}(t - t_0)^2 \tag{1-21}$$

obtained. Generally speaking, Eq. (1-20) and Eq. (1-21) each has three components, like in Eq. (1-1) and Eq. (1-7).

Lineal motion with constant acceleration:

Obviously, one dimensional motion or lineal motion, with constant acceleration is just a special case included in the discussion above. Let the linear path of the motion along x axis, if the initial condition known as $v = v_0$, $x = x_0$ when $t = t_0$, we can use one dimensional integration from

$$\mathrm{d}v = a\mathrm{d}t, \quad \mathrm{d}x = v\mathrm{d}t$$

Or we can simply take the x component of Eq. (1-20) and Eq. (1-21) to describe the motion, the same results will be obtained as

$$v = v_0 + a(t - t_0) \tag{1-22}$$

$$x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \tag{1-23}$$

Eliminate time t from above equations, we have another useful equation

$$v^2 = v_0^2 + 2a(x - x_0) \tag{1-24}$$

Note that:

(1) About the initial time: often, we choose $t_0 = 0$, which is the instant when we begin to observe the motion, and is not certainly the instant when the motion starts.

(2) About the sign of a, v and x: all quantities in Eq. (1-22) to Eq. (1-24) are algebraic numbers, whether they are positive or negative depends on their direction compared with the directions of the axes chosen in the coordinate system, if the direction is along the $+x$ direction, then it is positive; otherwise, negative.

Example 1-4 Projectile motion and Superposition principle.

An important example of motion under constant acceleration is the motion of a projectile: near the surface of the Earth, the pull of gravity gives a freely falling body a downward acceleration g of about 9.8 m/s^2. If we ignore air resistance, this is the only acceleration that a body experiences when launched from some initial position with some initial velocity $\boldsymbol{v}_0$. The motion mentioned above, is therefore the motion with constant vertical acceleration and zero horizontal acceleration, which is called projectile motion. Our projectile might be a golf ball, a base ball, a canon ball or a bullet, among many possibilities. We analyze in general case under no-atmosphere condition.

Solution (1) If we choose the x-y plane coincident with the plane defined by $\boldsymbol{v}_0$ and $\boldsymbol{a} = \boldsymbol{g}$, the y-axis directs upward and the origin O coincides with the initial position of the projectile, so that, we have $\boldsymbol{g} = -g\boldsymbol{j}$, $x_0 = 0$, $y_0 = 0$, at $t_0 = 0$. Suppose that $\boldsymbol{v}_0$ is at an angle of θ with the horizontal (that is x axis) as in Fig. 1-10, so the components of initial velocity are

$$v_{0x} = v_0 \cos\theta, \quad v_{0y} = v_0 \sin\theta \tag{1-25}$$

and the acceleration components are

$$a_x = 0, \quad a_y = -g \tag{1-26}$$

by integrating Eq. (1-18) and Eq. (1-19) for this case or substituting Eq. (1-25) and Eq. (1-26) into Eq. (1-22) and Eq. (1-23), we have the velocity at time t

$$v_x = v_0 \cos\theta, \quad v_y = v_0 \sin\theta - gt \tag{1-27}$$

and the position as a function of time

$$x = v_0 \cos\theta \cdot t, \quad y = v_0 \sin\theta \cdot t - \frac{1}{2}gt^2 \tag{1-28}$$

These results indicate that the motion of projectile can be regarded as the superposition of two independent rectilinear motions: a vertical motion with constant acceleration and a horizontal motion with constant speed, which is an application of the superposition principle of motion: **any motion can be regarded as superposition of several independent motions** that plays a very important role in physics.

(2) Some consequences of the motion of projectile:

(a) The equation of the path is obtained by eliminating time t from the two equations in Eq. (1-28)

$$y = x\tan\theta - \frac{gx^2}{2v_0^2 \cos^2\theta} \tag{1-29}$$

This is the equation of a parabola since both $\tan\theta$ and $g/(2v_0{}^2\cos^2\theta)$ are constants, and is also called trajectory as shown in Fig. 1-10.

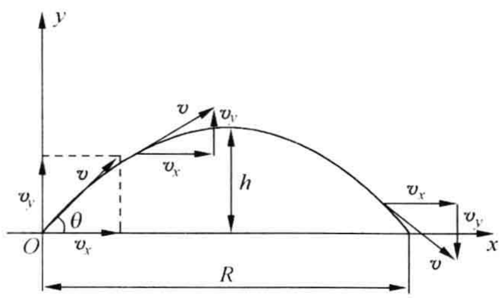


Fig. 1-10 Projectile motion

(b) The total time of flight can be obtained by making $y = 0$ in Eq. (1-28)

$$T = \frac{2v_0 \sin\theta}{g} \tag{1-30}$$

which is twice the time t_h required to reach the projectile's highest point, set $v_y = 0$ in Eq. (1-27), we have

$$t_h = v_0 \sin\theta / g$$

(c) The maximum height h is calculated by substituting t_h into the second function in Eq. (1-28)

$$h = \frac{v_0^2 \sin^2\theta}{2g} \tag{1-31}$$

(d) The range R is the horizontal distance covered and is found by substituting the value of flight time T into the first equation of Eq. (1-28)

$$R = \frac{2v_0^2 \sin\theta\cos\theta}{g} \tag{1-32}$$

You can prove that R has a maximum value at $\theta = 45°$, for a given magnitude of initial velocity.

Note that Eq. (1-29) ~ Eq. (1-32) hold only in the case that launch point and impact point are at the same level ($y = 0$).

You should be aware that above results about projectile motion are valid when (a) the range is small enough so that the curvature of the earth may be neglected, (b) the altitude is small enough so that the variation of gravity with height may be neglected, and (c) the initial velocity is small enough so that air resistance may be neglected. If the actual situation is beyond these restrictions the disagreement between our calculation and the actual situation of the projectile will become quite large. Fig. 1-11 is a rough sketch of the path of long-range projectile, which is not a parabola but an arc of an ellipse.

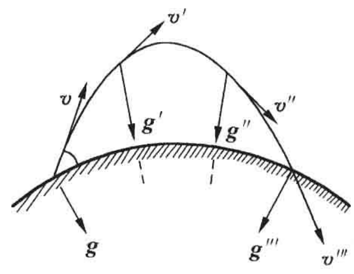


Fig. 1-11 The path of long range projectile is not a parabola

1. 3. 2 Sample of motion with variable acceleration

Example 1-5 A parachute with a sportsman falls from a helicopter, and its acceleration is $\boldsymbol{a} = \boldsymbol{g} - b\boldsymbol{v}$ (b is a constant) caused by both gravity and air resistance. Assume that $v_0 = 0$ at $t_0 = 0$, find the velocity and position as function of time.

Solution This problem can be regarded as an one dimensional motion with variable acceleration. Set the origin at the initial position of falling, the $+y$ axis downwards.

(1) From Eq. (1-13), we have

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = g - bv$$

Separate variables, rewrite the above as $\frac{\mathrm{d}v}{g - bv} = \mathrm{d}t$ and note that $v_0 = 0$ when $t = 0$, then make integration in both sides

$$\int_0^v \frac{\mathrm{d}v}{g - bv} = \int_0^t \mathrm{d}t$$

so that, the velocity at any time is given by

$$v = \frac{g}{b}(1 - \mathrm{e}^{-bt})$$

(2)From Eq. (1-8), we have $\mathrm{d}y = v\mathrm{d}t$ substitute above result into it, by integrating, the position function of time $y = \int_0^t \frac{g}{b}(1 - \mathrm{e}^{-bt})\mathrm{d}t = \frac{g}{b}\left(t + \frac{1}{b}\mathrm{e}^{-bt} - \frac{1}{b}\right)$ obtained.

1. 4 Circular Motion

1. 4. 1 Tangential and normal components of acceleration

Fig. 1-12 represents a particle moving in a circular path of radius R with center at O. Suppose the particle is at A with velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ at time t . Since $\boldsymbol{a}$ is pointing toward the concave side of the path, we can decompose it into two perpendicular components: a tangential component a_t called tangential acceleration, and a normal component a_n called normal acceleration. Each of these components has a well defined physical meaning. When the particle moves, its velocity may change, the change in magnitude is related to the tangential acceleration a_t, and the change in direction is related to the normal acceleration a_n.

In Fig. 1-12(a), draw the unit vector $\boldsymbol{\tau}$ tangent to the circle at A. The velocity is expressed as $\boldsymbol{v} = v\boldsymbol{\tau}$, thus the acceleration is

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v\boldsymbol{\tau}) = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} + v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} \tag{1-33}$$

To find the direction of $\mathrm{d}\boldsymbol{\tau}$, let us introduce the unit vector $\boldsymbol{n}$, normal to the circle at A and directing toward the center O. When the particle is at B after $\mathrm{d}t$, the unit vector tangent to the circle is $\boldsymbol{\tau}'$ and makes an angle $\mathrm{d}\theta$ with $\boldsymbol{\tau}$. From Fig. 1-12 (b) we see that $\mathrm{d}\boldsymbol{\tau} = \boldsymbol{\tau}' - \boldsymbol{\tau}$ has the direction parallel to $\boldsymbol{n}$. Because $|\boldsymbol{\tau}|=1$, so $|\mathrm{d}\boldsymbol{\tau}| = |\boldsymbol{\tau}|\mathrm{d}\theta$, that is

$$\mathrm{d}\boldsymbol{\tau} = \mathrm{d}\theta\,\boldsymbol{n} \tag{1-34}$$

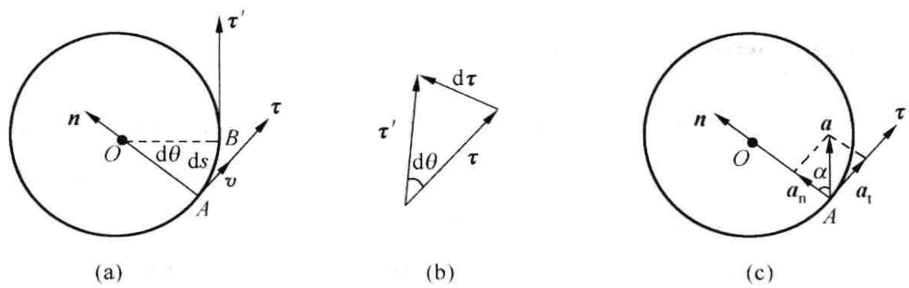


Fig. 1-12 Acceleration of circular motion

thus

$$\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\boldsymbol{n} = \frac{R\mathrm{d}\theta}{R\mathrm{d}t}\boldsymbol{n} = \frac{\mathrm{d}s}{R\mathrm{d}t}\boldsymbol{n} = \frac{v}{R}\boldsymbol{n} \tag{1-35}$$

Substituting Eq. (1-35) into Eq. (1-33), we have

$$\boldsymbol{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} + \frac{v^2}{R}\boldsymbol{n} = a_t\boldsymbol{\tau} + a_n\boldsymbol{n} \tag{1-36}$$

or

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t}, \quad a_n = \frac{v^2}{R} \tag{1-37}$$

in which a_t called as tangential acceleration along to tangent direction of the curve at A is proportional to the time rate of change in magnitude of the velocity, while a_n called as normal acceleration along to normal direction of the curve at A indicates how the direction of the velocity changes .

The magnitude of $\boldsymbol{a}$ is

$$a = \sqrt{a_t^2 + a_n^2} \tag{1-38}$$

The direction of $\boldsymbol{a}$ is represented by the angle α formed by $\boldsymbol{a}$ and $\boldsymbol{n}$ as shown in Fig. 1-12(c)

$$\tan\alpha = \frac{a_t}{a_n} \tag{1-39}$$

1. Uniform circular motion

In this case, the magnitude of velocity is a constant v, or

$$\frac{dv}{dt} = 0$$

so

$$a_t = 0, \quad a = a_n = \frac{v^2}{R}$$

which means that the velocity changes only in direction.

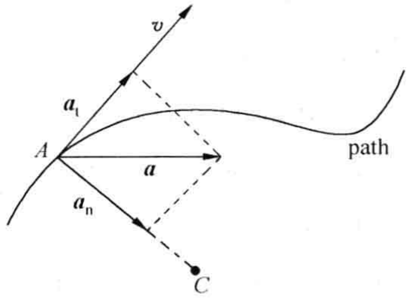


Fig. 1-13 The general case of curvilinear motion

2. General curvilinear motion

In this case, the path is not a circle, we can generalize Eq. (1-37) as

$$a_t = \frac{dv}{dt}, \quad a_n = \frac{v^2}{\rho} \qquad (1\text{-}40)$$

where ρ is the radius of curvature at A as shown in Fig. 1-13. C is called the center of curvature, then $\rho = AC$.

1.4.2 Angular variables in circular motion

Suppose particle P moves in a circular path with the center O, in x-y plane as shown in Fig. 1-14. The position vector $\boldsymbol{r}$ changes only in direction not in magnitude, which remains constant of R—the radius of the circle. Choose a reference line, say Ox, note that position vector $\boldsymbol{r}$ is at an angle θ to Ox, and θ is varies with time corresponding to the position vector's changing, so, specify the angular position as a function of time, that is

$$\theta = \theta(t) \qquad (1\text{-}41)$$

which describes the circular motion. If the arc length corresponding to θ (Fig. 1-14) is s, thus

$$\theta = s/R$$

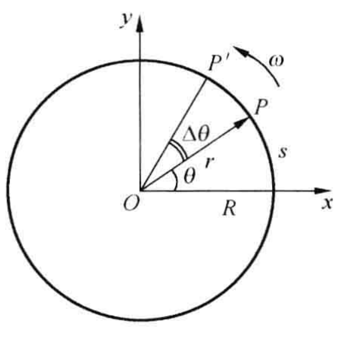


Fig. 1-14 Angular position

The unit of an angle defined in this way is given by radians, or simply given by rad, and

$$1 \text{ revolution} = 360° = 2\pi \text{ rad}$$

If the particle moves along counter clock wise direction, thus $\theta > 0$, otherwise $\theta < 0$.

1. Angular displacement

Suppose that, the particle is at angular position θ_1 at time t_1, and at angular position θ_2 at time t_2, the angular displacement that occurs during the time interval $\Delta t = t_2 - t_1$, is defined as

$$\Delta\theta = \theta_2 - \theta_1$$

the specification of the sign of $\Delta\theta$ is the same as that of angular position.

2. Angular velocity

We define the average angular velocity of the particle to be

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

The instantaneous angular velocity ω is the limit of this ratio as Δt is made to approach zero. thus

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{1-42}$$

If we know $\theta=\theta(t)$, we can find the angular velocity ω by differentiation. The unit of angular velocity is rad /s in SI.

3. Angular acceleration

Let ω_1 and ω_2 be the angular velocities at time t_1 and t_2, respectively, thus $\Delta\omega=\omega-\omega_1$ is the change in the angular velocity that occurs during the time interval $\Delta t= t_2-t_1$. The average angular acceleration of particle is defined from $\bar{\beta} = \Delta\omega/\Delta t$. The instantaneous angular acceleration β is the limit of this quantity as Δt is made to approach zero. That is

$$\beta = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{1-43}$$

If ω and β have the same sign ($\omega>0, \beta>0$ or $\omega<0$, $\beta<0$), which means that ω is increasing with time; otherwise, ω is decreasing with time. The unit of angular acceleration is rad/s^2 in SI system.

4. Motion with constant angular acceleration

In pure translation, motion with a constant linear acceleration is an important special case. The component equations in the series of Eq. (1-22) to Eq. (1-24), for example, the x component equations hold for such motions.

In circular motion, the case of constant angular acceleration is also important and an equivalent set of equations holds for this case. We simply write them down from the corresponding linear equations, by substituting equivalent angular quantities for the linear ones in Eq. (1-22) to Eq. (1-24):

$$\theta \rightarrow x, \quad \omega \rightarrow v, \quad \beta \rightarrow a$$

Then the set of equations of motion with constant angular acceleration is given by

$$\begin{aligned} &\omega = \omega_0 + \beta(t - t_0) && \text{(a)} \\ &\theta = \theta_0 + \omega_0(t - t_0) + \frac{1}{2}\beta(t - t_0)^2 && \text{(b)} \\ &\omega^2 = \omega_0^2 + 2\beta(\theta - \theta_0) && \text{(c)} \end{aligned} \tag{1-44}$$

The method that we use here is called analogy which plays a very important part in physics so that we shall use it quite often in the following parts of this course.

1.4.3 The relations between linear and angular variables

Since we can use the linear variables s, v and a, and the angular variables θ, ω, and β to describe a circular motion of a particle respectively, there must be some relations between them. The two sets of variables are connected by R, the perpendicular distance from the particle to the center of the circular path, from the definition of radian, which is $\theta = s/R$ in Fig. 1-14, we have

$$s = \theta R \tag{1-45}$$

Differentiating Eq. (1-45) with respect to time leads to

$$\frac{ds}{dt} = \frac{d\theta}{dt}R = \omega R$$

that is

$$v = \omega R \tag{1-46}$$

taking time differentiation of Eq. (1-46) leads to

$$\frac{dv}{dt} = \frac{d\omega}{dt}R = \beta R$$

That is the tangential component of linear acceleration

$$a_t = \beta R \tag{1-47}$$

Substituting Eq. (1-46) into Eq. (1-37), we have the normal acceleration

$$a_n = \frac{v^2}{R} = \omega^2 R \tag{1-48}$$

The magnitude of acceleration

$$a = \sqrt{a_t^2 + a_n^2} = R\sqrt{\beta^2 + \omega^4} \tag{1-49}$$

The direction of $\boldsymbol{a}$ can be determined from

$$\tan\theta = \frac{a_t}{a_n} = \frac{\beta}{\omega^2} \tag{1-50}$$

The series equations above will be applied in chapter 5.

Example 1-6 A particle is moving along a circular path of radius R, the path function is given by $s = v_0 t - \frac{1}{2}bt^2$ (v_0 and b are positive constants). Find

(1) a_t and a_n at any time t;

(2) at what time when $|\boldsymbol{a}| = b$?

Solution (1) From Eq. (1-11)

$$v = \frac{ds}{dt} = v_0 - bt$$

We have

$$a_t = \frac{dv}{dt} = -b, \quad a_n = \frac{v^2}{R} = \frac{(v_0 - bt)^2}{R}$$

(2) let $a = \sqrt{a_t^2 + a_n^2} = b$, that means $a_n = 0$, so that $t = v_0/b$.

1.5 Relative Motion

We mentioned in section1-1 that motion is relative — the descriptions of the position, velocity and acceleration of a particle depend on the frame of reference in which these quantities are measured. In the situation shown in Fig. 1-15, we assume that frame B, say a train coach, moves with respect to frame A, say the earth, their x and y axes remain parallel to each other. Two observers are watching a moving particle P, for example, a car. $\boldsymbol{r}_{PA}$ and $\boldsymbol{r}_{PB}$ are the position vectors of P with respect to frame A and B, respectively. $\boldsymbol{r}_{AB}$ is the position vector of origin O' of frame B with respect to origin O of frame A. From the vector triangle, we have

$$\boldsymbol{r}_{PA} = \boldsymbol{r}_{PB} + \boldsymbol{r}_{BA} \tag{1-51}$$

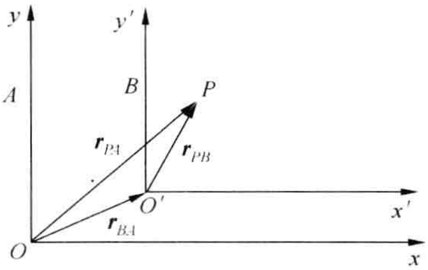


Fig. 1-15 Frame B is moving with respect to frame A

Take the time derivative of Eq. (1-51)

$$\frac{d\boldsymbol{r}_{PA}}{dt}=\frac{d\boldsymbol{r}_{PB}}{dt}+\frac{d\boldsymbol{r}_{BA}}{dt} \tag{1-52}$$

We obtain

$$\boldsymbol{v}_{PA}=\boldsymbol{v}_{PB}+\boldsymbol{v}_{BA} \tag{1-53}$$

Which means that the velocity of P measured by the observer in frame A is equals to the velocity of P measured by the observer in frame B plus the velocity of frame B measured by frame A. The term $\boldsymbol{v}_{BA}$ is the velocity of frame B with respect to frame A. Eq. (1-53) is usually called **theorem of velocity addition.**

Take the time derivative of Eq. (1-53), we have

$$\frac{d\boldsymbol{v}_{PA}}{dt}=\frac{d\boldsymbol{v}_{PB}}{dt}+\frac{d\boldsymbol{v}_{BA}}{dt}$$

That is

$$\boldsymbol{a}_{PA}=\boldsymbol{a}_{PB}+\boldsymbol{a}_{BA} \tag{1-54}$$

Eq. (1-54) is usually called **theorem of acceleration addition.**

Example 1-7 The gulf stream current has a velocity of 4. 8 km/h in direction due north. The captain of a motorboat wants to travel on a straight course from city M due east to city N as shown in Fig. 1-16(a). This boat has a speed of 18 km/h relative to the water. In what direction must he head his boat? What will be its speed relative to the shore?

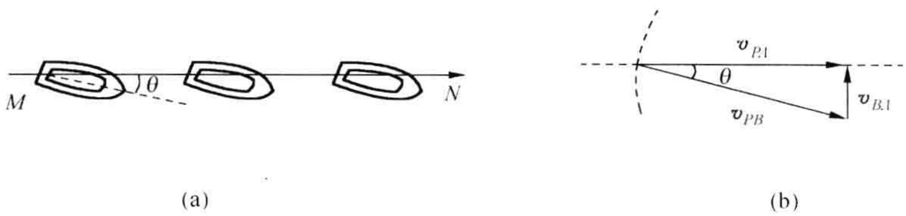


Fig. 1-16 For Example 1-7

Solution Suppose the shore is frame A, the stream is frame B, and the boat is the particle P. using Eq. (1-53), the speed of the boat with respect to the shore represented by v_{PA}, then the speed of the water relative to the shore by v_{BA}, and the speed of the boat relative to the stream by v_{PB}, that is

$$\boldsymbol{v}_{PA}=\boldsymbol{v}_{PB}+\boldsymbol{v}_{BA}$$

we have known that v_{PA} points due east and v_{BA} is due north, so the vector triangle is a right triangle as in Fig. 1-16(b), the angle between v_{PB} and the eastward direction is given by $\sin\theta = v_{BA}/v_{PB}$, so that

$$\theta=\arcsin\frac{v_{BA}}{v_{PB}}=\arcsin\frac{4.8}{18}=15^{\circ}$$

Thus, the boat must be head 15° south of east, the speed of the boat relative to the shore is then $v_{PA}=v_{PB}\cos\theta=18\times\cos15^{\circ}=17(\text{km/h})$.

Questions

1-1 Explain the difference between

(1) $|\Delta \boldsymbol{r}|$ and Δr;

(2) $\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$;

(3) $\left|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|$ and $\frac{\mathrm{d}v}{\mathrm{d}t}$. Draw some sketches and gave some examples.

1-2 The position function is given by $x = x(t)$, $y = y(t)$. Some students calculate $r(t) = \sqrt{x^2(t)+y^2(t)}$ first, then take time derivation for $v=\frac{\mathrm{d}r}{\mathrm{d}t}$, then again, for $a=\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$. Is this the correct way? Why? Give the correct procedure.

1-3 Take some examples and illustrate the motion of a particle for the following cases:

(1) $a_t = 0$;

(2) $a_n = 0$;

(3) $a = 0$;

(4) $a_t = 0$ and a = a constant, at any time.

1-4 What is the position of a projectile where

(1) it has the maximum normal acceleration;

(2) it has the minimum curvilinear radius, what is the value of this minimum if v_{0x} is known?

1-5 The path of a planet is an ellipse as shown in Fig. 1-17, the acceleration is always points to the focus where is the sun. Does the speed is increasing or decreasing when the planet passes points M and N.

1-6 A trajectories are shown in Fig. 1-18 for a football kicked from ground level. Rank the paths according to (a) time of flight; (b) initial vertical velocity component; (c) initial horizontal velocity component; (d) speed at launch, the greatest first (ignore the air resistance).

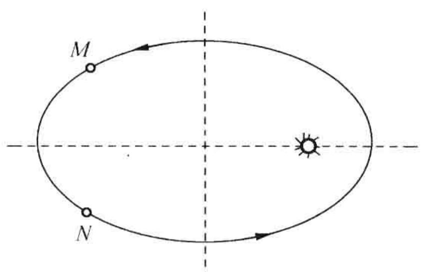


Fig. 1-17 For question 1-5

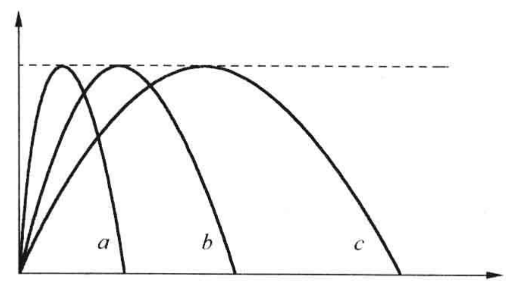


Fig. 1-18 For question 1-6

1-7 A person aims at a toy monkey hanging on a tree with an air-gun, at the moment when he shoots while the toy just falls freely from rest . Explain: why the bullet always shot the falling toy monkey?

1-8 A boy sitting in a railroad car moving at constant velocity, throws a ball straight up into the air.

(1) Will the ball behind him? fall back into his hands ?

(2) What will happen if the car accelerates forward as the ball just thrown ?

(3) What is the path observed (a)by the boy; (b)by a person standing on the ground near the car?

1-9 Whether or not does the following state of motion exist?

(1) $v = 0$, $a \neq 0$ at a given instant;

(2) $v \neq 0$, $a = 0$;

(3) speed = a constant while velocity is changing;

(4) velocity= a constant vector while speed is changing;

(5) velocity in east while acceleration in south;

(6) acceleration is a constant vector while the direction of the motion is continuously changing.

1-10 Suppose that you seat in a car traveling along a straight way with a known speed in the rain with no wind. You'll find that the rain drops strike the windshield forming an angle with the vertical, and measure it. Make a sketch to calculate the speed of the rain related to the ground?

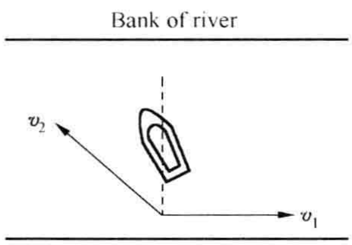


Fig. 1-19 For question 1-11

1-11 A motor boat is headed cross a river. Suppose the velocity of the stream current is v_1, the speed of the boat is v_2 with respect to the water ($v_2 > v_1$). If

(1) the boat is headed tilted in a upstream direction to get a resultant velocity directly toward the opposite bank as shown in Fig. 1-19;

(2) the boat is headed pointing directly at the opposite bank, but reaches the other bank in a downstream direction. In which way (a) the path will be the shortest, (b) the time taken will be the shortest?

Problems

1-1 A particle moves so that its position as a function of time in SI is $\boldsymbol{r}(t) = \boldsymbol{i} + 4t^2\boldsymbol{j} + t\boldsymbol{k}$. Write expressions for (1) its velocity and (2) its acceleration as function of time.

1-2 A man of h in height pulls a sled placed over ground by H as shown in Fig. 1-20, if the man runs with $\boldsymbol{v}_0$, find the velocity and acceleration of the sled.

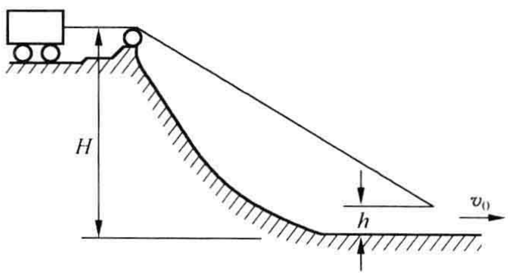


Fig. 1-20 For problem 1-2

1-3 The position function of a particle is given by $\boldsymbol{r} = 2t\boldsymbol{i} + (2-t^2)\boldsymbol{j}$ (SI). Find

(1) the path funetion of the particle;

(2) the position vector at $t=0$ and $t=2$ s;

(3) the displacement from $t=0$ to $t=2$ s.

1-4 The position vector of a particle is given by $\boldsymbol{r} = 2t^2\boldsymbol{i} + (5t-1)\boldsymbol{j} - t^3\boldsymbol{k}$ (SI), find

(1) the displacement from $t_1=1$s to $t_2=2$ s;

(2) the average velocity during the same time interval as in (1);

(3) the instantaneous velocity at t_1 and t_2.

1-5 A particle is moving in x-y plane, its position function is known as

$$x = \sqrt{3}\cos\frac{\pi}{4}t \text{ and } y = \sin\frac{\pi}{4}t \text{(SI)}$$

find

(1) path function, draw the path curve in x-y plane;

(2) the expressions of velocity and acceleration;

(3) the magnitude and direction of the position vector, velocity and acceleration at $t=1$ s.

1-6 A particle moves along a linear path, it's velocity is known as $v = t^3 - 2t^2 + 1$(m/s), if $x=2$ m when $t=1$ s, find

(1) the acceleration and position function;

(2) the acceleration, velocity and position of the particle at $t=2$ s.

1-7 The acceleration of a body moving along x-axis is $a = (4x-2)$ m/s^2 where x is in meter. It is given that $v_0 = 10$ m/s at $x_0 = 0$ m, find the velocity at any other position.

1-8 A particle moves with a constant speed $v=5$m/s, the angle between velocity and x-axis is t (equal to the value of time) rad. The initial position is that $x=0$, $y=5$ m at $t=0$. Find the position function; draw the path in x-y plane.

1-9 A block undergoes back and forth motion along x-axis as shown in Fig. 1-21, and its acceleration $a=-kx/m$ (m is mass, k is spring constant). Suppose $x=A$ when $v=0$, find the velocity at any other positions.

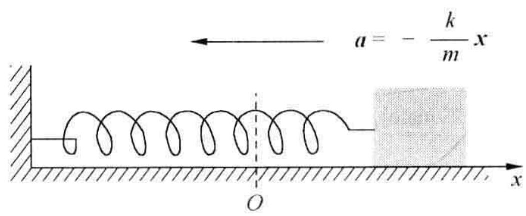


Fig. 1-21 For problem 1-9

1-10 A balloon rises from the ground with initial speed v_0 and initial position $y_0=0$, it is known that its $v_x=by$ (b is a positive constant) due to wind. Find

(1) position vector of the balloon (chose the $+x$ points right);

(2) the function of x coordinate relative to y coordinate;

(3) the function of tangential acceleration a_t and normal acceleration a_n relative to y.

1-11 Known that the acceleration of a moving particle is given by $\boldsymbol{a}=(6\boldsymbol{i}+4\boldsymbol{j})$ m/s^2, $v_0=0$(m/s) and $\boldsymbol{r}_0=10\boldsymbol{i}$ (m) at $t=0$. Calculate

(1) the velocity and position vector at any time;

(2) the path function in x-y plane and sketch the path.

1-12 A elevator is moving up with acceleration of 1. 22 m/s^2, a screw drops down from the ceiling when the speed of the elevator is of 2. 44 m/s. The distance between the ceiling and the floor is 2. 74 m as shown in Fig. 1-22. Calculate

(1) how long does the screw take before it impacts to the floor?

(2) what is the distance does the screw pass with respect to the fixed column outside the elevator.

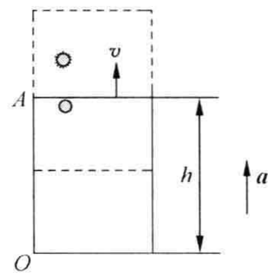


Fig. 1-22 For problem 1-12

1-13 A man throws a ball with a velocity of 32 m/s at an angle of 40° with the ground. Find the velocity and position of the ball after 3 s. Also find the range and the time required for the ball to return to the ground.

1-14 A sportsman is jumping up over a pit with a motorcar as shown in Fig. 1-23 in which $v_0=65$ m/s, $\theta=22.5°$, $H=70$ m. Calculate

(1) the width of the pit;

(2) the time required for the motorcar to impact to the other side of the pit;

(3) the magnitude and direction of the impact velocity.

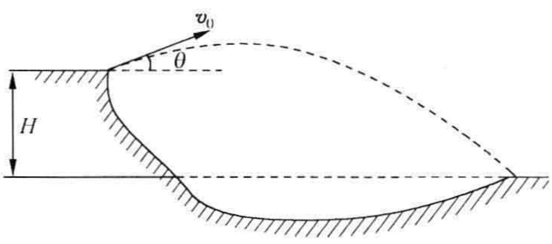


Fig. 1-23 For problem 1-14

1-15 The kicker on a football team can give the ball an initial speed of 25 m/s. Within what angular range must he kick the ball if he is to just score a field goal from a point 50m in front of the goalposts whose horizontal bar is 3. 44 m above the ground?

1-16 A wheel of a grindstone rotates about its central axis, has a constant angular acceleration $\beta=0.35$ rad/s^2, and an initial angular velocity $\omega_0=-4.6$ rad/s. Calculate

(1) at what time t_1 will the grindstone come momentarily to rest?

(2) at what time will the grindstone have turned (in the positive direction) through five revolutions from its initial angular position?

1-17 The Earth rotates on its axis with the radius R of 6370 km. Calculate the speed and acceleration of a point whose location is at φ degree latitude as shown in Fig. 1-24, for

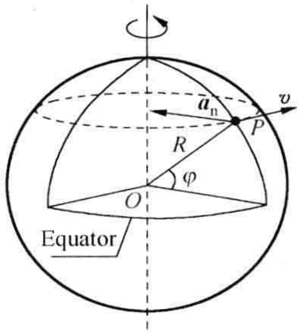


Fig. 1-24 For problem 1-17

(1) Beijing $\varphi_1 = 39°57'$;

(2) Guangzhou $\varphi_2 = 23°00'$ respectively.

1-18 A particle is moving in a circle of radius 20 cm, known that $a_t = 5\ \text{cm/s}^2$, if $v_0 = 0$ at $t = 0$, find the time required for

(1) $a_n = a_t$;

(2) $a_n = 2a_t$.

1-19 A particle undergoes circular motion of $r = 10$ m, the angular acceleration $\beta = \pi\ \text{rad/s}^2$, suppose that $v_0 = 0$ at $t = 0$, find

(1) angular speed;

(2) a_n and a_t;

(3) $\boldsymbol{a}$ (magnitude and direction) at $t = 1$ s.

1-20 A projectile is thrown horizontally with an initial speed $v_0 = 15$ m/s. Calculate a_n and a_t at $t = 1$s.

1-21 A particle is moving along a circular path of radium R, known that the path function is given by $s = bt + 0.5ct^2$ (b and c are positive constants). Find

(1) a_t and a_n at any time t;

(2) at what time when $a_t = a_n$;

(3) how many revolutions has the particle completed during the time interval of $t = 0$ to $t = 2$ s ?

1-22 A train travels due south at 30m/s (relative to ground) in a rain that is blown toward the south by the wind. The path of each raindrop makes an angle of 22° with the vertical, as measured by an observer stationary on the earth. An observer seated in the train, however, sees perfectly vertical tracks of rain on the windowpane. Determine the speed of the raindrops relative to the earth.

1-23 A man swims cross a river of 100m width with the speed of 1.50m/s relative to the water, he finds himself having been brought along the direction of downstream for 50m when he reaches the opposite bank. Find

(1) the time he needs to swim cross the river;

(2) the speed of the water with respect to the bank;

(3) the actual velocity of the man with respect to the bank?

1-24 Two cars A and B are traveling with the speed $v_A = 80$ km/h and $v_B = 100$ km/h, respectively, calculate the velocity of A relative to B in the following situations:

(1) A and B travel in same direction;

(2) A and B travel in opposite direction;

(3) v_A and v_B form an angle of 60°.

1-25 The compass in a plane indicates that it is headed due east, its air speed indicator reads 215 km/h. A steady wind of 65 km/h is blowing due north.

(1) What is the velocity of the plane with respect to the ground?

(2) If the pilot wishes to fly due east, what must be he heading? That is, what must the compass read? Then, what will be the speed of the plane relative to the ground?

1-26 A aircraft travels due east from city A to city B, then back to A again. Suppose the distance between A and B is L, the speed of wind is u, the speed of the plane relative to the wind is v'. Show:

(1) if $u = 0$, the total flight time $t_0 = 2L/v'$;

(2) if u is due east, the total flight time $t_E = t_0 \Big/ \left(1 - \frac{u^2}{v'^2}\right)$;

(3) if u is due north, the total flight time $t_N = t_0 \Big/ \sqrt{\left(1 - \frac{u^2}{v'^2}\right)}$;

(4) in (2) and (3) we must assume $u < v'$, why?

Chapter 2

Dynamics—Newton's Laws of Motion

Dynamics is the study of forces and their effects on the motions of bodies. In this and the next two chapters we will focus on the motions of particles. In this chapter we shall see that the cause of acceleration is a force exerted by some external agent or environment. The fundamental properties of forces and the relationship between force and acceleration are given by Newton's three laws of motion. The first of these laws describes the natural state of motion of a free body on which no external forces are acting, whereas the other two laws deal with the behavior of bodies under the influence of forces.

Newton's mechanics predicts results that agree with experiments in highly accurate within a broad range of phenomena for macroscopic motions. However, it is necessary to point out that for the problems in which the speeds of the particles involved are an appreciable fraction of the speed of light, we must replace Newton's mechanics by Einstein's special theory of relativity while for the problems on the scale of atomic structure (for example, the motions of electrons within atoms), we must replace Newton's mechanics by quantum mechanics. We now view Newton's mechanics as an important, special case of these two.

2.1 Newton's Laws, Force, and Inertial Reference Frame

2.1.1 Newton's first law

Everyday experience suggests that a stationary object remains at rest unless some outside force acts on it. A cup on a table will stay there until someone takes it away. A car parking on the lot will be there until the driver start it. Less obvious is the statement that an object in motion will continue in motion until an external force exerts on it and changes the motion. For example, a baseball that if kicked twice, first on the ground, second on the ice, would roll much father on ice before stopping than on the ground. This is because the horizontal interaction called friction between the ball and the ground is much greater than that between the ball and ice. This leads to the idea that a ball on a perfectly frictionless horizontal plane would stay in motion forever. These ideal are formulated in Newton's first law of motion:

A body at rest remains rest and a body in motion continues to move at constant velocity unless acted upon by an external force.

First of all, this law reveals an important property of all bodies that is the tendency to maintain their original state of motion, or maintain at rest, which is called inertia so that the first law is called the law of inertia. Image what will happen when you seat in an moving automobile while suddenly accelerated or stop, to understand that everything in the world obeys the law of inertia. All matter has inertia. The concept of mass, introduced later, is a measure of a body's inertia.

Furthermore, the first law of motion contains a qualitative definition of the concept of force, or at least one aspect of the force concept, as "which changes the state of motion of a body". The friction force causes the ball slow down; the gravitational force makes a projectile body bent its path; and the satellites have centripetal acceleration toward the earth, which are produced by the gravitational pull of the earth, and so on.

Finally, the first law defines by implication what is known as inertial reference frame. It is obvious that if this law is valid in one given reference frame, then it can not be valid in a second reference frame that has an accelerated motion relative to the first. Those special reference frames in which the law is valid are called inertial reference frames. How do we know whether or not a given reference frame is inertial? We can only tell by making a test: take a free body, that is, a body isolated from all external forces, and observe its motion, if the body maintains its state of uniform motion, then the reference frame is inertial. For example, in Fig. 2-1, in the reference frame of an accelerating train leaving the train station, a ball initially at rest on the frictionless table of the train has a "spontaneous" acceleration toward the rear of the train observed by the man in the train, in contradiction to Newton's first law, therefore the train is not an initial reference frame; but in the reference frame of the ground, the ball remains at rest before and after the train accelerates, and obeys the first law, so that the ground is a inertial reference frame.

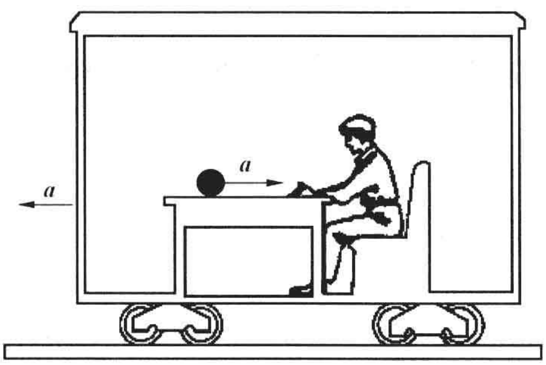


Fig. 2-1 A test to tell a reference frame is inertial or not

The problem we mostly concern is what reference frames in practical use for everyday measurements are inertial? The most commonly used reference frame is the one attached to the earth. Although crude experiments indicate that this reference frame is inertial (for example, the test shown in Fig. 2-1), more precise experiments show that this reference is not exactly inertial, because the earth rotates about its axis with a centripetal acceleration on the order of 10^{-2} m/s^2 variable at different location. For most practical purposes, we can neglect this small acceleration and so we can regard a reference frame attached to the ground as inertial to a good approximation.

It is proved that the reference frame that moves relative to an inertial reference frame with a constant speed along a straight line is also an inertial reference frame.

2.1.2 Newton's second law

Newton's second law establishes the relation between the force acting on a body and the acceleration caused by the force. To find this relation experimentally, we can exert different forces on different bodies and exert several forces on same body etc. All experiments and observations having been done so far can be summarized in a simple vector equation:

$$\boldsymbol{F} = m\boldsymbol{a} \tag{2-1}$$

It means: An external force acing on a body causes an acceleration which is in the direction of the force and has a magnitude inversely proportional to the mass of the body.

The mass defined by Eq. (2-1) called inertial mass which is the measurement of the inertia of a body. If a certain force exerts on deferent bodies, the body with larger mass gains a smaller acceleration, which indicates that its inertia is larger, and its state of motion is relative harder to be changed. Put in the other way, the acceleration produced by a certain force is inversely proportional to the mass of the body that the force acts on. On the other hand, for a certain body of mass m, the acceleration gained is proportional to the external force.

In the SI system, the unit of mass is kilogram, and the unit of force is Newton, 1 Newton $= 1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$. It is the adoption of SI system of units that makes the factor equal to one in Eq. (2-1).

To understand Newton's second law, we must note what physics meanings it includes.

(1) Newton's second law is valid only in particles or the objects that can be regarded as particles.

(2) If several external forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\cdots$ act simultaneously on a body, then the acceleration is the same as that produced by a single fore given by

$$\boldsymbol{F}_{\text{net}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots = \sum \boldsymbol{F}_i$$

The single force that has the same effect as the combination of the individual forces is called the net force or the resultant force which is the vector sum of the individual forces. In term of net force, Newton's second law, becomes

$$\boldsymbol{F}_{\text{net}} = \sum \boldsymbol{F}_i = m\boldsymbol{a} \tag{2-2}$$

Eq. (2-2) can be interpreted as follows: each force acting alone produces its own acceleration ($\boldsymbol{a}_1 = \boldsymbol{F}_1/m$, $\boldsymbol{a}_2 = \boldsymbol{F}_2/m$, $\cdots$); and all the forces acting together produce a net acceleration $\boldsymbol{a}$, which, according to Eq. (2-2), is simply the sum of these individual accelerations.

$$\boldsymbol{a} = \frac{1}{m}(\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots) = \sum \boldsymbol{a}_i \tag{2-3}$$

This is called the **principle of superposition of forces** and is equivalent to the assertion that each force produces acceleration independently of the presence or absence of other forces.

(3) The relation between the net force and the acceleration holds instantaneously. This means that they change simultaneously, or, the instantaneous acceleration corresponds only to the value of net force in the same instant, once the force vanishes the acceleration becomes zero simultaneously.

(4) Eq. (2-2) can be rewritten (dropping the subscript of net) as

$$\boldsymbol{F} = m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \tag{2-4}$$

Like all vector equations, Eq. (2-4) is equivalent to two sets of scalar equations:

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z \tag{2-5}$$

or

$$\sum F_x = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \quad \sum F_y = m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \quad \sum F_z = m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} \tag{2-6}$$

If the path of motion is given, project Eq. (2-2) and Eq. (2-3) on tangential direction and normal direction of the path respectively, leads

$$\sum F_\mathrm{t} = ma_\mathrm{t} = m\frac{\mathrm{d}v}{\mathrm{d}t}, \quad \sum F_\mathrm{n} = ma_\mathrm{n} = m\frac{v^2}{r} \tag{2-7}$$

These equations relate every component of the net external force acting on a body to the corresponding component of the acceleration of that body, and represent the most useful form for a mathematical statement of Newton's second law. Eq. (2-4), Eq. (2-6) and Eq. (2-7) are also called differential functions of motion. By integrating, we can calculate the velocity and position functions to determine the motion.

(5) The acceleration in Eq. (2-1) ~ Eq. (2-7) is measured by the observer in an inertial reference frame, which means Newton's second law is valid only in inertial reference frames.

Finally, we note that Newton's second law includes the formal statement of Newton's first law as a special case. That is, if no force acts on a body, Eq. (2-1) tell us that the body will not be accelerated. This is not to trivialize Newton's first law; its role in determining the set of reference frames in which Newton's mechanics holds, justifies its status as a separate law.

2.1.3 Newton's third law

Everyday experiences tell us that forces come in pairs. If a hammer exerts a force on a nail, the nail exerts an equal but oppositely directed force on the hammer. If you kick a brick wall, the wall pushes back at you. The situation has been summed up with the gentle words: "you can not touch without being touched".

Fig. 2-2 represents the general situation formally, let body A exert a force $\boldsymbol{F}_{BA}$ on body B; experiment shows that body B then exerts a force $\boldsymbol{F}_{AB}$ on body A. These two forces are equal in magnitude and oppositely directed, that is

$$\boldsymbol{F}_{AB} = -\boldsymbol{F}_{BA} \tag{2-8}$$

This is the quantitative expression of Newton's third law, put it in words: **Whenever body A exerts a force on body B, body B also exerts a force of equal magnitude and opposite direction on body A.**

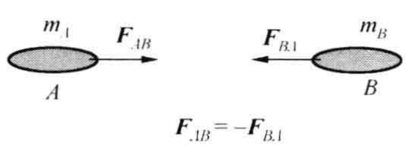


Fig. 2-2 Newton's third law

Note that the order of the subscripts in Eq. (2-8). $\boldsymbol{F}_{AB}$ is the force exerted on body A by body B. The mutual interaction involves two forces. Commonly, one of them (it does not matter which) is called the action force, and the other member of the pair is then called the reaction force. Every time you see a force, a good question is: "where is its reaction force?". Usually, Newton's third law is called "action-reaction law".

To apply Newton's third law correctly, you must be aware of the following aspects:

(1) As Fig. 2-2 shows, the two members of an action-reaction pair always act on different bodies so that they can not possibly cancel each other. If two forces act on the same body they are not an action-reaction pair, even though they may be equal and in opposite direction.

(2) The action-reaction pair of forces act on the interaction bodies simultaneously and vanish simultaneously too.

(3) The action-reaction pair of forces belong to same category of force, that is, if the action force is gravitational, or elastic, or frictional force, thus the reaction force is the gravitational, or elastic, or frictional force also.

(4) Some action-reaction forces exist by contact, for example, you walk by pushing backward on the ground; the reaction of the ground then pushes you forward. This kind of reaction plays a crucial role in machines that produce locomotion by pushing against the ground, water, or air, and so on. However, some other kinds of force exist even if the two interaction bodies are not in direct contact, so the forces between them must bridge the intervening empty space. For instance, the earth pulls on everything on it by means of gravity then the third law requires that everything on the earth pull on the earth with an opposite force (its effective point of application is the center of the Earth) but equal magnitude, this reaction force is also gravity.

2.2 The Forces in Mechanics and the Fundamental Forces in Nature

2.2.1 The general properties of forces in mechanics

1. Weight

Weight is the pull of Earth gravity. Consider a body of mass m in free fall near the surface of the earth. Under the influence of gravity, this body will be accelerated downward with an acceleration $\boldsymbol{g}$. According to Newton's second law, the gravitational force that causes this acceleration must be $\boldsymbol{F} = m\boldsymbol{a} = m\boldsymbol{g}$, it has the magnitude of mg and the direction pointing downward as $\boldsymbol{g}$ does. This force is what we call the weight. Usually we denote the weight by the vector symbol $\boldsymbol{G}$, thus

$$\boldsymbol{G} = m\boldsymbol{g} \tag{2-9}$$

In which $\boldsymbol{g}$ is called the gravitational acceleration. If the body is not in free fall but is held in a stationary position by same supports, then the weight is still the same as given by Eq. (2-9), however, the supports prevent the body from downward motion.

The weight of a body and the mass of a body are often confused because at any point near the earth surface they are proportional to each other. Being the measurement of the inertia of a body, mass, a scalar, is an intrinsic property, and is the same on the earth surface, in an orbiting satellite on Mars, or on the Moon. Weight, a vector, is not an intrinsic property of the body, it depends on the location because $\boldsymbol{g}$ varies with different altitude.

2. Elastic forces

A body is said to be elastic if it suffers a deformation (no matter how small) when a compressing or stretching force is applied to it and returns to its original shape when the force is removed. Even bodies normally regarded as rigid, such as a steal ball, a wooden block, etc., are somewhat elastic. The force with which a body resists deformation and tends to return its original shape, is called restoring force which acts on the other body in contact with it.

(1) Normal force is the perpendicular force with which each surface presses on the other, its direction always points the opposed one. The magnitude of the normal force depends on the extent of compression between each other and the properties of the material.

(2) Tension at the end of a cord is the force with which the cord pulls on what is attached to it. Tension is always along the cord, and points the direction of contraction of the cord. If the mass of a string can be neglected, applying Newton's second law, we can prove that the tension acting on any segment of the cord is equal to the tension at the end, which exerts on the attached body.

(3) Restoring force of a spring with which an elastic body opposes whatever pulls on it often obeys a simple empirical law known as Hooke's law: **the magnitude of the restoring force is directly proportional to the deformation.** Hooke's law is only an approximate and is often a quite good approximation, provided that the deformation is within the elastic limit of the material. If a spring is stretched beyond its elastic limit, it will suffer a permanent deformation and not snap back to its original shape when released.

Fig. 2-3 shows an important special case of a coil spring. In Fig. 2-3(a), a light spring is in its relaxed state, the coordinate x axis has an origin coinciding with its equilibrium position, and the left end of the spring is fixed to a rigid support. If we apply a stretching or compressing force to the right end, then we can measure the deformation of the spring by the displacement that the right end undergoes relative to its initial position. In Fig. 2-3(b), a positive value of x corresponds to an elongation of the spring, and a negative value corresponds to a compression in Fig. 2-3(c), obviously, x is nothing but the change in the length of the spring. In this case, Hooke's law can be written as

$$\boldsymbol{F} = -k\boldsymbol{x} \tag{2-10}$$

where k is the spring constant, and is a positive number characteristic of the spring, a stiff spring has a high value of k, and a soft spring has a lower value of k. The unit of k is N/m.

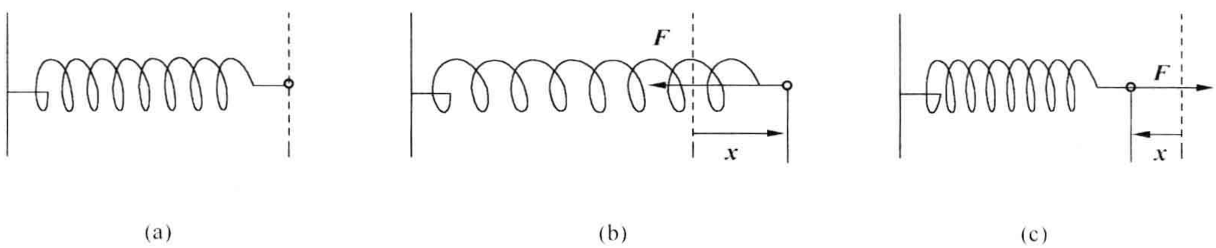


Fig. 2-3 (a) A spring relaxed, the origin O coincides with the equilibrium position; (b) spring stretched $x>0$, then $\boldsymbol{F}<0$; (c) spring compressed $x<0$, then $\boldsymbol{F}>0$

Eq. (2-10) indicates that the restoring force is proportional to the magnitude of the displacement of the spring and points in the opposite direction of displacement refer to the coordinate system above.

3. Frictional forces

(1) Kinetic friction exists between dry solid surfaces moving across each other at relative slow speeds. Fig. 2-4 shows the general case in which a block of steal slides on a table top of wool. Frictional force f_k exerted on the block by the table, pointing the opposite direction of motion and resisting the motion, will gradually slow the block down and ultimately stop it. From the Newton's third law, the block also exerts an opposite friction f'_k on the top of the table. Experiment shows that the magnitude of the kinetic friction between un-lubricated, dry surfaces sliding one over the other is proportional to the normal force pressing the surface together and is independent of the area of contact and of the relative speed. Put it into mathematical form as

$$f_k = \mu_k N \tag{2-11}$$

Where μ_k is the coefficient of kinetic friction, a constant characteristic of the materials involved. Friction force is parallel to the contact surface and always in the direction opposite to the relative motion. This simple law is quite a good approximation for a wide range of materials (and is at its best for metals) when the speeds are not at high or low extremes.

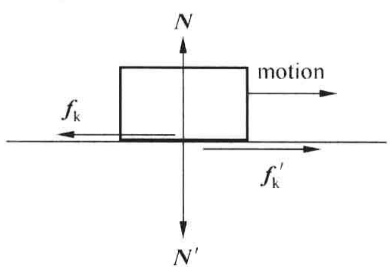


Fig. 2-4 Interaction between the moving block and the table surface

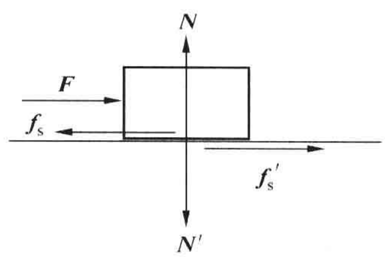


Fig. 2-5 Interaction between the tabletop and the block held at rest

(2) Static friction exists between two surfaces at rest. Fig. 2-5 shows the general situation in everyday experience: If a external force $\boldsymbol{F}$ is exerted against the side of a steel block initially at rest on a wooden tabletop, the block will not be moved, this is because, in response, a static frictional force $\boldsymbol{f}_s$ arises pointing to the opposite direction and exactly cancels the external force. If $\boldsymbol{F}$ is increased, $\boldsymbol{f}_s$ will automatically increase until the external force is sufficiently large to overcome the frictional force and to accelerate the body. The value that the static friction attains when the external force is just about to start the motion (so called "breakaway" point), is the maximum of the static friction. Experiment shows that the magnitude of the maximum force of friction between un-lubricated dry surfaces at rest with respect to each other is proportional to the normal force and independent of the (macroscopic) area of contact. Mathematically

$$f_{s\,max} = \mu_s N \tag{2-12}$$

where μ_s is the coefficient of static friction which depends on the materials. The direction of f_s

is parallel to the surface, so as to oppose the external force that tends to move the body. If the external force is less than the critical value of "breakaway" point, then the static friction is also less than f_{smax}. In general, we have

$$f_s \leqslant \mu_s N \quad (2\text{-}13)$$

For most material $\mu_s > \mu_k$ and therefore the maximum static friction force is larger than the kinetic friction force.

(3) Fluid friction drag force: If a body moves through a fluid, a friction like drag force will act on it to retard its motion. The object may be a cannon ball sinking in the ocean, a jet plane in flight, a ball falling in oil etc.. When the body moves at a relative slow velocity through a fluid, the force of the friction may be well represented by Stokes' law

$$\boldsymbol{F}_f = -k\eta \boldsymbol{v} \quad (2\text{-}14)$$

the coefficient k depends on the shape of the body. For example, in the case of a sphere of radius r, a close examination reveals that

$$k = 6\pi r \quad (2\text{-}15)$$

so

$$\boldsymbol{F}_f = -6\pi r \eta \boldsymbol{v} \quad (2\text{-}16)$$

which means that the drag force is proportional to the velocity and opposed to it. The coefficient η depends on the frictional force between different fluid layers moving with different velocity, which is called viscosity and η is called the coefficient of viscosity. In Eq. (2-15) the unit of k is meter, thus in Eq. (2-16) the unit of η must be $N \cdot s/m^2$. On the other hand, the fluid exerts a buoyant force on the moving body, which according to Archimedes' principle, is equal to the weight of the fluid displaced by the body. If m_f is the mass of fluid displaced by the body, so that the upward buoyant force is

$$\boldsymbol{B} = -m_f \boldsymbol{g}$$

the net downward force is $(m - m_f)g$, in this case the Newton's second law becomes

$$(m - m_f)\boldsymbol{g} - 6\pi r \eta \boldsymbol{v} = m\boldsymbol{a} \quad (2\text{-}17)$$

The three forces act on the ball as shown in Fig. 2-6 .

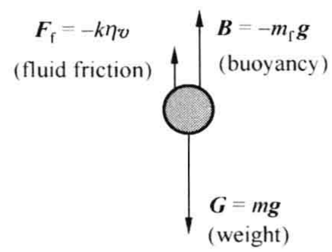


Fig. 2-6 The forces on a body falling in fluid

2.2.2 The fundamental forces in nature

At the fundamental level, all the forces mentioned above fall into two categories: the gravitational force of which weight is an example and the electromagnetic force for the others. Beyond above two kids of forces, there are two other forces known to occur in nature: the strong force and the weak force.

1. Gravitational force

1) Newton's law of universal gravitation

- **Every particle attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.**

The magnitude of the gravitational force that two particles of masses M and m separated by a distance r exert on each other is

$$F = G_0 \frac{Mm}{r^2} \tag{2-18}$$

where G_0 is a universal constant called as the gravitational constant and in metric units its value is

$$G_0 = 6.67 \times 10^{-11} \mathrm{N \cdot m^2/kg^2} \tag{2-19}$$

The direction of the force on each particle is toward the other particle along the straight line between the two particles; therefore the two forces are of equal magnitude and opposite direction, form an action-reaction pair.

The gravitational force has some remarkable characteristics: first, it does not require any contact between the interacting particles. In reaching from one remote particle to another the gravitational force somehow bridges the empty space between the particles, this is called action-at-a-distance. The gravitational force is the weakest among the four forces.

Second, the gravitational force between two particles is unaffected by the presence of intervening masses, in the other words, it is completely independent of the presence of other particles. This consequently leads that the net gravitational force between two bodies is merely the vector sum of the individual forces between all the particles making up the bodies, that is, the gravitational force obeys the principle of superposition. It can be proved that the net gravitational force between two spherical bodies acts just as though the mass of each body were concentrated at the center of its respective sphere. This important result is called Newton's theorem.

2) Gravitational force on a particle by Earth

As an example to apply Newton's theorem, the magnitude of gravitational force exerted by the earth on a particle above its surface is given by

$$F = G_0 \frac{M_e m}{r^2} \tag{2-20}$$

in which M_e is the mass of Earth, m is the mass of the particle, and r is the distance from the center of the earth to the particle.

For the special cases that the particle is at the surface of the earth (almost all of the cases involved in our course), let R_e represent the radius of the earth, from Eq. (2-20), we have

$$F = G_0 \frac{M_e m}{R_e^2} \tag{2-21}$$

The consequent acceleration of the particle is therefore

$$a = \frac{G_0 M_e}{R_e^2}$$

Which is a constant for all particles at (or near) the surface of the earth, called the acceleration of gravity, and labeled as g, so

$$g = \frac{G_0 M_e}{R_e^2} \tag{2-22}$$

We usually call F in Eq. (2-21) as weight G, that is, weight $\boldsymbol{G} = m\boldsymbol{g}$.

At a distance r ($r > R_e$) from the center of the earth, the acceleration of gravitation becomes

$$a = \frac{G_0 M_e}{r^2} \tag{2-23}$$

It is obvious that the acceleration of gravitation decreases with the altitude above the surface of the earth.

The mass defined by Eq. (2-21) is called the mass of gravitation while the mass defined by Eq. (2-1) is called the mass of inertia. If we choose proper units, it can be proved that these two masses of the same particle equal each other. In the following, we'll make no difference between these two kinds of mass.

2. Electromagnetic force

Electromagnetic force is an attraction or repulsion between electric charges. We shall discuss this kind of forces in detail later in the second part of our course. The electric and magnetic forces once considered to be separated are now grouped together because they are closely related. The magnetic force is nothing but an extra electric force that acts whenever charges are in motion.

Like the gravitational force, electromagnetic force is also action-at-a-distance force, because it bridges the empty space between the charged particles. Another similarity of electric force compared with the gravitational force is that it is also inversely proportional to the square of distance between the interacting charged particles.

All such forces, including the tension in a cord, restoring force in a spring, the frictional forces, the drag force and so on, at their deepest level, are noting but electromagnetic force between the charged particles in the atoms of one body and those in the atoms of another.

3. Strong force

The strong force acts mainly within the nuclei of atoms. It binds together the constitution of protons and neutrons and serves as a"glue" that holds the atomic nucleus together. This nuclear force is called "strong" because it is the strongest of the four fundamental forces. It can be either attractive or repulsive. The strong force will push the protons apart if they come too near to each other and it will pull them together if they begin to drift too far apart.

4. Weak force

Finally, the weak force only manifests itself in certain kinds of radioactive decay reactions, which involve the spontaneous break up of a particle into several other particles. This force is called "weak" because it is very weak. The distance over which a force acts is called its range. Table 2-1 lists the strength and range of the four fundamental forces.

Table 2-1 The four fundamental forces

Force	Acts on	Strength/N	Range
gravitational	all masses	10^{-34}	infinite
weak	most elementary particles	10^{-2}	less than 10^{-17} m
electromagnetic	electric charges	10^{2}	infinite
strong	nuclear particles	10^{4}	10^{-15} m

2.3 Fundamental Quantities, Units, and Dimensions

Physics is based on measurement. Before we measure something, we must first set up a standard and assign a unit for each quantities to be measured. There are fundamental (or basic) and derived quantities and units. Physicists recognize four fundamental independent quantities: length, mass, time and electric current, we deal with the former three in mechanics. The other quantities called as derived quantities, can be related to these basic quantities by their definitions, expressed as some combination of them. By means of these defining relations, the units of all derived quantities are represented in terms of the units of the four basic quantities. Different selection of fundamental units forms different system of units. International system of units, abbreviated SI and popularly known as the metric system, is commonly used in the world. In mechanics **the fundamental units in SI are meter, second and kilogram for length, time and mass, respectively.** The derived mechanical units are defined in terms of them, for example, the unit of force, Newton, is represented as kg · m/s^2, according to Newton's second law of $\boldsymbol{F} = m\boldsymbol{a}$ and the definition of acceleration.

Meter, with the development of science, for higher precision, redefined in 1983 is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second, instead of the standard of the wavelengths of a particular-red light emitted by atom of Kr-86 adopted in 1960 that ones was instead of the original standard of the platinum-iridium bar.

Second, base on the cesium clock was adopted as the international standard in 1967, that is: one second is the time occupied by 9 192 631 770 vibrations of the light (of a special wavelength) emitted by a cesium-133 atom. The precise time signals are continuously announced by radio station WWV, Fort Collins, Colorado of United States, which is called Greenwich Mean Time.

Kilogram, abbreviated kg, the first SI standard of mass, is defined as the mass of the international prototype of the kilogram, a platinum-iridium cylinder kept at the international Bureau of Standards near Paris. Accurate copies have been sent to standardizing laboratories in other countries and the masses of other bodies can be found by the equal-arm balance method.

To meet the need for a better and much smaller mass standard in science, the second mass standard has been developed. It is the carbon-12 atom which by international agreement, has been assigned a mass of carbon-12 unified atomic mass (abbr. u), also called Atomic Mass Unit. The relation between the two SI mass standards is

$$1\text{u} = 1.6605402 \times 10^{-27}\text{kg}$$

The uncertainty of it is about ± 10 units in the last 2 places of decimals.

Dimension is a very important concept. The SI meter, millimeter, kilometer, the English units of inches, feet, etc., all of these units are said to have dimensions of length, symbolized by L. Likewise, all different time units, such as seconds, minutes, hours, etc., are said to have dimensions of time, symbolized by T. The kilogram and all other mass units have dimensions of mass, symbolized by M. From these three fundamental quantities, we can express all various mechanical derived quantities by means of combinations of L, T and M. For instance, volume, such as cubic meter, has dimension of L^3. Mass density is defined as mass per unit volume, so that, has dimensions of M/L^3. The formula that expresses a quantity in terms of a combination of the fundamental quantities, is called the dimension of that quantity. The dimensions of some quantities in mechanics we discussed before are given in the following:

$$[v] = LT^{-1}, \quad [a] = LT^{-2}, \quad [F] = MLT^{-2}$$
$$[\omega] = T^{-1}, \quad [\beta] = T^{-2}, \quad [\theta] = L^0 \text{(no dimension)}$$

A quantity in a square brackets, say [F], represents the dimension of that quantity. Numerical factors have no dimensions. Another way to express the dimension of a quantity is simply putting the symbol dim before it, for example, dim $F = MLT^{-2}$, dim $\omega = T^{-1}$ etc..

Dimensional analysis is important in understanding physics and in solving physics problems. Physical equations must be dimensionally consistent. In other words, every term in the two sides of a equation must be dimensionally the same. For instance, if someone give an equation $F = mv$, we analyze the equation dimensionally, the dimension of the left side is MLT^{-2}, but the right side MLT^{-1}, obviously, the equation is not correct. Dimensional analysis is a useful way to catch careless errors in physics calculation. Furthermore, dimensional analysis can give physicists some hint to find, even, to discover some new laws of physics.

2.4 Applying Newton's Laws of Motion

To analyze the forces exerted on the object and to find its motion are an important practice in the study of university physics. The procedure consists of following steps:

(1) Select a body to which Newton's laws will be applied. Analyze the process of its motion, classify it and use specific method to find the solution.

(2) Draw a free-body diagram, be sure all the forces acting on the chosen body, including both contact forces such as normal force, tension, friction, and noncontact forces such as gravity and so on.

(3) Set up a coordinate system, determine components of forces and accelerations with respect to the axes, and label them in algebraic symbols. If more than one body is involved, the above steps must be carried out for each body, and express the geometrical relationships, if any, between the motions in algebraic form.

(4) Write the component equations of Newton's second law for each body, find the unknown quantities. Analyze the results if necessary.

Example 2-1 A box m_1 of 0.4 kg and a box m_2 of 0.2 kg are connected by a light rope passing over a frictionless pulley slide on two inclines A and B which form 60° and 30° angles with the horizontal respectively as shown in Fig. 2-7(a). The coefficient of kinetic friction μ_k between the boxes and the inclines is 0.4. Find the acceleration of the boxes.

Solution First, select the two boxes and analyze the forces exerted on; there are four forces acting on each of the two masses: the weight acting downward, the normal force exerted by the plane at right angles to the plane, the tension in the string and the retarding force due to kinetic friction along the attached surfaces. Assume that m_1 slides downward along A while m_2 up along B.

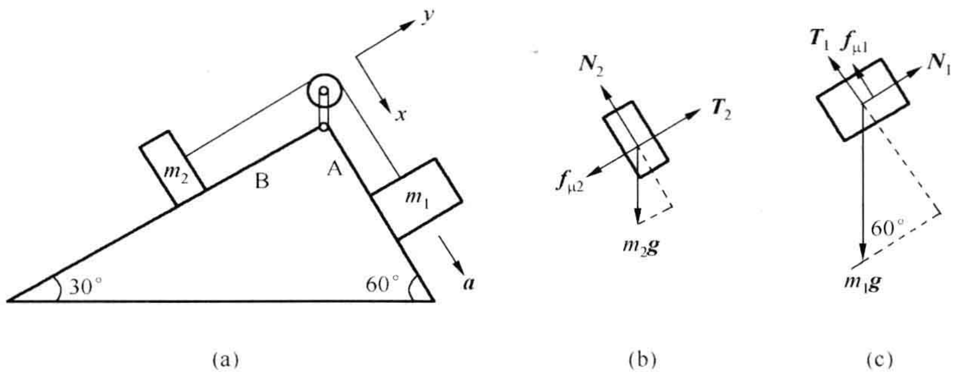


Fig. 2-7 For Example 2-1

Draw a free-body diagram. Set up coordinates so that x axis parallel to the inclines, mark every force on each body and their x, y components as shown in Fig. 2-7(b) and Fig. 2-7(c).

According Newton's second law, write down the x, y components equations for each body as followings:

For m_1

$$m_1 g \sin 60° - T_1 - f_{\mu 1} = m_1 a$$

$$N_1 - m_1 g \cos 60° = 0$$

and

$$f_{\mu 1} = \mu N_1 = \mu m_1 g \cos 60°$$

We have

$$m_1 g \sin 60° - T_1 - \mu m_1 g \cos 60° = m_1 a \quad ①$$

For m_2

$$T_2 - m_2 g \sin 30° - f_{\mu 2} = m_2 a$$

$$N_2 - m_2 g \cos 30° = 0$$

and

$$f_{\mu 2} = \mu N_2 = \mu m_2 g \cos 30°$$

We have

$$T_2 - m_2 g \sin 30° - \mu m_2 g \cos 30° = m_2 a \quad ②$$

Because the mass of the rope can be neglected, so that, $T_1 = T_2$ in magnitude, but oppositely, adding Eq. ① and Eq. ② gives

$$m_1 g \sin 60° - m_2 g \sin 30° - \mu (m_1 g \cos 60° + m_2 g \cos 30°) = (m_1 + m_2) a \quad ③$$

so that the acceleration of the masses obtained as

$$a = \frac{g}{m_1 + m_2}[m_1 \sin 60° - m_2 \sin 30° - \mu (m_1 \cos 60° + m_2 \cos 30°)] \quad ④$$

Substitute all data into Eq. ④, the magnitude of acceleration is given as

$$a = \frac{9.8 \times}{0.4 + 0.2}\left[0.4 \times \frac{\sqrt{3}}{2} - 0.2 \times \frac{1}{2} - 0.4 \times (0.4 \times \frac{1}{2} + 0.2 \times \frac{\sqrt{3}}{2})\right] \text{m/s}^2 = 1.59 \text{ m/s}^2 \quad ⑤$$

You can find the magnitude of the tension T in the rope by substituting all data including $a = 1.59$ m/s^2 into Eq. ① or Eq. ② easily.

If we assumed m_1 would ascend, the above analysis would lead to a negative acceleration indicating that it descends actually, in this case we should change the signs of some forces, check the procedure and the result again.

Example 2-2 A reindeer drags a flatcar-sledge with mass $m_1 = 60$ kg on which a crate of mass $m_2 = 40$ kg is placed, moves forward on the snow coved ground. Suppose that the force $\boldsymbol{F}$ exerted on the sledge by the reindeer makes an 30°angle with the horizontal as shown in Fig. 2-8(a), the coefficient of kinetic friction between m_1 and the ground is $\mu_k = 0.15$ while the coefficient of static friction between m_1 and m_2, $\mu_s = 0.3$.

(1) If no relative motion between m_1 and m_2, find the expression of the force F related to the acceleration.

(2) What is the maximum acceleration before m_2 starts to slide relative to m_1? Calculate the corresponding value of F.

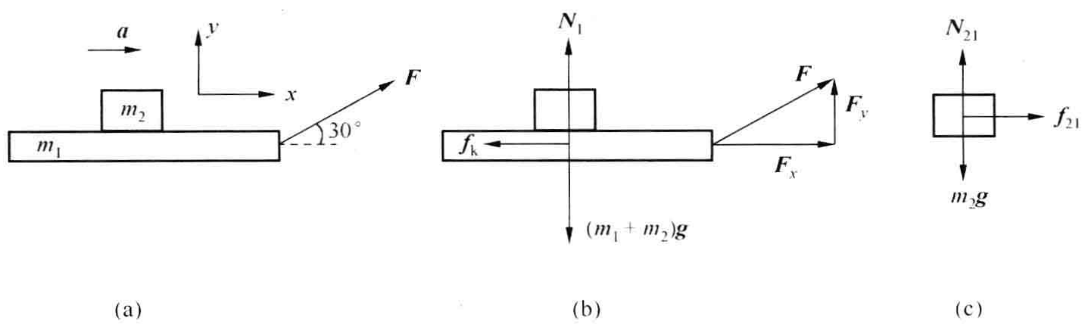


Fig. 2-8 For Example 2-2

Solution (1) We use blocks m_1 and m_2 to represent the flatcar and the crate respectively. If no relative motion between m_1 and m_2, we can treat $(m_1 + m_2)$ as a whole. Draw a free body diagram, set the x axis of the coordinate pointing right, and project every force into the axes, let $\boldsymbol{a}$ be the common acceleration along positive x direction showing in Fig. 2-8(b). Apply the component equations of Newton's second law, we have

the x component $$F\cos 30° - f_k = (m_1 + m_2)a \quad ①$$

the y component $$F\sin 30° + N_1 - (m_1 + m_2)g = 0 \quad ②$$

from Eq. ②

$$N_1 = (m_1 + m_2)g - F\sin 30°$$

and

$$f_k = \mu_k N_1 = \mu_k (m_1 + m_2)g - \mu_k F\sin 30° \quad ③$$

substituting Eq. ③ into Eq. ① yields

$$F\cos 30° - \mu_k (m_1 + m_2)g + \mu_k F\sin 30° = (m_1 + m_2)a$$

so that the relation between $\boldsymbol{F}$ and $\boldsymbol{a}$ is

$$F = \frac{(m_1 + m_2)(\mu_k g + a)}{(\cos 30° + \mu_k \sin 30°)} \quad ④$$

Put the known data into Eq. ④

$$F = [100 \times (1.47 + a)/0.941]\text{N}$$

If the sledge moves at constant velocity, $a = 0$, then $F_0 = 156.2$ N at least.

(2)Choose m_2 as the object and draw its free-body diagram as in Fig. 2-8(c). It is that the static frictional force exerted on m_2 by m_1 produces the acceleration a for m_2. The static friction will increase with the increase of the acceleration until it reaches its maximum value $f_{s\max}=\mu_s N_{21}$ which is corresponding to the maximum acceleration of $a_{\max}$. If the acceleration was over $a_{\max}$ then m_2 would start to slide relative to m_1. Applying Newton's second law to m_2 for the moment just before it slides, we have two component equations:

the x component $\quad f_{21} = m_2 a_{\max}$ ⑤

the y component $\quad N_{21} - m_2 g = 0$ ⑥

and

$$f_{21} = f_{s\max} = \mu_s N_{21} = \mu_s m_2 g \qquad ⑦$$

combining Eq. ⑤ and Eq. ⑦ yields

$$a_{\max} = \mu_s g$$

substituting all data, we have

$$a_{\max} = 0.3 \times 9.8 = 2.94(\text{m/s}^2)$$

which is the maximum acceleration before m_2 starts to slide relative to m_1.

Put $a_{\max}$ into Eq. ④ the corresponding force F is therefore

$$F_{\max} = \frac{100 \times (1.47 + 2.94)}{0.941}\text{N} = 468.6\text{N}$$

Example 2-3 In Fig. 2-9(a), a incline of mass M is placed on a table and a block of mass m is put on M. Suppose that all contact surfaces are frictionless and angle θ is known. Find the acceleration of M, and the acceleration of m with respect to M.

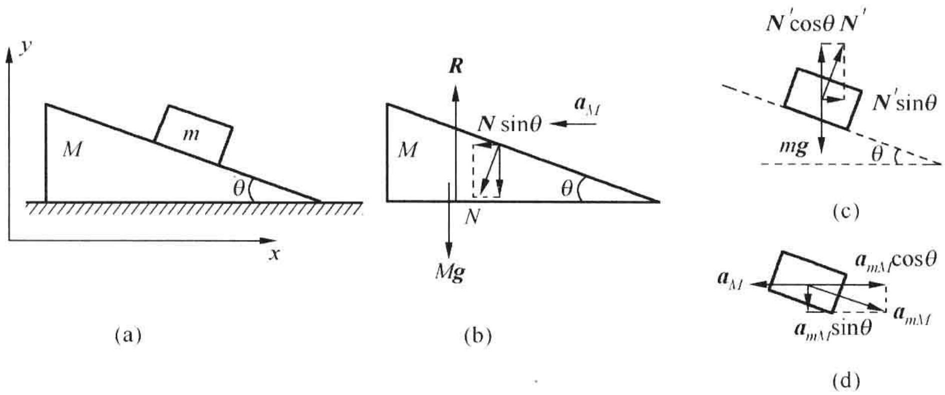


Fig. 2-9 For example 2-3

Solution We have two objects to be analyzed. Find all forces exerted on M and m, and draw free-body diagrams for each one as in Fig. 2-9(b) and (c). The possible motion of M: suppose a_M with respect to table is towards the left caused by the horizontal component of normal force N exerted by m. The possible motion of m: as m moves together with M, it slides down relative to M with a_{mM}, therefore, the acceleration of m is the vector sum of a_M and a_{mM}. Set x-y coordinates, decompose all forces, a_m and a_{mM} into x, y axes as in Fig. 2-9 (b), (c) and (d), note that

$$\boldsymbol{a}_m = \boldsymbol{a}_{mM} + \boldsymbol{a}_M$$

x-component of a_m $\quad a_{mx} = a_{mM}\cos\theta - a_M$

y-component of a_m $\quad a_{my} = a_{mM}\sin\theta$

Write the component equations of Newton's second law for m and M, respectively, we have

x-component for m $\qquad N'\sin\theta = m(a_{mM}\cos\theta - a_M)$ ①

y-component for m $\qquad N'\cos\theta - mg = -ma_{mM}\sin\theta$ ②

x-direction for M $\qquad -N\sin\theta = -Ma_M$

or

$$N\sin\theta = Ma_M \quad ③$$

and

$$N = N' \quad ④$$

The number of equations is equal to the number of variables, so we obtain

$$a_M = \frac{mg\ \sin\theta\ \cos\theta}{m\ \sin^2\theta + M}$$

$$a_{mM} = \frac{(m+M)g\ \sin\theta}{m\ \sin^2\theta + M}$$

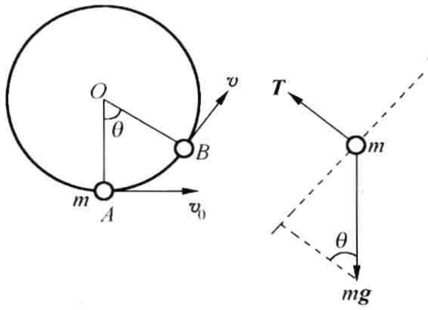


Fig. 2-10 For Example 2-4

Example 2-4 A small ball of mass m is attached by a thin cord, moving along a circle of radius R about a fixed center O in the vertical plane as shown in Fig. 2-10. Suppose that the ball is passing through the bottom point A with initial speed $\boldsymbol{v}_0$. Find

(1) the speed as a function of angular position;

(2) the tension in relation to angular position.

Solution (1) The forces exerted on the small body are weigh $m\boldsymbol{g}$ and the tension $\boldsymbol{T}$. Applying Newton's second law, we have $m\boldsymbol{g} + \boldsymbol{T} = m\boldsymbol{a}$, and the two components of it are:

normal direction $\qquad T - mg\cos\theta = m\dfrac{v^2}{R}$ ①

tangential direction $\qquad -mg\sin\theta = m\dfrac{dv}{dt}$ ②

and

$$\frac{dv}{dt} = \frac{dv}{Rd\theta}\cdot\frac{Rd\theta}{dt} = \frac{vdv}{Rd\theta} \quad ③$$

Substituting Eq. ③ into Eq. ② yields

$$\frac{v}{R}\frac{dv}{d\theta} = -g\sin\theta \quad ④$$

separate variables in Eq. ④ and rearrange it, we have

$$v\,dv = -Rg\ \sin\theta\,d\theta$$

By integrating, and note when $t = 0$, $\boldsymbol{v} = \boldsymbol{v}_0$, so that we have

$$\int_{v_0}^{v} v\ dv = -Rg\int_0^{\theta}\sin\theta\ d\theta$$

so

$$v = \sqrt{v_0^2 + 2Rg(\cos\theta - 1)} \quad ⑤$$

This is the speed in relation to angular position θ.

(2) Substituting Eq. ⑤ into Eq. ① and rearranging it yields

$$T = m\left(\frac{v_0^2}{R} - 2g + 3g\cos\theta\right) \quad ⑥$$

You can make the analysis about the variety of T with the change of θ. Furthermore, what is the initial speed needed in order to finish one circle, which means the tension $T \geqslant 0$ as the ball just passes the top position.

Example 2-5 A small ball of mass m and radius r is moving descend through a viscous fluid from rest along a straight line . Find

(1) the velocity of the ball as a function of time;

(2) the terminal speed. The coefficient of viscosity η of the fluid is known.

Solution (1) This is the case shown in Fig. 2-6. From Stokes' law Eq. (2-14) to Eq. (2-17), let $+y$ direction downward, applying Newton's second law for the ball in a viscous fluid:

$$(m - m_{\mathrm{f}})\boldsymbol{g} - 6\pi r \eta v = m\boldsymbol{a}$$

In which $F=(m-m_{\mathrm{f}})g$ is a constant for the given ball, $k=6\pi r$, rewrite above equation as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F - k\eta v$$

so that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{k\eta}{m}\left(v - \frac{F}{k\eta}\right)$$

Separating variables and integrating, while considering that $v=0$ at $t=0$, we have

$$\int_0^v \frac{\mathrm{d}v}{v - F/(k\eta)} = -\frac{k\eta}{m}\int_0^t \mathrm{d}t$$

That is

$$\ln\left(v - \frac{F}{k\eta}\right) - \ln\left(-\frac{F}{k\eta}\right) = -\frac{k\eta}{m}t$$

or

$$\frac{v - F/(k\eta)}{-F/(k\eta)} = \mathrm{e}^{-k\eta t/m}$$

Rewriting above equation yields the speed of the ball as the function of time

$$v = \frac{F}{k\eta}(1 - \mathrm{e}^{-\frac{k\eta}{m}t}) \qquad ①$$

(2) The exponential term decreases rapidly soon becoming negligible. Let $t \to \infty$, the speed becomes constant:

$$v_t = \lim_{t\to\infty}\frac{F}{k\eta}(1 - \mathrm{e}^{-\frac{k\eta}{m}t}) = \frac{F}{k\eta} \qquad ②$$

which is the terminal speed. Suppose the mass densities of the ball and the fluid are ρ and ρ' respectively, substitute them and $k=6\pi r$ as well as $m=4\pi r^3\rho/3$ into Eq. ②, the terminal speed is then

$$v_t = \frac{(\rho - \rho')g}{6\pi r\eta}\cdot\frac{4\pi}{3}r^3 = \frac{2}{9}\cdot\frac{\rho - \rho'}{\eta}\cdot gr^2$$

obtained.

2.5 Galilean Transformations

In section 1.5, relative motion, we have discussed the situation that two observers moving with respect to each other describe the same motion of a particle, and obtained the relations between the positions, velocities and accelerations indicated by Eq. (1-51)~Eq. (1-54).

Now we discuss the transformations between two inertial reference frames moving relative to one another. Consider two coordinate systems S and S' which is moving with constant velocity $\boldsymbol{u}$ along x axis relative to S as shown in Fig. 2-11. Assume that O and O' coincide at $t=0$. so $OO' = ut$, then we have

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{u}t \tag{2-24}$$

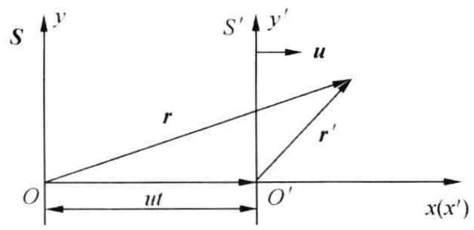


Fig. 2-11 Two frames moving relative to each other

Additionally, there is an assumption of

$$t' = t \tag{2-25}$$

where t and t' are the times measured by S and S' respectively. The Components of space-time coordinates are

$$\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{2-26}$$

Eq. (2-26) for reference frames in uniform motion relative to one another is called Galilean transformation, which is lie at the foundations of Newton's theory of space and time, it means that time is absolute and therefore the space is also absolute in classical mechanics. The Galilean transformation is therefore valid as long as Newton's theory holds. That means when we are dealing with the problem of velocity much smaller compared with the speed of light, Eq. (2-26) is valid. For the situation of high speed comparable with the speed of light, Galilean transportation is proved not be valid and will be replaced by Lorentz transportation that is the subject of special theory of relativity in modern physics of this course.

The Galilean transportation for velocity and acceleration obtained by taking differentiation of Eq. (2-26), because of uniform relative speed $\boldsymbol{u}$, we have

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u} \tag{2-27}$$

$$\boldsymbol{a}' = \boldsymbol{a} \tag{2-28}$$

This means the acceleration does not depend on the reference frames which are in motion respect to one another with a constant velocity. In Newton's theory, force $\boldsymbol{F}'$ and mass m measured in frame S' all remain unchanged, thus, It can be seen that the form of Newton's second law

$$\boldsymbol{F}' = m\boldsymbol{a}'$$

$$\boldsymbol{F} = m\boldsymbol{a}$$

remains unchanged also, it follows that the form of the conservation law of momentum remains unchanged as well as the other mechanical laws in classical mechanics. In brief, mechanical laws are equivalent for all inertial frames, there is no an absolute inertial frame, which is called as Galilean relativity theorem.

2.6 Non-inertial Frame and Inertial Force

As stated above Newton's laws hold only for inertial frames. The reference frame moving accelerated relative to an inertial frame is a non-inertial frame. For instance, the accelerated train in Fig. 2-1, for the ball on the frictionless surface in the train, is a non-inertial frame, as well as an accelerating elevator, a car suddenly making a turn, an accelerating vehicle for the passenger in it etc.. Particularly when we investigate the atmospheric circulation or other motion in large scale, the influence by the earth rotating can not be neglected, in these cases we take the earth as an non-inertial frame. Moreover, in view of such situation it is interesting and significant that if we introduce a hypothetical force or a new concept — inertial force, then Newton's law renews in form and still can be applied to solve the problems in non-inertial frame conveniently.

2.6.1 Inertial force

In the case shown in Fig. 2-1 of section 2-1, the train moves with acceleration $\boldsymbol{a}_r$ respect to the station (an inertial frame), the observer in the train find the ball has a "spontaneous" acceleration $\boldsymbol{a} = -\boldsymbol{a}_r$ toward the rear of the train without any external force exerted on it, which violates Newton's law, but, if we introduce an assumed force—inertial force $\boldsymbol{F}_i = -m\boldsymbol{a}_r$ that exerts on the ball, then the form of the Newton's second law could be used to explain the phenomenon, that is, $\boldsymbol{F}_i = m\boldsymbol{a}$, $\boldsymbol{a} = -\boldsymbol{a}_r$ here $\boldsymbol{a}$ represented the acceleration of the ball with respect to the train.

For general case of the motion in non-inertial system, suppose that a non-inertial frame translational moves with acceleration $\boldsymbol{a}_r$ related to an inertial frame, an object of mass m exerted by external force $\boldsymbol{F}$, moves with acceleration $\boldsymbol{a}$ related to the non-inertial frame. According to the theorem of acceleration addition, the acceleration $\boldsymbol{a}'$ with respect to the inertial frame is

$$\boldsymbol{a}' = \boldsymbol{a} + \boldsymbol{a}_r$$

and the Newton's second law in the inertial frame is

$$\boldsymbol{F} = m\boldsymbol{a}' = m(\boldsymbol{a} + \boldsymbol{a}_r) \tag{2-29}$$

in view of the non-inertial frame the inertial force on the object is defined as

$$\boldsymbol{F}_i = -m\boldsymbol{a}_r \tag{2-30}$$

substitute Eq. (2-30) to Eq. (2-29), we have

$$\boldsymbol{F} + \boldsymbol{F}_i = m\boldsymbol{a} \tag{2-31}$$

It indicates that the sum of the external force and the inertial force, the total effective force exerted on the object equals to the product of mass and the acceleration with respect to the non-inertial frame, which is so called the Newton's second law renewed in non-inertial reference frames.

Note that (1) The magnitude of an inertial force equals the product of mass and acceleration a_r of the non-inertial frame related to the inertial frame, but in the opposite direction of $\boldsymbol{a}_r$.

(2) Inertial force is a suppositious force, it has noting to do with mutual interaction between two objects, but a kinematical effect, there is no exerting side and reaction side either. So, having introduced inertial force in a non-inertial frame, Newton's second law renews in form but the third law does not.

Example 2-6 Acceleration meter. In Fig. 2-12, a block-spring system on a frictionless surface of a table fixed in a train accelerated moves along $+x$ direction. An observer in the train finds the block at rest when the spring extends by x_0. Suppose that the mass m of the block and the spring constant k are known, to find the acceleration a_r of the train related to the station.

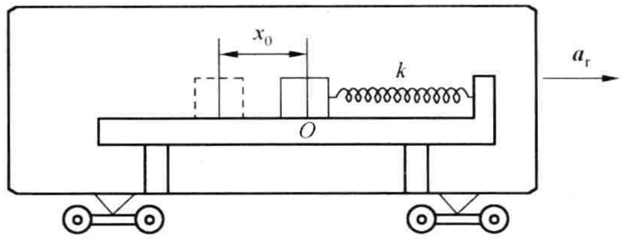


Fig. 2-12 For Example 2-6

Solution The train is a non-inertial frame. In view of the observer in it, the acceleration of the block relative to the train is zero, the inertial force on the block $\boldsymbol{F}_i = -m\boldsymbol{a}_r$ pointing to the left, from Hook's law the external force $\boldsymbol{F} = -k\boldsymbol{x}_0$ pointing to the right, note that $x_0 < 0$, by applying Eq. (2-31), we have

$$-kx_0 - ma_r = 0$$

then $a_r = -\dfrac{k}{m}x_0$ obtained.

This result indicates that we can use the extension of the spring to measure the acceleration of the train, conversely, if a_r, m known, k can be measured as well.

Example 2-7 Apparent weight in a freely falling elevator. Assume that a passenger of mass m stand on a platform scale in an elevator shown in Fig. 2-13. What would be the scale reading if the elevator falls freely?

Solution In this case the freely fall elevator is a non-inertial frame with acceleration $\boldsymbol{g}$. The scale reading equals to the magnitude of the normal force $\boldsymbol{N}$ on the passenger by the scale, which is the weight that the passenger would read from the scale display, this quantity is often called the apparent weight.

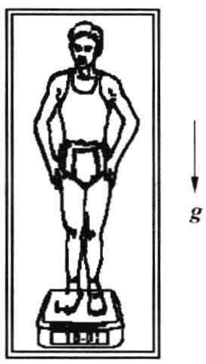


Fig. 2-13 For Example 2-7

The only external force on the passenger is the gravitational force $\boldsymbol{G} = m\boldsymbol{g}$. In view of the observer in the falling elevator the inertial force is $\boldsymbol{F}_i = -m\boldsymbol{g}$ upward directed, and the passenger is keeping stationary respect to the cab, in other words, the acceleration $\boldsymbol{a}$ related to the cab is zero. According to $\boldsymbol{F} + \boldsymbol{F}_i = m\boldsymbol{a}$, we have

$$-N + mg - mg = 0$$

so

$$N = 0$$

Thus, the freely falling scale reads zero and the passenger concludes that he is weightless. This is the same weightlessness that astronauts of a spacecraft in orbit around the earth experiences. The feeling of weightlessness arises not because gravity has ceased to act — it hasn't but because the vehicle and its occupant are both accelerating toward the center of the earth at the same rate. In the matter of fact, it is gravitation that keeps the spacecraft in orbit around earth. Actually, in a space shuttle, the astronauts are "weightless"— they would float above any scales attached to the floor.

2.6.2 Inertial Centrifugal force

A block of mass m on a frictionless disc while attached to the fixed central axis by a string of radius r shown in Fig. 2-14, when the block is moving together with the disc of the same angular speed relative to the ground what is the inertial force on the block?

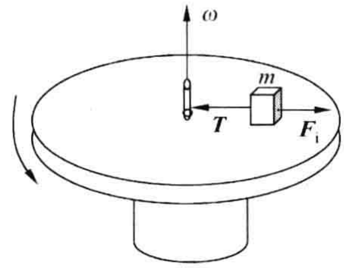


Fig. 2-14 A rotating frame and inertial centrifugal force

In this case, first, in view of the observer on the ground, the inertial frame, the centripetal force needed to keep the block in uniform circular motion is the tension force $\boldsymbol{T}$ pulling on the block from the string inward along the radial direction. From Newton's second law

$$T = mr\omega^2$$

While, in view of the observer on the rotating disc, the non-inertial frame, the block keeps at rest with respect to the disk, and an inertial force $\boldsymbol{F}_i$ on the block together with the tension force keep the balance, applying Eq. (2-31)

$$\boldsymbol{T} + \boldsymbol{F}_i = 0$$

So that

$$F_i = -m\omega^2 r \qquad (2\text{-}32)$$

which is directed outward along the radial axis as shown in Fig. 2-14. This inertial force is called as inertial centrifugal force. Note that although the tension force and the centrifugal force are in same magnitude and in opposite direction but not a pair of action-reaction force.

The centrifugal effect is an objective reality in rotating system, though it is a kind of hypothesis force introduced in non-inertial frame. And this effect wide utilized in all fields, from centrifugal cream separator in milk industry, centrifugal whizzer in chemical laboratory, to separation of uranium 235 from a kind of uranium gaseous compound in atomic reactor as well. The most common centrifugal machine is an automatic washer used in thousands of households.

2.6.3 Coriolis force

Finally we discuss so-called Coriolis force. If an object of mass m moves with respect to a rotating frame, besides the centrifugal force, there is another kinematical effect—a Coriolis force would be on the moving object. It can be proved that the Coriolis force $\boldsymbol{F}_C$ is represented by

$$\boldsymbol{F}_C = 2m\boldsymbol{v}' \times \boldsymbol{\omega} \qquad (2\text{-}33)$$

Which is the cross product of two vectors, $\boldsymbol{v}'$ is the velocity of the object relative to the rotating frame while $\boldsymbol{\omega}$ is the angular velocity of the rotating frame with respect to inertial frame, say, a frame fixed on the ground. The both directions of $\boldsymbol{\omega}$ and $\boldsymbol{F}_C$ are determined by the Right-handed screw rule shown in Fig. 2-15. The thumb points to the direction of $\boldsymbol{\omega}$, when one's 4 fingers curve along the way of rotation, for example, if rotating counterclockwise, then $\boldsymbol{\omega}$ pointing upward. The direction of Coriolis force is pointed by the thumb, when one's 4 fingers curve from $\boldsymbol{v}'$ to $\boldsymbol{\omega}$ as shown in Fig. 2-15, perpendicular to the plane determined by $\boldsymbol{v}'$ and $\boldsymbol{\omega}$, and $\boldsymbol{F}_C$ is on the right side of $\boldsymbol{v}'$, makes the motion deviate to right hand direction. The magnitude of Coriolis force is given by

$$F_C = 2mv'\omega\sin\theta \qquad (2\text{-}34)$$

θ is formed by $\boldsymbol{v}'$ and $\boldsymbol{\omega}$. $\theta = 90°$ in Fig. 2-15 where $F_C = 2mv'\omega$.

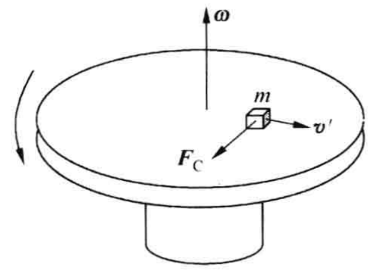


Fig. 2-15 The direction of Coriolis force in rotating frame

Considering the effect of Coriolis force in a rotating system, we can interpret many phenomena by the influence of earth's rotation. From Eq. (2-33), it is deduced that $\boldsymbol{F}_C$ is always on the right side of the motion at north hemisphere, bud on the contrary at south hemisphere. In weather forecasting TV program, the whirling in atmosphere is counterclockwise in satellite cloud diagram in north hemisphere, while such whirling is clockwise in south hemisphere. And also, because of the influence of Coriolis force the right shore of the river is more washed out than the left shore in north hemisphere.

Finally, the Foucault's pendulum provides one of the most dramatic demonstrations that the earth is not an inertial frame, but is rotating. The observation on the earth show that the orientation of the plane of swing gradually rotates clockwise, which conversely gives the proof of earth's rotating. Because of the Coriolis force is always on the right side of $\boldsymbol{v}'$, so that the real trial of the pendulum bob drifts sideways that makes the plane of swing rotate. Fig. 2-16 shows the situation that if Foucault's pendulum were placed at the North pole. In Beijing astronomy observatory, the plane of Foucault pendulum with 10 meters length rotates one round every 37 hours and 15 minutes.

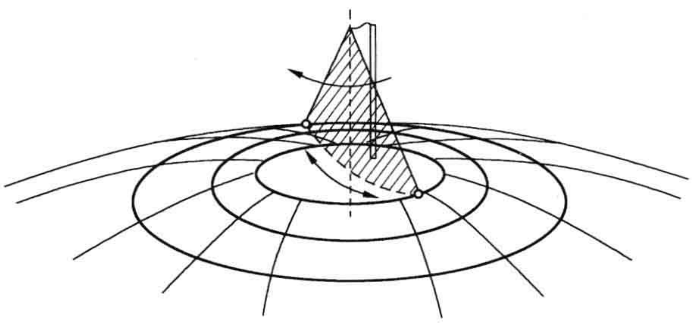


Fig. 2-16 A Foucault's pendulum is placed at the North pole of the earth, tends to swing in a fixed plane, actually, the earth rotates under it

Questions

2-1 Why no matter how hard you push the wall of a carriage in which you are, it's never move forward at all?

2-2 The weight of a certain bowling ball, say, is 71N on the earth's surface, 27N on the Mars, and 12N on the Moon; while its mass is 7.2 kg in the all three places. Explain.

2-3 Is certainly the frictional force acting on a body always pointing to the opposite direction of its motion? Explain using some examples.

2-4 The following statement is true, explain it: Two teams are having a tug-of-war, the team that pushes harder (horizontally) against the ground wins.

2-5 While rounding a curve at high speed, a motorcycle rider leans the motorcycle toward the center of the curve. Why?

2-6 Why are the train road beds and highway banked on curves?

2-7 How could you measure the coefficient of static friction between a small metal block and a wood plate on which the block put at rest in the beginning? Draw a sketch and describe the possible experiment you design.

2-8 A weight is hung by a cord from the ceiling of an elevator. From the following condition, choose the one in which the tension in the cord will be greatest or least: (a) elevator at rest; (b) it rising with uniform speed; (c) it descending with decreasing speed; (d) it descending with increasing speed.

2-9 Two identical weights are attached by a cord over two identical pulleys. In the beginning, they were at same height, then, give a horizontal speed to one of them so that it begins to swing about its equilibrium position, as show in Fig. 2-17. whether or not the two weights remain balance? If not, why?

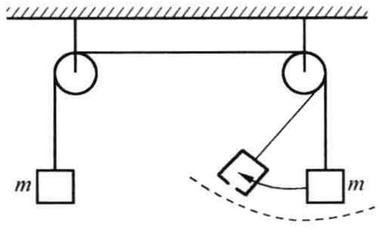


Fig. 2-17 For Question 2-9

2-10 A passenger in the front seat of a car find himself sliding toward the door as the driver make a sudden left turn. Describe the forces on the passenger and on the car at this instant if the motion is viewed from a reference frame (1) attached to the earth (2) attached to the car.

2-11 In a roller coaster with a full vertical loop, when a men is just passing its top position, some students give two statements:

(1) there are three forces including weight, tension, and centripetal forces acted on him;

(2) beside above three downward forces, there is a upward centrifugal force to balance them. Are they correct? If not, give correct answer.

2-12 Review Example 2-3, could you choose non-inertial reference frame to solve it? if so, how do you analyze the forces on the block and write the dynamic equations for it?

2-13 Draw a sketch of the earth, like Fig. 2-16, using Eq. (2-33), show that $\boldsymbol{F}_C$ is always on the right side of the motion at north hemisphere, on the contrary, it is on the left side at south hemisphere.

Problems

2-1 A 40 kg slab rests on a frictionless floor. A 10kg block rests on the top of it (Fig. 2-18). The coefficient of static friction between the slab and the block is 0.6, whereas the kinetic coefficient is 0.4. The 10 kg block is acted by a horizontal force of 100N. What are the resulting acceleration of (1) the block and (2) the slab, respectively?

2-2 A men of mass $m=72.2$ kg stands on a platform scale in an elevator cab(Fig. 2-13). Find the scale reading for the following cases: (1) if the cab is at rest or moving with constant speed; (2) if it has an upward acceleration of 3.20 m/s^2; (3) it has a downward acceleration of -3.20 m/s^2; (4) if the cab broken, so that the cab falls freely; (5) what would happen if the cab were pulled (downward) with an acceleration of -12.0 m/s^2.

2-3 A truck travels with a speed of 54km/h on a road, and a 50 kg crate is placed on the floor of the truck. Suppose no relative motion between the crate and the floor, find the frictional force exerted on the crate in the following cases: (A) The path is a straight line, (1) the speed is increasing with an acceleration of 2 m/s^2; (2) the speed is decreasing with an acceleration of -1.6m/s^2. (B) The path is an arc of radius 250 m, problem (1) and (2) are the same as those in (A).

2-4 At a highway, a curve of radius R is banked at an angle of θ as shown in Fig. 2-19. If the driver of a car does not wish to rely on friction, at what a speed of v_0 should he take this curve? if $v > v_0$ or $v < v_0$ in what direction of the frictional force will be?

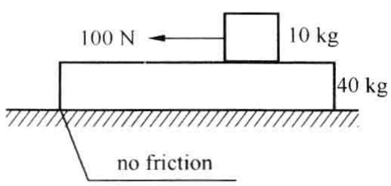


Fig. 2-18 For problem 2-1

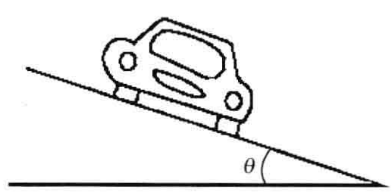


Fig. 2-19 For problem 2-4

2-5 Two masses, $m_1=1.65$ kg and $m_2=3.30$ kg, attached by a mass-less rod paralleled to the incline on which both slide, as shown in Fig. 2-20, travel down along the plane with m_1 trailing m_2. The angle of the incline is $\theta=30°$, the coefficient of kinetic friction between m_1 and the incline is $\mu_1=0.226$; between m_2 and the incline is $\mu_2=0.113$. Calculate

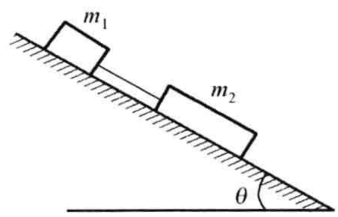


Fig. 2-20 For problem 2-5

(1) the tension in the rod linking the masses;

(2) the common acceleration of the two masses;

(3) how would the answers to above questions be changed if m_2 trailing m_1?

2-6 A particle of mass 10 kg, subject to a force $F=(12t+40)$N, moves in a straight line. At time $t=0$ the particle is at $x_0=5$m, with velocity $v_0=6$ m/s. Find its velocity and position at any later time.

2-7 Fig. 2-21 shows a conical pendulum, its bob of mass m whirls around in a horizontal circle at constant speed v at the end of a cord whose length, measured to the center of the bob, is L. The cord makes an angle θ with the vertical. As the bob swing around, the cord sweeps out the surface of a cone. Find the period of the pendulum, that is, the time τ for the bob to make one complete revolution.

2-8 A kind of rotor is found in mane amusement parks. It is a hollow cylindrical room that can be made to rotate around a central vertical axis, as show in Fig. 2-22. A person stands up on the floor against the wall. The rotor starts rotating and gradually increases its speed until, at a certain critical speed, the floor is dropped away, revealing a deep pit. The person does not fall but remains pinned to the wall. μ_s between the wall and the person's clothing is 0.4, and the radius R of the rotor is 2.1m.

(1) What force supports the person so that he doesn't drop?

(2) At what minimum rotational speed is it safe to drop the floor?

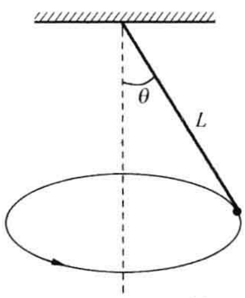


Fig. 2-21 For problem 2-7

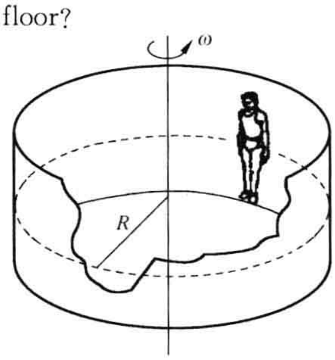


Fig. 2-22 For problem 2-8

2-9 In the system shown in Fig. 2-23, suppose that the cord and pulley are all mass-less, neglect the friction between M and the table surface, and the friction between m_1 and M.

(1) What is the horizontal force exerted on M so that m_1 doesn't move relative to m_2.

(2) When the system moves, what is the normal force acted on the table by M? Suppose $m_1 > m_2$.

2-10 A small block of mass m slide on a horizontal frictionless surface as it travels around the inside of a hoop of radium R (Fig. 2-24). The μ_k between the block and the wall is known; therefore, the speed v of the block decreases. In terms of m, R, μ_k and v, find expressions for

(1) the frictional force on the block and its tangential acceleration;

(2) the time required to reduce the speed of the block from an initial value v_0 to $v_0/3$.

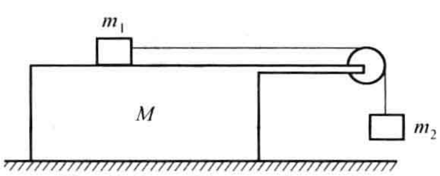


Fig. 2-23 For problem 2-9

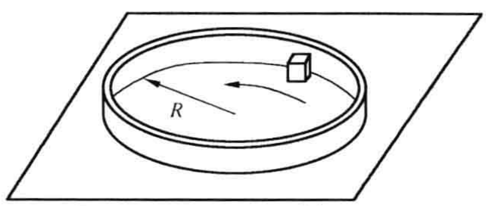


Fig. 2-24 For problem 2-10

2-11 A truck starts from rest at time $t=0$ and accelerates uniformly, achieving a speed of 20 m/s in 10 s. A small package of mass 5.0 kg is initially located 3.0 meter from rear of the truck. The package starts to slide at time $t=0$, and the μ_k between the package and the truck floor is 0.15 (Fig. 2-25).

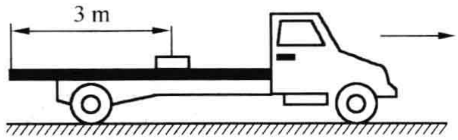


Fig. 2-25 For problem 2-11

(1) Find the horizontal acceleration of the package relative to the reference of the ground.

(2) Determine the time after $t=0$ when the package reaches the rear edge of the truck.

(3) Determine the horizontal component of the velocity of the package as it strikes the ground.

2-12 Body D, which has a mass of 12 kg (Fig. 2-26), is on a smooth conical surface ABC and is spinning about the axis EE' with an angular velocity of 12 r/min, $\theta=30°$, $L=1.5$m. Calculate

(1) the linear velocity of the body;

(2) the tension in the thread and the reaction of the surface on the body;

(3) the angular velocity necessary to reduce the reaction of the surface on the body to zero.

2-13 A small ball of mass m initially at A, slides on a smooth circular surface ADB of radius r (Fig. 2-27). When the ball is at point C with angular position α relative to the initial position, show the angular velocity and the force exerted on the ball by the surface.

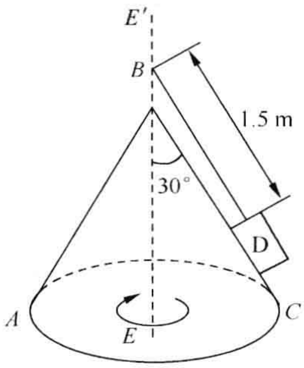


Fig. 2-26 For problem 2-12

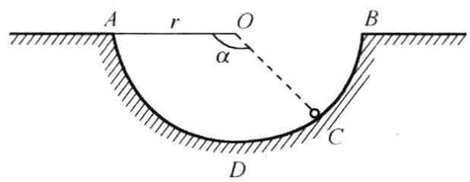


Fig. 2-27 For problem 2-13

2-14 A pilot in a fast jet aircraft loops the loop. Assume that the aircraft maintains a constant speed of 200 m/s and the radius of the loop is 1.5 km.

(1) At the bottom of the loop, what is the apparent weight that the pilot feels? Express the answer as a multiple of his normal weight.

(2) what is his apparent weight at the top of the loop?

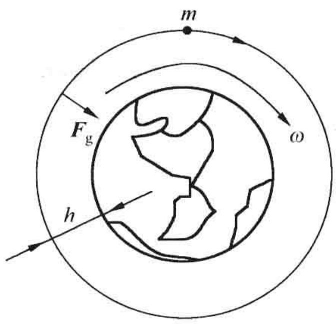
(3) at which position it is possible to experience the feeling of "weightless"?

2-15 Consider a satellite in a circular orbit concentric and coplanar with the equator of the earth (Fig. 2-28). At what height h from the surface of the earth, will the satellite appear to remain stationary when viewed by observers fixed on the earth? Suppose that the rotation of the satellite in its orbit is in step with the spin of the earth. The mass of the earth $M_e = 5.98\times10^{24}$ kg.

2-16 Assuming that a body is launched with a vertical velocity v_e from the surface of the earth, if we want the body reach a destination at which the earth's gravitational force no long effects it so that it will coast along with zero velocity without being pulled back to the earth, that launching velocity is called escape velocity, find it, starting from Eq. (2-20) ($R_e=6.37\times10^6$ m).

2-17 In the system shown in Fig. 2-29, m_1, m_2, and m_3 are known, suppose that the cord and pulleys are mass-less, neglect all possible friction, find the acceleration of each body and the tension in the cord.

2-18 In the system shown in Fig. 2-30, $M_A=1.0$ kg, $M_B=2.0$ kg, $M_C=3.5$ kg, $\theta=30°$, the coefficient of kinetic friction between A and the incline is $\mu_{k1}=0.1$, between A and B is $\mu_{k2}=0.80$. Suppose that the cord and pulley are mass-less, ignore the possible friction on the axis. Find the acceleration of B relative to A.



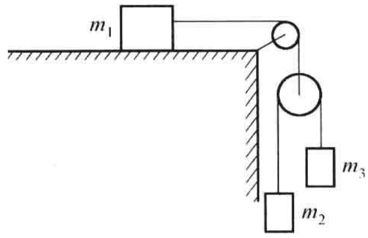
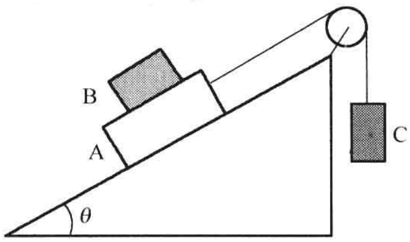


Fig. 2-28 For problem 2-15 Fig. 2-29 For problem 2-17

2-19 A flexible chain of length l and mass m is initially placed at rest on a smooth frictionless surface ABC with the distance $l-a$ from B to D (Fig. 2-31), the angle of the incline is θ. Calculate

(1) the speed of the chain when the end D just arrives at the point B;

(2) the time required from the beginning of the motion to the moment when the whole chain just leaves the horizontal part of the surface.



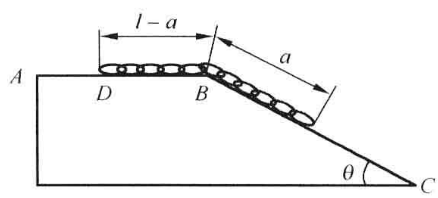


Fig. 2-30 For problem 2-18 Fig. 2-31 For problem 2-19

2-20 A small ball bearing is released from rest and drops through viscous medium. The retarding force acting on the ball bearing has magnitude kv, where k is a constant depending on the radius of the ball and the viscosity of the medium, and v is the bearing's velocity. Find the terminal velocity acquired by the ball bearing and the time taken to reach a speed of half a terminal velocity.

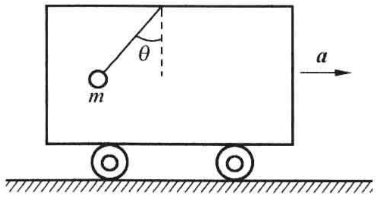
2-21 Consulting Example 2-7, use elevator as a non-inertial reference frame, solve problem 2-2 again.

2-22 A pendulum suspended from the roof of a coach moving along a straight rail with acceleration a, someone at the coach observed that the cord of the pendulum forms an angle θ with the vertical and measured it (Fig. 2-32), how could he calculate the acceleration by taking the coach as reference frame.

2-23 The system shown in Fig. 2-33, placed in an elevator moving up with an acceleration $a = 0.2g$, body A and B have equal mass m, A is on the smooth horizontal surface, B is hanged and connected to A by a light cord over a pulley. Neglect the possible friction and the masses of the cord and pulley, take the elevator as reference frame, find

(1) the acceleration of the bodies related to the elevator;

(2) the tension in the cord.



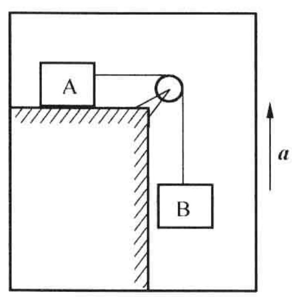


Fig. 2-32 For problem 2-22 Fig. 2-33 For problem 2-23

2-24 A steal block of mass m placed on the surface of a disc table rotating about the fixed central axis. The static friction coefficient between the block and the table is μ_s, the distance from the block to the center of the disc is r. In order to keep the block without sliding relative to the table, what is the maximum angular speed (use Fig. 2-14 and the rotating frame)?

Chapter 3

Work and Energy

In many important problems encountered in physics, the force on a particle is known as a function of position $\boldsymbol{F}(\boldsymbol{r})$ that leads to the definition of two new concepts: work and energy. You will find that these powerful methods will enable us to solve problems with remarkable ease. But work and energy are the first steps on the trail to an universal law—so far known no exceptions, the Law of conservation of energy.

Newtonian mechanics does fail when we apply it to particles moving at speed comparable to the speed of light, yielding there to Einstein's special theory of relativity. It also fails when we apply it to motions of electrons in atom, yielding in that case to quantum physics. The law of conservation of energy as well as the other two conservation laws, however, hold in all these domains.

The other conservation laws involving mechanical quantities are the conservation of momentum and angular momentum, which is the core of the following two chapters. Conservation laws play an important role in the world of matter. Physicists apply all conservation laws throughout the study of the objective world especially the microscopic field of physics.

3.1 Work

3.1.1 Work done by a constant force

Consider that a constant force $\boldsymbol{F}$ acts on a particle moving in one dimension, Fig. 3-1 shows the general case in which the force vector and the displacement vector makes a constant angle θ with each other. Then the work done by this force on the particle as it moves through a displacement $\boldsymbol{S}$, is defined as

$$W = (F\cos\theta)S \tag{3-1}$$

In words, we say that work is equals to the displacement times the component of the force along the displacement. Thus only the component of the force along the displacement products work. The work is positive or negative depending on the angle θ. If $\theta < 90°$, the work is positive; if $\theta > 90°$, the work is negative; if the force is parallel or anti-par-

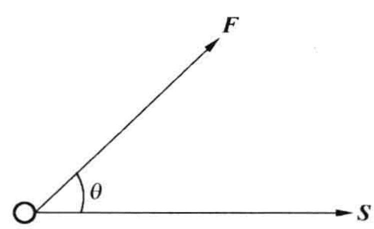


Fig. 3-1 A constant force $\boldsymbol{F}$ acts on a particle undergoing a displacement $\boldsymbol{S}$

allel to the displacement, then, the work is simply FS or $-FS$, respectively. If the force is perpendicular to the direction of motion($\theta=90°$), then the work is zero. This means that neither the normal force $\boldsymbol{N}$ on the body sliding on a surface nor the centripetal force on a body in circular motion does any work—both of them are always perpendicular to the instantaneous motion.

Work is a scalar, although the two quantities involved in its definition, force and displacement, are vectors. We can write Eq. (3-1) more compactly in vector form, as a scalar (or dot) product, thus

$$\boldsymbol{W}=\boldsymbol{F}\cdot\boldsymbol{S} \tag{3-2}$$

This equation is identical with Eq. (3-1) in every respect. The SI unit of work is called joule.

1 joule=1 J=1 N · m=1 kg · m^2/s^2. The dimension of work, $[W]=ML^2T^{-2}$.

3.1.2 Work done by a variable force

Let us now consider a particle moving along a curvilinear path under the action of a force $\boldsymbol{F}$ which is not a constant but a variable vector, that is, both of the magnitude and direction of the force depends on the position of the particle, as shown in Fig. 3-2.

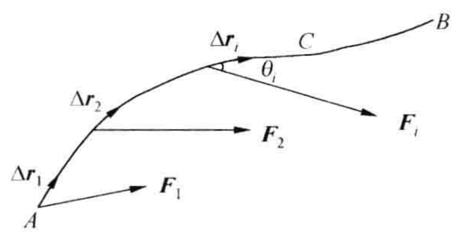


Fig. 3-2 A variable force acts on a particle moving along a curvilinear path c

What work is done by this force as the particle on which it act moves from an initial point A to a final point B? To find out it let us divide the path into a number of intervals of width $\Delta\boldsymbol{r}$, corresponding to a number of displacement $\Delta\boldsymbol{r}_1$, $\Delta\boldsymbol{r}_2$, $\Delta\boldsymbol{r}_3$, …, $\Delta\boldsymbol{r}_i$, …. We choose $\Delta\boldsymbol{r}$ small enough so that we can regard every interval as a very short straight line and we can take the force as reasonably constant over that interval. Therefore, the element work ΔW_i done by $\boldsymbol{F}_i$ over any particular interval $\Delta\boldsymbol{r}_i$ is given by Eq. (3-1) or Eq. (3-2):

$$\Delta W_i=\boldsymbol{F}_i\cdot\Delta\boldsymbol{r}_i=F_i\cos\theta_i\Delta r_i$$

To find the total work done by the variable force as the particle moves from A to B, we add up all work elements, that is

$$W=\sum\Delta W_i=\sum\boldsymbol{F}_i\cdot\Delta\boldsymbol{r}_i$$

This is an approximation to the actual value of the work. According to the concept of calculus you have learned, we can make the approximation better by reducing $\Delta\boldsymbol{r}$ to infinite small, thus we get more intervals, let the width of $\Delta\boldsymbol{r}$ approach zero, the number of intervals then becoming infinite large. We then have, as an exact result,

$$W=\sum_{\lim\Delta\boldsymbol{r}_i\to 0}\boldsymbol{F}_i\cdot\Delta\boldsymbol{r}_i \tag{3-3}$$

As you have known in your calculus course, this limit is exactly what we mean by the curvilinear integral of the function $\boldsymbol{F}(\boldsymbol{r})$ along the path C between the limits A and B. Thus, Eq. (3-3) becomes

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B F\cos\theta \, \mathrm{d}s \tag{3-4}$$

where $\mathrm{d}s$ is the magnitude of $\mathrm{d}\boldsymbol{r}$, θ is the angle between $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$. If we know the function $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$, we can write the elementary work as $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, and carry out the integration with the limits A and B, then find the work.

Special case (a) if $\boldsymbol{F}$ = a constant vector over the entire path (Fig. 3-3), Eq. (3-4) becomes

$$W = \boldsymbol{F} \cdot \int_A^B \mathrm{d}\boldsymbol{r} = \boldsymbol{F} \cdot (\boldsymbol{r}_B - \boldsymbol{r}_A) = \boldsymbol{F} \cdot \overrightarrow{AB} = \boldsymbol{F} \cdot \boldsymbol{S} \tag{3-5}$$

It is the same as Eq. (3-2) that means Eq. (3-5) holds also in the case of curvilinear motion of a particle under the action of a constant force.

Special case (b) if $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ are along the same straight line, say, x axis, thus

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = \int_A^B F(x)\mathrm{d}x \tag{3-6}$$

Geometrically, this value is equal to the area that lies under the $F(x)$ curve between the limits A and B. If $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ in opposite direction, the work is negative.

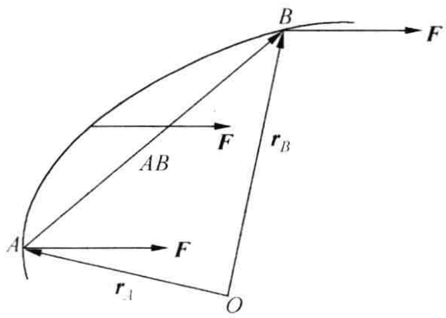


Fig. 3-3 Displacement over a curvilinear path

3.1.3 Work done by resultant force

If the particle moves under the action of several forces, say, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, ⋯, the resultant force $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots$, the work done by the resultant is then

$$\begin{aligned} W &= \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_A^b (\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots) \cdot \mathrm{d}\boldsymbol{r} \\ &= \int_A^B \boldsymbol{F}_1 \cdot \mathrm{d}\boldsymbol{r} + \int_A^B \boldsymbol{F}_2 \cdot \mathrm{d}\boldsymbol{r} + \int_A^B \boldsymbol{F}_3 \cdot \mathrm{d}\boldsymbol{r} + \cdots \\ &= W_1 + W_2 + W_3 + \cdots \end{aligned} \tag{3-7}$$

which means that the work done by the resultant force is equal to the algebraic sum of the works done by every force acting on the particle.

3.1.4 Power

In practical applications, especially in connection with machines and engineering, it is important to know the rate at which work is done. If an amount of work ΔW is carried out in a time interval Δt, the average power for that interval is defined to be $\bar{P} = \Delta W / \Delta t$. The instantaneous power P is defined by

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{3-8}$$

that is power is the work done per unit time during a very small time interval $\mathrm{d}t$. The SI unit of power is the joule per second, or watt (W) that is named after James Watt, who greatly improved the rate at which steam engines could do work. 1 watt = 1W = 1J/s. The dimension of power is $[P] = \mathrm{ML^2T^{-3}}$.

Considering $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$ and $\boldsymbol{v} = d\boldsymbol{r}/dt$ Eq. (3-8) becomes

$$P = \frac{dW}{dt} = \frac{\boldsymbol{F} \cdot d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \boldsymbol{v} \tag{3-9}$$

and then the power is defined also as the scalar product of force times velocity.

Example 3-1 A 45 kg block of ice slides down on an incline of 1.5 m long and 0.9 m high. A worker pushes up on the ice parallel to the incline so that it slides down at constant speed. The coefficient of friction between the ice and the incline is 0.10. Find

(1) the force exerted by the worker;

(2) the work done by the worker on the block;

(3) the work done by gravity on the block;

(4) the work done by the surface of the incline on the block;

(5) the work done by the resultant force on the block.

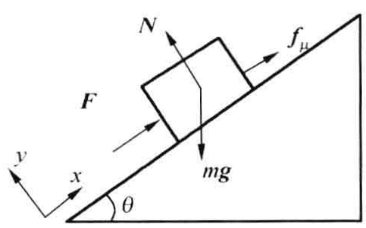


Fig. 3-4 For Example 3-1

Solution (1) Draw a sketch including all forces acting on the block and set up the coordinates as shown in Fig. 3-4. We have two equilibrium equations in x and y directions respectively:

$$F - mg\sin\theta + f_\mu = 0 \quad ①$$

$$N - mg\cos\theta = 0 \quad ②$$

and

$$f_\mu = \mu N = \mu mg\cos\theta \quad ③$$

substituting Eq. ③ into Eq. ① and rearranging it gives

$$F = mg(\sin\theta - \mu\cos\theta)$$

in which

$$\sin\theta = 0.9/1.5 \approx 0.6, \quad \cos\theta = \sqrt{1-0.6^2} = 0.8$$

Substituting the known data:

$$F = 45 \times 9.8 \times (0.6 - 0.1 \times 0.8)\text{N} = 229.3 \text{ N}$$

(2) The work done by the worker

$$W_F = \boldsymbol{F} \cdot \boldsymbol{S} = F\cos180^\circ S = -229.3 \times 1.5 \text{ J} = -343.9 \text{ J}$$

(3) The work done by weight

$$W_G = \boldsymbol{G} \cdot \boldsymbol{S} = mg\cos(90^\circ - \theta)L = mgh = 45 \times 9.8 \times 0.9 = 396.9(\text{J})$$

(4) The work performed by the surface on the incline is the work done by frictional force:

$$W_\mu = \boldsymbol{f}_\mu \cdot \boldsymbol{S} = f_\mu \cos180^\circ L = -\mu mg\cos\theta L = -0.1 \times 45 \times 9.8 \times 0.8 \times 1.5 = -52.9 \text{ (J)}$$

(5) The total work done by the net force:

$$W_F + W_G + W_\mu = 0$$

Which is expected because the speed is uniform so that

$$\sum \boldsymbol{f}_i = 0, \quad \sum W_i = 0$$

Example 3-2 Work done by a spring. As an important example of variable force. consider a spring of force constant k fixed on one end and attached with a block on the other end, see Fig. 3-5, set the origin of coordinate axis Ox at its relaxed position O. How much work does the spring force do when it changes from an initial state in which its extension is x_i to a final state in which its extension is x_f. Suppose there is no frictional force between the block and the horizontal surface on which it moves.

Solution With respect to our origin O of x axis, the extension of the spring is x, therefore, Hooke's law can be written as $F(x) = -kx$, here x is just the displacement of the end the spring, the restoring force $\boldsymbol{F}(x)$ is always points in the opposite direction of displacement. The element work done by the spring force is

$$dW = F(x)dx = -kx\,dx$$

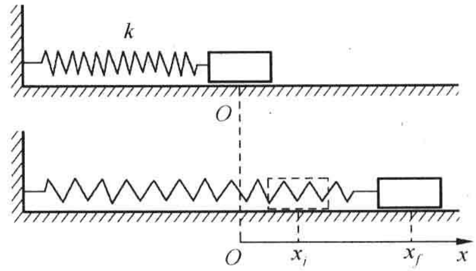


Fig. 3-5 For Example 3-2

the total work from state x_i to state x_f is then

$$W = \int_{x_i}^{x_f} F(x)dx = \int_{x_i}^{x_f} -kx\,dx = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 \quad (3\text{-}10)$$

Discussion

(1) If the spring is initially in its relaxed state($x_i=0$) and is stretched (or compressed) to an extension x, the absolute value of work done by the spring force (elastic force) is $W = \frac{1}{2}kx^2$, whose sign is different in the cases as the following.

(2) If the spring is stretched, $x>0$, when $x_f>x_i$, then $W<0$, otherwise $W>0$.

(3) If the spring is compressed, $x<0$, when $|x_f|>|x_i|$, $W<0$, otherwise $W>0$.

(4) Suppose that the position of the spring's end changes in the sequence of x_i—0—x_i—x_f, by using Eq. (3-10), we obtain the total work

$$W = \left(\frac{1}{2}kx_i^2 - 0\right) + \left(0 - \frac{1}{2}k(x_i)^2\right) + \left(\frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2\right) = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

which means that the work done by a restoring force of a spring depends only on the initial and final position rather than the path.

(5) Suppose that there exists friction between the surface and the block, the coefficient of friction is μ. How about the work done by frictional force: (a) from x_i along right to x_f; (b) in the same sequence as in question (4), what conclusion will be made, which leas to solve by yourself.

Example 3-3 Work done by the universal force of gravitation

In Fig. 3-6, the object of mass M and the object of mass m interact on each other with a universal force of gravitation

$$\boldsymbol{F} = -\frac{G_0 Mm}{r^3}\boldsymbol{r}$$

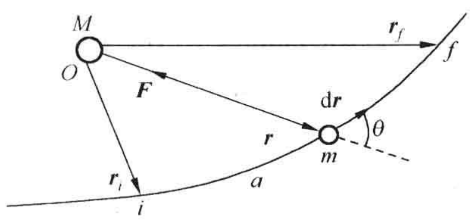


Fig. 3-6 For Example 3-3

here G is the force constant. Assume that $M \gg m$ so that the object M could be regarded as at rest and as the origin of our coordinate, $\boldsymbol{r}$ is therefore the position vector from M to m. Calculate the work done by the universal gravity on object m when it moves from point i to point f along path iaf.

Solution From Eq. (3-4), $dW = \boldsymbol{F} \cdot d\boldsymbol{r} = F\cos(\pi - \theta)dr = -Fdr$

$$W = -G_0 Mm \int_{r_i}^{r_f} \frac{dr}{r^2} = G_0 Mm\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \quad (3\text{-}11)$$

Like in Example 3-2, if the path is ibf rather than iaf, the result will be all the same. Therefore, the work done by the universal force of gravitation is independent of the path too.

3. 2 Kinetic Energy and the Law of Kinetic Energy

We now consider what effect will be caused when a force has done an amount of work on a particle? Suppose a particle moves along a curvilinear path under action of net force $\boldsymbol{F}$ as shown in Fig. 3-7.

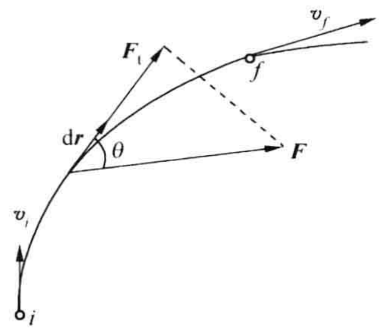


Fig. 3-7 The tangential component of net force

The elementary work done by the net force during the displacement $\mathrm{d}\boldsymbol{r}$ is

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = F\cos\theta\ \mathrm{d}s$$

where $F\cos\theta = F_t$, the tangential component of the net force. Recall that the tangential equation of Newton's second law is

$$F_t = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Consider that $\mathrm{d}s = v\,\mathrm{d}t$, therefore

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = F\cos\theta\ \mathrm{d}s = m\frac{\mathrm{d}v}{\mathrm{d}t}v\mathrm{d}t = mv\mathrm{d}v$$

By integrating, we find the total work done in moving the particle from i to f is

$$W = \int_i^f F\cos\theta\,\mathrm{d}s = \int_i^f mv\mathrm{d}v = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \qquad (3\text{-}12)$$

in which v_f is the particle's velocity at f and v_i is the particle's velocity at i. The result Eq. (3-12) indicates that no matter how the force varies with position and no matter what the path is followed by the particle, the value of the work W done by the force is always equal to the difference between the value of the quantity $\frac{1}{2}mv^2$ at the end and its value at the beginning of the path. This important quantity is defined as kinetic energy of the particle, and designated by E_k, therefore

$$E_k = \frac{1}{2}mv^2 \qquad (3\text{-}13)$$

Eq. (3-12) can then be expressed in the form

$$W = E_{kf} - E_{ki} = \Delta E_k \qquad (3\text{-}14)$$

which is called **the law of kinetic energy of a particle, or in words as the change in the kinetic energy of a particle is equal to the total work done on that particle by all the forces that act on it .**

If the work $W>0$, then the final kinetic energy E_k is larger than that initial kinetic energy E_{ki} that is, the kinetic energy increases by $\Delta E_k = E_{kf} - E_{ki}$; otherwise, when $W<0$, the particle overcomes the external force and does some work by decreasing its kinetic energy.

Kinetic energy is obviously measured in the same unit as work, i. e. in Joules in SI, and has the same dimension as work, i. e. $[E_k] = ML^2T^{-2}$.

The law of kinetic energy is not a new independent law of classical mechanics, it is derived directly from Newton's second law. It gives us a new way to look at familiar problems and makes the solution of certain kinds of problem much easier.

Example 3-4 Apply the law of kinetic energy to solve the first problem in Example 2-4.

Solution From Fig. 3-8, we know that only the tangential component of weight, f_t does work, and $f_t = mg\sin\theta$, when the angular position changes from θ to $\theta + d\theta$. making an angular displacement $d\theta$, as shown in Fig. 3-8, the elementary work done by the tangential force then is

$$dW = f_t ds = -mg\sin\theta R d\theta$$

the total work is

$$W = -mgR\int_0^\theta \sin\theta\, d\theta = mgR(\cos\theta - 1)$$

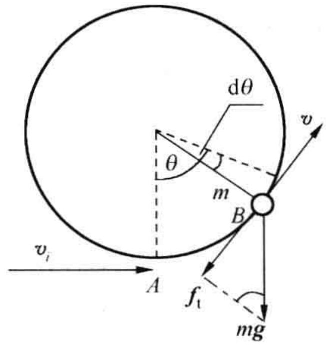


Fig. 3-8 For Example 3-4

Applying the law of kinetic energy to the ball, because $v_B = v$, we have

$$mgR(\cos\theta - 1) = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

so that

$$v = \sqrt{v_0^2 + 2gR(\cos\theta - 1)} = \sqrt{v_0^2 - 2gR(1 - \cos\theta)}$$

This result is the same as that in Example 2-4. But we have simplified the procedure by means of applying the law of kinetic energy.

Example 3-5 A 2 kg particle is at rest in the beginning, then an external force $F = 12t(\mathrm{N})$ acts on it, the direction of the force does not change. Find the work done by the force in the first 2 seconds and the power at $t = 1$ s and $t = 2$ s.

Solution From the Newton's second law, the acceleration of the particle is

$$a = \frac{F}{m} = \frac{12t}{2} = 6t \quad (\mathrm{SI}) \qquad ①$$

The velocity is therefore

$$v = \int a\, dt = \int_0^t 6t dt = 3t^2 \quad (\mathrm{SI}) \qquad ②$$

Using the law of kinetic energy, Eq. (3-14), the work done by the external force is

$$W = E_{k2} - E_{k0} = \frac{1}{2}mv_2^2 - 0 = \frac{9}{2}mt^4 \quad (\mathrm{SI}) \qquad ③$$

Substituting $m = 2$ kg and $t = 2$ s we have

$$W = \frac{9}{2} \times 2t^4 = 9t^4 = 9 \times 2^4 = 144(\mathrm{J})$$

The power is

$$P = \boldsymbol{F} \cdot \boldsymbol{v} = Fv = 12t \times 3t^2 = 36t^3 \quad (\mathrm{SI})$$

$$P_1 = 36\ \mathrm{W}, \quad P_2 = 36 \times 2^3 = 288\ \mathrm{W}$$

Another way to find the work is integration of the work done by the varying force:

$$v = \frac{dx}{dt}, \quad dx = v dt = 3t^2 dt$$

so that

$$W = \int \boldsymbol{F} \cdot d\boldsymbol{r} = \int 12t \times 3t^2 dt = 9t^4 \Big|_0^2 = 144(\mathrm{J})$$

3.3 Conservative Force and Potential Energy of Weight

If you pull back a small block attached on a spring, you do work but no kinetic energy appears. However, you can generate some energy in the form of a oscillating block-spring—by simply releasing the block. It makes sense to say that the kinetic energy of the spring-block system is in some sense "stored" in the stretched spring, which suggests that the energy is hidden away but has the potential to reappear in kinetic form. This kind of energy is named potential energy that could also be called configuration energy, because the system on which the work is done—the spring in our example, stores potential energy by changing its configuration, say, being stretched or compressed.

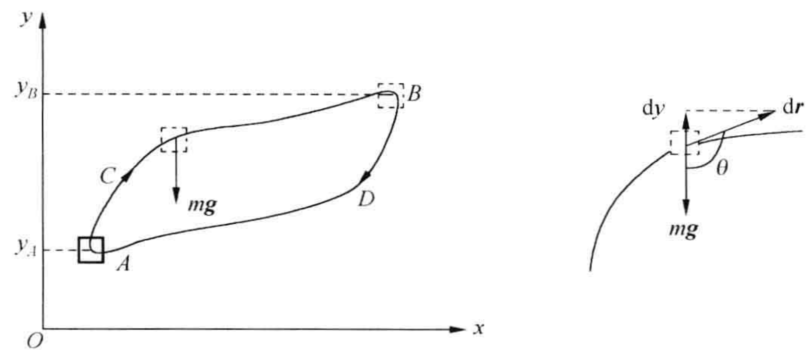
Before we introduce three kinds of potential energy, it is important to introduce a concept of conservative force. We now start from the Gravitational force (or weight) near the surface of the earth.

1. Work done by gravity near the surface of the earth, Conservative force

Let us consider only displacements near the earth surface, so that variations of gravitation force with distance from the earth's center can be neglected. The downward gravitational force on the body is then constant and equal to the weight $\boldsymbol{G} = m\boldsymbol{g}$. We call this force as constant gravitational force or weight in the followings.

Suppose that a body of mass m starts from point A at elevation y_A and moves up to point B at elevation y_B along some arbitrary path ACB as shown in Fig. 3-9(a). Fig. 3-9(b) is an enlarged view of a small portion of the path. The element work done by the gravitational force is

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = m\boldsymbol{g} \cdot \mathrm{d}\boldsymbol{r} = mg\cos\theta \mathrm{d}r$$



(a) (b)

Fig. 3-9 The work done by weight

in Fig. 3-9(b)

$$\cos\theta \mathrm{d}r = -\cos(180° - \theta)\mathrm{d}r = -\mathrm{d}y$$

so that

$$\mathrm{d}W = -mg\mathrm{d}y \tag{3-15}$$

the total work is then

$$W_{ACB} = -mg\int_{y_A}^{y_B}\mathrm{d}y = -mg(y_B - y_A) = -mgh \tag{3-16}$$

here

$$h = y_B - y_A$$

Note that the path ACB is arbitrarily chosen, therefore, we can make conclusion that **the work done by gravitational force depends only on the initial and final positions (configurations) and not on the path.**

It is the distinctive property of gravitational force in doing work. If the body moves down from point B back to point A along another arbitrary path BDA, the work done by weight then equal to

$$W_{BDA} = -mg\int_{y_B}^{y_A} dy = -mg(y_A - y_B) = mgh \tag{3-17}$$

If the body makes a "round trip", the path is a closed loop $ACBDA$, then the work done by weight is

$$W_{ACBDA} = W_{ACB} + W_{BDA} = 0 \tag{3-18}$$

which leads to another expression of the distinctive property of Gravity mentioned before:

The work done on a body by gravitational force along any closed path is zero.

The force whose work depends only on the initial and final positions (configurations) is called a conservative force. So that, gravitational force is a kind of conservative force.

2. Potential energy of weight

Since the term mgy in Eq. (3-16) and Eq. (3-17) is a function of position y, moreover, the work done by gravity is equal to the value of the change of this function. Furthermore, this value is a constant for the given position A and B, no matter how is the path connecting A and B. Therefore, we can define the value of this work as the change of gravitational energy from the initial position i to the final position f.

Let E_{pi} and E_{pf} represent the weight potential energy of a body at position i and position f respectively, thus

$$W_{if} = E_{pi} - E_{pf} = -(E_{pf} - E_{pi}) \tag{3-19}$$

Which means that the work done by gravity acting on a moving body from the initial position to the final position equals to the decrement of gravitational potential energy (or the negative value of increment of that potential energy).

To determine the value of gravitational potential energy of a body at a given point we must chose a reference position (point or a plane) where the gravitational potential energy is zero. The most convenient reference is the surface of the earth (the ground). In Eq. (3-19), let $E_{pf} = E_{pg} = 0$, that is the position f is chosen as the surface of the earth, position g, we have

$$W_{ig} = -(E_{pg} - E_{pi}) = E_{pi} \tag{3-20}$$

that is, if ground is chosen to be the reference of zero potential energy, the gravitational potential energy at position i equals to the work done by the gravity during the changing in position from point i to the ground.

If the elevation of a given position above the ground is h, the gravitational potential energy of the body at that point is therefore

$$E_p = mgh \tag{3-21}$$

with which we have been so familiar before.

It is necessary to point out that the change in gravitational potential energy of a body between two given positions (or configurations) is of absolute significance in all possible different zero reference levels while the particular value of the potential energy at a given position is of relative significance.

We should also recognize that gravitational potential energy belongs to the system including the body and the earth, interacting on each other with a pair of gravitational forces.

Not only the constant force of gravitation but also the other several kinds of forces are in the category of conservative forces as long as the work done by the force is independent of the path and consequently has a corresponding function of potential energy associated with it i. e. the work equals to the difference between the final and the initial values of the function of that potential energy. The principles discussed above hold for all of such cases as the elastic force produced by a extended spring in Example 3-2 as well as the universal force of gravitation between any pair of objects in Example 3-3.

3.4 Elastic Potential Energy and Universal Gravitational Potential Energy

According to the previous argument about conservative force, we are now ready to define another two potential energies in the same way of introducing potential energy of weight.

3.4.1 Elastic potential energy

Eq. (3-10), the result of Example 3-2, the work done by a spring on the block when the spring changes from the initial state of extension x_i to the final state of extension x_f along an arbitrary path in x axis,

$$W = \frac{1}{2}kx_i^2 - \frac{1}{2}x_f^2$$

indicates that elastic (or restoring) force produced by a extended spring is a conservative force, therefore, we can define Eq. (3-10) as the negative value of increment of elastic potential energy of the spring block system, that is

$$W = -\int_{x_i}^{x_f} kx\,\mathrm{d}x = -\left(\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right) = -(E_{pf} - E_{pi}) \tag{3-22}$$

Then we chose the relaxed position (origin of the coordinate in Fig. 3-5, i. e. $x=0$) as the reference point of zero elastic potential, in which state the spring has no extension, so that, when the extension (displacement) is x, the elastic potential energy of the system equals then to the work done by the spring from position x to the position $x = 0$, that is

$$E_p = -\int_x^0 kx\,\mathrm{d}x = \frac{1}{2}kx^2 \tag{3-23}$$

which means that the positive value of the work done by the spring on the block equals to the decrement of the potential energy during that process. In the opposite process, as the block moves from origin ($x=0$) to the x position, the negative work done by the spring on the block will be the increment of the potential energy stored in the spring, in the other words, the capacity of the spring to perform work will then increase. By solving the problem 3-18 in the end of this chapter, you will convince yourself that the change in elastic potential energy has an absolute meaning, but the value of that energy at a given position is a relative one which depends on the reference of the zero potential chosen.

3.4.2 Universal gravitational potential energy

The work done by the universal gravitational force on a body that moves along an arbitrary path from the initial position r_i to the final position r_f, in Example 3-3 (Fig. 3-6) is given by Eq. (3-11)

$$W = -\int_{r_i}^{r_f} \frac{G_0 Mm\,\mathrm{d}r}{r^2} = G_0 Mm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

Which indicates that the universal gravitational force is also a conservative force. Once again we can define the amount of the work as the negative value of increment of the universal gravitational potential energy during that process, that is

$$W = -\left[\left(-\frac{G_0 Mm}{r_f}\right) - \left(-\frac{G_0 Mm}{r_i}\right)\right] = -(E_{pf} - E_{pi}) \qquad (3\text{-}24)$$

If we take $E_p = 0$ at $r = \infty$ as the reference of zero potential, the mutual gravitational potential energy of the system $(M+m)$ is therefore equal to the work done by the force when it moves from position r to $r = \infty$, that is

$$E_p = -\int_r^{\infty} \frac{G_0 Mm}{r^2}\mathrm{d}r = -\frac{G_0 Mm}{r} \qquad (3\text{-}25)$$

from which the expression of gravitational potential energy near the surface of the earth, Eq. (3-21) i. e. the expression of the weight potential energy can be deduced, we leave it in problem 3-15. You will find that the potential energy of weight can be treated as a special case of universal gravitational potential energy.

Additional to the above, the Coulomb's force between charged particles is conservative so that associated with a potential function, which we shall deal with in chapter 6. And the force between atoms in an molecule is conservative too. You can find the corresponding potential energy by solving problem 3-16.

3.5 Conservation of Mechanical Energy

3.5.1 Kinetic energy of a system

Consider a system included particle 1 of mass m_1 and particle 2 of mass m_2, suppose that the external force $\boldsymbol{F}_1$ exerts on m_1 and the external force $\boldsymbol{F}_2$ on m_2, meanwhile, m_1 and m_2 exert action-reaction forces f_{21} and f_{12} on each other as shown in Fig. 3-10.

Our plan is to deduce the relation between the change in total kinetic energy of the system and the work done on the system. Let v_{1i} and v_{2i}; v_{1f} and v_{2f} represent the velocities of the particles at the initial state and at the final state respectively. Applying the law of kinetic energy to particle m_1 and m_2, we have

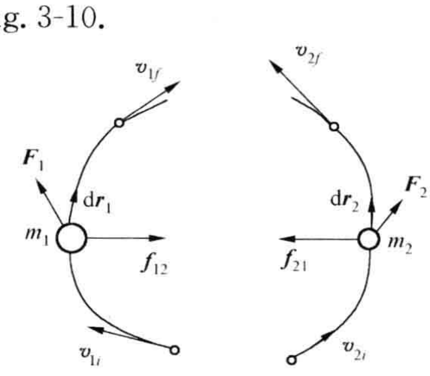


Fig. 3-10 Two particles system

$$\int_i^f (\boldsymbol{F}_1 + \boldsymbol{f}_{12}) \cdot \mathrm{d}\boldsymbol{r} = \frac{1}{2}m_1 v_{1f}^2 - \frac{1}{2}m_1 v_{1i}^2$$

$$\int_i^f (\boldsymbol{F}_2 + \boldsymbol{f}_{21}) \cdot d\boldsymbol{r} = \frac{1}{2}m_2 v_{2f}^2 - \frac{1}{2}m_2 v_{2i}^2$$

Rewrite them as

$$W_{ex1} + W_{in1} = E_{k1f} - E_{k1i}$$
$$W_{ex2} + W_{in2} = E_{k2f} - E_{k2i}$$

add the two sides up respectively, and use

$$W_{ex} = W_{ex1} + W_{ex2}, \quad W_{in} = W_{in1} + W_{in2}$$
$$E_{kf} = E_{k1f} + E_{k2f}, \quad E_{ki} = E_{k1i} + E_{k2i}$$

We have

$$W_{ex} + W_{in} = E_{kf} - E_{ki} \tag{3-26}$$

which means that the total work done by the external and internal forces equals to the change in the total kinetic energy of the system from the initial state to the final state. We call Eq. (3-26) as the kinetic energy law of a system. This conclusion can be extended to the system including more than two bodies.

3.5.2 Work—energy theorem

It is possible that the internal forces include either conservative ones or non conservative ones, so that we divide the work done by the internal forces into two parts

$$W_{in} = W_{conin} + W_{nonin}$$

thus, Eq. (3-26) con be rewritten as

$$W_{ex} + W_{conin} + W_{nonin} = E_{kf} - E_{ki} \tag{3-27}$$

On the other hand, we have defined that the work done by conservative internal force equals the negative value of the change in the corresponding potential energy between the two states, that is

$$W_{conin} = -(E_{pf} - E_{pi})$$

Substituting this equation into Eq. (3-27) and rearranging it, we have

$$W_{ex} + W_{nonin} = (E_{pf} + E_{kf}) - (E_{pi} + E_{ki}) \tag{3-28}$$

The sum of total kinetic energy and potential energy is called mechanical energy of the system, labeled as E, so Eq. (3-28) can be written as

$$W_{ex} + W_{nonin} = E_f - E_i \tag{3-29}$$

Which means that **the sum of the work done by the external forces and the non conservative internal forces equals to the increment of the mechanical energy of the system from initial state to final state. This conclusion is called the work-energy theorem.**

3.5.3 Conservation law of mechanical energy

The most important special case that we concern is that from Eq. (3-29), work-energy theorem, if $W_{ex} + W_{nonin} = 0$, then

$$E_f = E_i = \text{constant} \tag{3-30}$$

Eq. (3-30) is a statement of **the law of conservation of mechanical energy for a system**, which indicates that if no external force and non conservative force act on the particles in the system or the work done by them is all zero, the mechanical energy of the system remains constant.

In the other words, if only internal conservative forces do work on the system, its mechanical energy will never change.

Let the left side of Eq. (3-28) equals zero and rearrange it, we get an equivalent form of Eq. (3-30)

$$E_{pf} - E_{pi} = -(E_{kf} - E_{ki})$$

or

$$\Delta E_p = -\Delta E_k \tag{3-31}$$

Which is another expression of the law of conservation of mechanical energy, it tell us that every change in the kinetic energy of the system is accompanied by an equal but opposite change in the potential energy of it. For instant, a block attached to a spring oscillates back and forth on a frictionless horizontal surface, in Fig. 3-5, suppose the magnitude of the maximum displacement is x_0. The energy shuttles back and forth between the spring and the block as they do work on each other. When the block passes the equilibrium position from left or right, in one cycle, the energy is all kinetic, while when reaches the position x_0 or $-x_0$, it is all potential. At intermediate positions the energy is shared by these two forms, but the sum E, the total mechanical energy of the system, remains constant at all times. As long as the oscillation continues, the cycle of energy transformation repeats itself.

Example 3-6 (1) Once again solve the same problem of Example 2-4: A small ball of mass m is attached by a thin cord, moving along a circle of radius R about a fixed center O in the vertical plane (Fig. 3-11). Suppose that the ball is passing through the bottom point A with the initial speed v_0. find the speed as a function of angular position.

(2) Prove that in order to let the ball complete one circle, the initial speed $v_0 \geqslant \sqrt{5gR}$ must be satisfied (neglect air resistant).

Solution (1) First, we choose the ball and Earth as our system, therefore, the tension being a external force which does zero work on the ball. The only force doing work is the conservative force, weight, so that the mechanical energy of this system is conserved.

Second, choose the bottom position A as the reference level of zero weight potential energy, choose the initial state as the ball passed through A, the final state as the ball reaches angular position θ. Applying the law of energy conservation, we have

$$E_i = \frac{1}{2}mv_0^2 + 0$$

$$E_f = \frac{1}{2}mv^2 + mgR(1-\cos\theta)$$

from

$$E_i = E_f$$

we have the speed at position θ

$$v = \sqrt{v_0^2 - 2gR(1-\cos\theta)}$$

Taking advantage of application of conservation of mechanical energy, which involves only energy of two states rather than the particular process in the motion, we solve this problem in the simplest way compared with those used in Example 2-4 (the Newton's second law) and Example 3-4 (the law of kinetic energy).

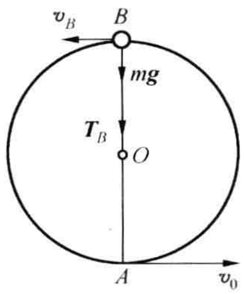


Fig. 3-11 For Example 3-6

(2) Choose state B as the final state when the ball passes the uppermost point as shown in Fig. 3-11. Because $E_A = E_B$, we have

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_B^2 + 2mgR$$

$$v_B^2 = v_0^2 - 4gR \qquad ①$$

on the other hand, the resultant force acting on the ball is $\boldsymbol{T}_B$ and $\boldsymbol{mg}$, the normal component equation of Newton's second law is then

$$T_B + mg = m\frac{v_B^2}{R}$$

The tension

$$T_B = m\frac{v_B^2}{R} - mg \qquad ②$$

Substitute Eq. ① into Eq. ②, we obtain

$$T_B = \frac{m}{R}(v_0^2 - 5gR)$$

Note that $T_B \geqslant 0$ is the critical condition that the ball can pass through point B and finish one circle, so that the minimum value of v_0

$$v_{\min} = \sqrt{5gR} \qquad ③$$

Example 3-7 Escape velocity of a projectile. Find the minimum launch velocity that a projectile, say, a spaceship requires in order to escape the earth's gravity.

Solution By "escape" we mean that the projectile's speed and kinetic energy approach zero as the distance from it to the center of earth, $r \to \infty$. Because that in the system of projectile + Earth, gravitational force is the only force doing work, so the mechanical energy of the system remains constant.

Take the initial position of the spaceship to be at the earth's surface and the final position at the infinity. The minimum launch speed is called the escape velocity, and corresponds to $v_f = 0$, We have

$$E_{ki} = \frac{1}{2}mv_i^2, \quad E_{kf} = \frac{1}{2}mv_f^2 = 0$$

$$E_{pi} = -\frac{G_0 M_e m}{R_e}, \quad E_{pf} = \lim_{r\to\infty}\left(-\frac{G_0 M_e m}{r}\right) = 0$$

From Eq. (3-30), $E_i = E_f$, the law of conservation of mechanical energy requires

$$\frac{1}{2}mv_i^2 - \frac{G_0 M_e m}{R_e} = 0$$

Solving this equation for the launch speed, we get

$$v_{esc} = \sqrt{\frac{2G_0 M_e}{R_e}} = \sqrt{2gR_e}$$

Using $R_e = 6.37\times10^6$ m, $g = 9.8$ m/s^2, the escape velocity $v_{esc} \approx 1.12\times10^4$ m/s$=11.2$ km/s obtained, which is also called the second universal velocity.

Example 3-8 In Fig. 3-12, a block of mass m attached with a spring of stiff constant k, is placed on a horizontal frictional surface. When the block is at the equilibrium position, an initial speed is given so that the block can reach the maximum displacement $x_0 = 0.05$ m. Suppose that $m=0.1$ kg, $k=20$ N/m, the coefficient of kinetic friction between the block and the surface is $\mu = 0.2$.

(1) What is the initial speed v_0 ?

(2) As the block moves back and passes $x = 0$ again, what is the speed v_{01} ?

(3) What is the magnitude of next maximum displacement x_{01}?

(4) What is the speed when the block leaves $x = x_{01}$ and reaches $x = -0.01$ m?

Solution (1) Choose the system including the block and the spring so that the elastic force is a conservative internal force, the frictional force becomes a external force which does negative work, therefore, the total energy is not conserved.

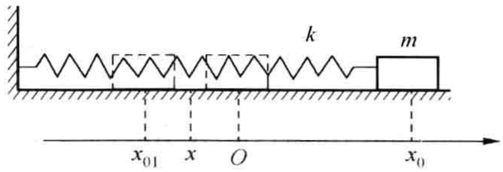


Fig. 3-12 For Example 3-8

Let the equilibrium position $x=0$ as the zero reference of the elastic potential energy. We apply the work-energy theorem

$$W_{fri} = E_f - E_i$$

from $x=0$ to $x=x_0$ the work done by the frictional force is

$$-\mu m g x_0 = \frac{1}{2}kx_0^2 - \frac{1}{2}mv_0^2$$

so that

$$v_0 = \sqrt{\frac{kx_0^2 + 2\mu m g x_0}{m}}$$

substituting the known data, we get

$$v_0 = \sqrt{0.696} = 0.834(\text{m/s})$$

(2) When the block passes $x=0$, again, we have

$$-2\mu m g x_0 = \frac{1}{2}mv_{01}^2 - \frac{1}{2}mv_0^2$$

$$v_{01} = \sqrt{v_0^2 - 4\mu g x_0} = 0.551 \text{ m/s}$$

(3) As the block reaches the next maximum displacement x_{01} (on the left), we have

$$-\mu m g\,|x_{01}| = \frac{1}{2}kx_{01}^2 - \frac{1}{2}mv_{01}^2$$

or

$$\frac{1}{2}kx_{01}^2 + \mu m g x_{01} - \frac{1}{2}mv_{01}^2 = 0$$

Solve this equation and take the positive answer, we get $|x_{01}| = 0.0304$m.

If took two states being $x = x_0$ and $x = x_{01}$, the result would be the same as above.

(4) Leaving $x=-0.0304$m, when the block passes $x = -0.01$m, the equation becomes

$$-\mu m g(|x_{01}| - |x|) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 - \frac{1}{2}kx_{01}^2$$

Substituting the known data, the instantaneous speed at that moment $v=0.291$ m/s obtained.

From above examples, the procedure to apply the law of conservation of mechanical energy or the work-energy theorem can be concluded as following:

(1) Choose the system so that the conservative force being internal force associated with potential energy.

(2) Choose the zero reference of potential energy.

(3) Take two states (or configurations) of the motion, write down the kinetic energy and potential energy respectively.

(4) Apply conservation law of energy, if only the conservative force does work, otherwise apply the work-energy theorem to write an equation connecting the two states and solve it.

(5) Sometimes, Newton's second law or other relations are needed to find the answer.

3.6 The Conservation of Energy

3.6.1 Non-conservative force

Consider the spring-block system again, in Example 3-8, because the mechanical energy of that system is no longer conserved, we can except that the motion decreases continually in maximum displacement and eventually dies away. The frictional force is therefore called a non-conservative or dissipation force. If we choose our system including the horizontal plane holding the block so that the frictional force becomes an internal force, using Eq. (3-29), substituting W_{fri} instead of W_{nonin}, we have

$$W_{fri} = E_f - E_i = \Delta E \tag{3-32}$$

Where W_{fri} is negative because the motion stops at last, that is $E_f=0$ and $\Delta E < 0$.

3.6.2 Conservation of energy

What happens to the "missing" energy in above example? The experiments and daily experiences imply us a clue in the fact that both the block and the surface over which it slides become slightly warmer as the oscillations of the system die away. It is as if the kinetic energy of the directional motion of the sliding block was transformed into kinetic energy of the disordered random motion of the molecules that make up the block and the surface over which it slides. We call such energy as internal energy or thermal energy and represent it by E_{int}. The change in internal energy in that system can be defined as the negative value of the work done by the frictional force, that is

$$W_{fri} = -(E_{fint} - E_{iint})$$

or

$$W_{fri} = -\Delta E_{int} \tag{3-33}$$

combining Eq. (3-32) and Eq. (3-33) yields

$$\Delta E = -\Delta E_{int} \tag{3-34}$$

or

$$\Delta E + \Delta E_{int} = 0 \tag{3-35}$$

as a statement of **the law of conservation of energy**. This law is more comprehensive and general than the law of conservation of mechanical energy. We shall study this general conservation principle more fully in Chapter 11 in which thermal phenomena are involved.

You may already be aware that when we apply Eq. (3-29), the work-energy theorem, we have chosen a larger system, made up of the block with spring and the surface over which it moves, as shown in Fig. 3-13, rather than the block merely. The internal energy that is generated as the block comes to rest is shared between these two bodies. We call the system as an isolated system within which the energy transfers from one body to another, from one kind to another, in the other words, there is no energy transferred between the system and the outside. It turns out that in new situation involving perhaps electrical or magnetic phenomena we can always identify new quantities like internal energy that permit us to expand the scope of our definition of energy and to retain the law of conservation of energy in a more generalized form, thus, for an isolated system

$$\Delta E + \Delta E_{\text{int}} + (\text{changes in other forms of energy}) = 0 \qquad (3\text{-}36)$$

This generalized conservation of energy principle can be put in words as follows: **Energy may be transformed from one kind to another in an isolated system but it can not be created or destroyed, the total energy of the system remains constancy.**

This statement is a generalization of experience, so far not contradicted by any laboratory experiment or observation of nature. In the later chapters dealing with electromagnetism and thermodynamics, we shall study a number of new kind of energy transformations — from mechanical to electrical; from electrical to magnetic; from mechanical to thermal and so on.

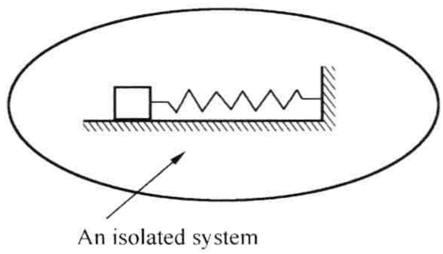


Fig. 3-13 An isolated system

Questions

3-1 Does kinetic energy depend on the reference frame of the observer in?

3-2 Consider that a box placed on the rough floor of a moving truck remains at rest relative to the floor, analyze the work done by the forces exerted on the box and the truck, observed on the ground in the following cases: (1) the truck travels up along a slope; (2) travels down along a slope; (3) moves with an acceleration on the level ground.

3-3 In a tug war, one team is slowly giving way to the other, what work is being done and by whom ?

3-4 Does the frictional force do work on a block placed on the top of a rotating table about its central axis, (1) if the angular speed remains constant; (2) if the angular speed is increasing. Suppose there is no relative motion between the block and the table.

3-5 Spring A and B are identical except that A is stiffer than B, that is, $k_A > k_B$. On which spring is more work expended if they are stretched (1) by the same amount and (2) by the same force?

3-6 A planet moves around the sun along an elliptic orbit as shown in Fig. 3-14. In the trip from the perihelion toward to aphelion, is the work done on the planet by the gravitation of the sun positive or negative? Answer the same question in the trip from the aphelion toward to perihelion. If the path is a perfectly circular orbit, does the sun do any work?

3-7 Explain, using work-energy concepts (1) how a child pumps a swing up to large amplitudes from a rest position. (2) A girl standing on frictionless roller stakes on a level surface, facing a rigid wall sets herself in motion backward by pushing against the wall. What is the work done on the girl? What is the work responsible for the increase in kinetic energy?

3-8 Air bags greatly reduce the chance of injury in a car accident. Explain how they do so in terms of energy transfers.

3-9 In Fig. 3-15, there is no friction between B and the table surface, a constant force ***F*** pulls B that moves. If A slides on top of B, whether or not the total work done by the pair of frictional forces equals zero before A leaves off B?

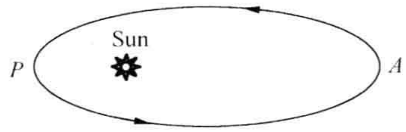


Fig. 3-14 For question 3-6

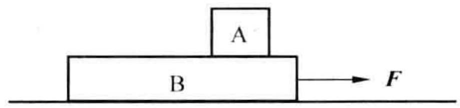


Fig. 3-15 For question 3-9

Problems

3-1 A small body of mass m acted upon by an external force moves from point A to point B in a horizontal table surface, the coefficient of kinetic friction is μ_k. Draw a sketch, calculate the work done by the kinetic friction, make a conclusion about work done by frictional force:

(1) The path is a half circle of radius R;

(2) the path is the diameter AB.

3-2 A block of mass $m=3.57$ kg is drawn a distance $d=4.06$m at constant speed along a horizontal floor by a rope exerting a constant force of magnitude $F=7.68$ N at an angle $\theta=15.0°$ above the horizontal. Compute

(1) the work done by the rope on the block;

(2) the work done by friction on the block;

(3) the total work done on the block;

(4) the coefficient of kinetic friction between block and floor.

3-3 Each of the two engines on a Boeing 767 aircraft develops a forward force of 1.97×10^5 N. When the airplane is flying at 250 m/s, what power does each engine develop?

3-4 A runner with mass 50.0 kg runs up the stairs to the 268 m-tall of the second ball of the Oriental Pearl TV Tower (with 468 m-tall) in Shanghai, in order to lift herself

(1) to the second ball in 20 minutes;

(2) to the top in the same time taken, what must be her average power output respectively?

3-5 A car of mass 1.5 t drives along a straight road with a constant frictional resistant 300 N between the car and the ground. The air resistance is about $1.8v^2$ N. Find the simultaneous power supported by the engine at the moment as the speed is 60km/h and the acceleration is 1 m/s^2.

3-6 A sled slides down from a hill of h on height along a curvilinear slope as shown in Fig. 3-16, then it continuously slides on the horizontal plane, eventually stops at the position x, the coefficient of kinetic friction between the sled and the surface in both cases is μ. Prove $\mu=h/x$.

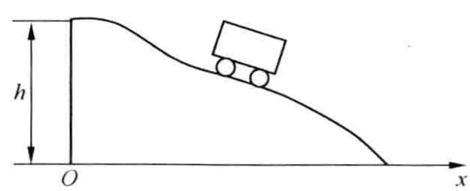


Fig. 3-16 For problem 3-6

3-7 In the same situation as shown in Fig. 1-9 for example 1-3, suppose that the magnitude of the force ***F***, exerting on the boat, is a constant. Find the work done by ***F*** pulling the boat from x_1 to x_2 ($x_1>x_2$ in the same coordinate system).

3-8 A force $F = 6\,t$ N (t refers to time) acts on a particle whose mass is 2.0 kg. If the particle starts from rest. Find

(1) the work done by the force during the first 2 s;

(2) the power at $t=2$ s.

3-9 A 10 kg brick moves along the positive x axis from rest. The net force acting on it as a function of its position as shown in Fig. 3-17.

(1) What is the work performed by the force as the brick moves ① from $x=0$ to $x=3$ m, ② from $x=3$ m to $x=10$ m;

(2) find the speed ① at $x=5$ m, ② at $x=8$ m.

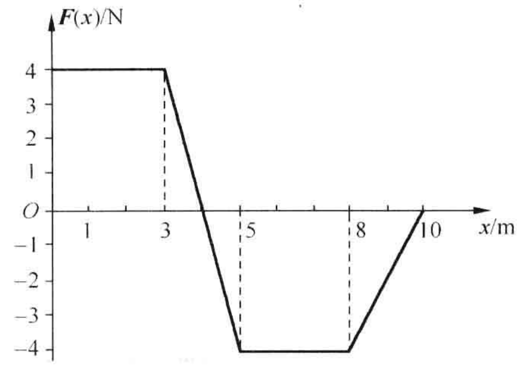


Fig. 3-17 For problem 3-9

3-10 A pendulum of length l fixed at one end of the cord, its bob is released from horizontal position. Find the speed of the bob when the cord makes an angle θ with the horizontal

(1) apply the law of kinetic energy;

(2) use Newton's second law.

3-11 A force acts on a 3.0 kg particle in such a way that the position of the particle as a function of time is given by $x=3t-4t^2+t^3$ (SI),

(1) find the work done by the force during the first 4.0 seconds;

(2) what is the instantaneous power at $t=1.0$ second and $t=4.0$ seconds.

3-12 Use the law of kinetic energy to solve the problem 2-19 (calculate the work done by weight first).

3-13 A semicircular shaped screen is fixed on a frictionless table surface, a block of mass m slides along the screen with a initial tangential speed v_0 when it enters one end of the screen as shown in Fig. 3-18, the coefficient of kinetic friction between the block and the screen is μ.

(1) Prove: as the block passes through another end of the screen, the work done by the friction is $W = \frac{1}{2}mv_0^2(e^{-2\mu\pi}-1)$;

(2) Why the radius of the semicircle has no effect on the answer?

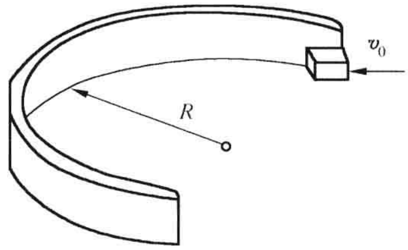


Fig. 3-18 For problem 3-13

3-14 What is the work done by the force of gravity on a 80 kg astronaut in a displacement from the surface of the earth (point A) to a point B whose altitude is 2 times of the earth's radius. Draw a sketch and calculate ($M_e = 5.98\times10^{24}$ kg, $R_e = 6.37\times10^6$ m).

3-15 From Eq. (3-25), deduce it to the expression of weight potential energy at height h from the surface of the earth ($h \ll R_e$), let the zero reference to be the surface of the earth.

3-16 The force between atoms in an oxygen molecule was called the Lennard-Jones force which is given as $F = F_0\left[2\left(\frac{\sigma}{r}\right)^{13} - \left(\frac{\sigma}{r}\right)^{7}\right]$ where $F_0 = 9.6\times10^{-11}$ N, $\sigma = 3.5\times10^{-10}$ m.

Using the definition of potential energy and picking $E_p(r_0)=0$ when $r_0 \to \infty$. Find the inter-atomic potential energy E_p, between two oxygen atoms and make a sketch of the potential function.

3-17 A 1.0 kg object is acted on by a net conservative force given by $F=-3.0\,x-5.0x^2$, where F is in Newton and x is in meter.

(1) What is the potential energy of the object at $x=2.0$ m (picking $E_p(x)=0$ when $x=0$)?

(2) If the object has a speed of 4.0 m/s in the negative x direction when it is at $x=5.0$ m, what is its speed as it passes through the origin?

3-18 Fig. 3-19 shows a spring-block system on a frictionless surface. Suppose that the extension of the spring is l_0, and choose this position as the reference of zero potential energy and the origin of x axis. Show that the change in elastic potential energy of the system between two given positions (states) is the same as that when the equilibrium position was chosen as the references of zero potential, but the particular value of potential energy at a given position is different for the two references of zero potential.

3-19 Using Fig. 3-20, show that the total work done by a pair of forces (action-reaction forces) exerted on particle 1 and 2 by particle 2 and 1 respectively, is given by $W = \int \boldsymbol{f}_2 \cdot d\boldsymbol{r}_{21}$ in which $d\boldsymbol{r}_{21}$ is the displacement of particle 2 relative to particle 1, or $W = \int \boldsymbol{f}_1 \cdot d\boldsymbol{r}_{12}$, in which $d\boldsymbol{r}_{12}$ is the displacement of particle 1 relative to particle 2. Make the conclusion about the total work done by a pair of forces.

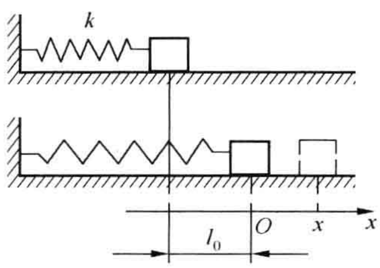


Fig. 3-19 For problem 3-18

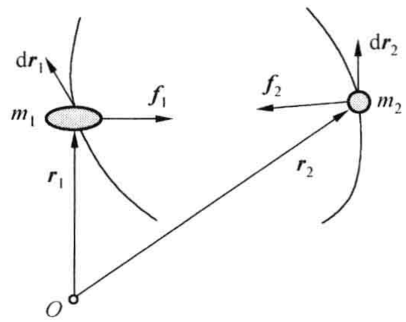


Fig. 3-20 For problem 3-19

3-20 A 3.2 kg block starts at rest and slides a distance d down a smooth 30° incline where it runs into a spring of negligible mass, see Fig. 3-21. The block slides an additional 21 cm before it is brought to rest momentarily by compressing the spring whose force constant is 430 N/m.

(1) What is the value of d?

(2) The speed of the block continues to increase for a certain interval after the block makes contact with the spring. What additional distance does the block slide before it reaches its maximum speed and being to slow down?

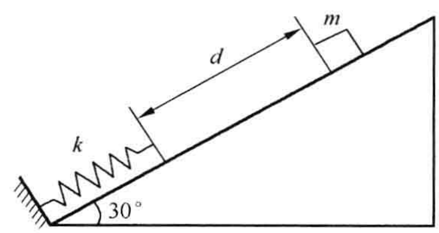


Fig. 3-21 For problem 3-20

3-21 A bungee-cord jumper of mass 61.0 kg is on a bridge 45.0 m above a river. The elastic bungee-cord has a relaxed length of $L=25.0$ m. Assume that the cord obeys Hooke's law, with a spring constant of $k=160$ N/m.

(1) If the jumper stops before reaching the water, what is the height h of her feet above the water at her lowest point (Draw a sketch first, treat the jumper as a particle)?

(2) Suppose the allowed maximum mass of the jumper is 100 kg, if the height h is the same as in above situation, what is the spring constant k needed ?

3-22 A 10 kg package is acted by a constant force $\boldsymbol{F}$ horizontally and moves up along a 30° incline, the coefficient of kinetic friction between the incline surface and the package is 0.2. After a time interval, the package has passed 3.0 m while its speed increases from 0.3 m/s to 2.0 m/s. Find the work done by the force and the magnitude of that force.

3-23 A small block of mass m slides along the frictionless loop-the-loop track shown in Fig. 3-22. the block is released from rest at point P.

(1) If $H=5R$, what is the normal force acting on it at point Q?

(2) At what height above the bottom of the block be released so that it is on the verge of losing contact with the track at the top of the loop?

(3) If $H=2R$, at what position above the bottom the block will lose contact with the track? What is the maximum height where the block can reach after it leaves off the track?

3-24 A heavy ball of 1.5kg is attached at the end of a cord, then released from rest at an angular position θ as show in Fig. 3-23. Change θ and observe the motion, it is found that the cord is not broken until $\theta=60°$. What is the critical value of tension in the cord?

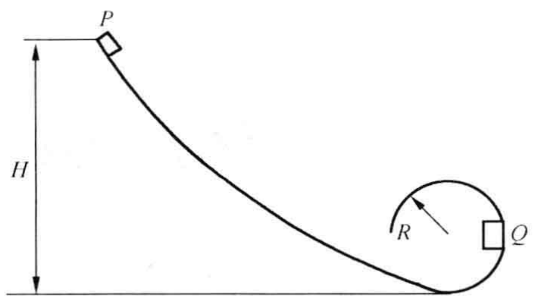


Fig. 3-22 For problem 3-23

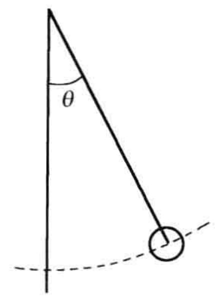


Fig. 3-23 For problem 3-24

3-25 Two plates of mass m_1 and m_2 are connected by a stiff spring as shown in Fig. 3-24. Can you press the upper plate down enough so that when it is released so that it will spring back and raise the lower plate off the table? If you can, what must the force $\boldsymbol{F}$ be?

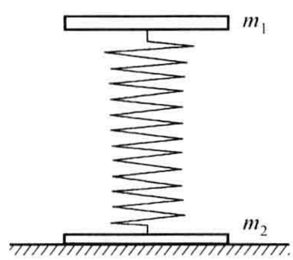


Fig. 3-24 For problem 3-25

3-26 A slider of mass m is connected with two springs (of force constant k_1 and k_2) to the ends of an air track as shown in Fig. 3-25(a). Suppose that the springs are in relax state at beginning and the slider is at position O, then the one end of the track is raised quickly to form an angle α with the horizontal as shown in Fig. 3-25(b). Find

(1) the distance from the lowest point where the slider can reach to point O?

(2) the maximum speed that the slider can have?

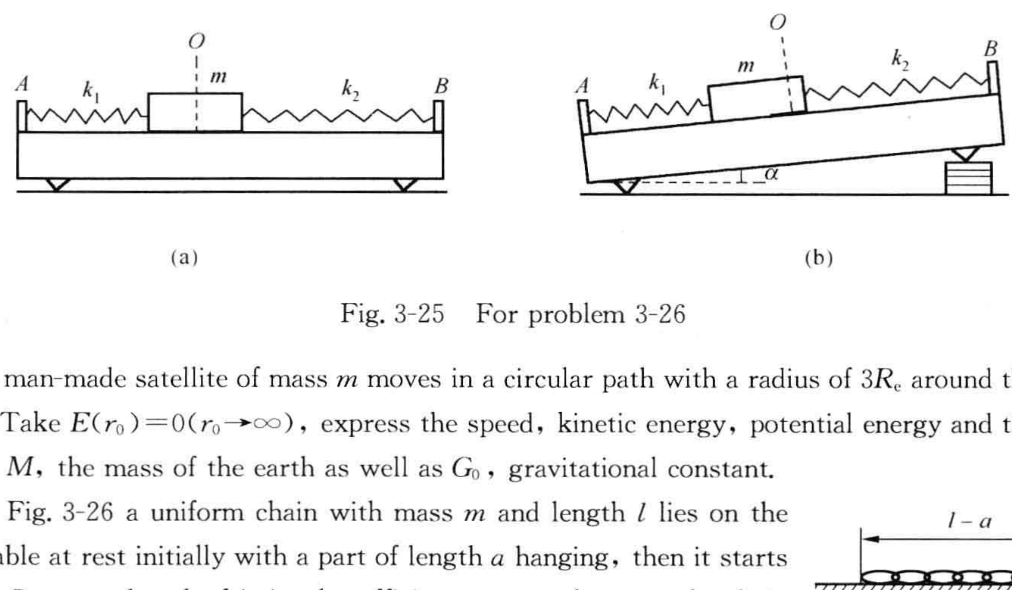


Fig. 3-25 For problem 3-26

3-27 A man-made satellite of mass m moves in a circular path with a radius of $3R_e$ around the earth whose radius is R_e. Take $E(r_0)=0(r_0\to\infty)$, express the speed, kinetic energy, potential energy and the total energy by m, R_e and M, the mass of the earth as well as G_0, gravitational constant.

3-28 In Fig. 3-26 a uniform chain with mass m and length l lies on the surface of a table at rest initially with a part of length a hanging, then it starts sliding down. Suppose that the frictional coefficient constant between the chain and the surface is μ. Find

(1) the work done by the frictional force as the whole chain just leaves the table;

(2) the speed at the same moment.

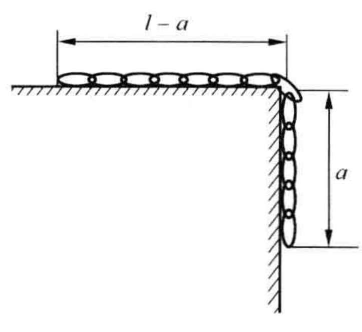


Fig. 3-26 For problem 3-28

Chapter 4

Momentum

The linear momentum, or briefly called momentum of a particle is the product of that particle's velocity and mass. Linear momentum deserves special attention because, like energy, it is conserved under certain conditions. To provide a basis for the law of conservation of linear momentum we will introduce the concept of linear impulse and use it to reformulate Newton's second law. In this new formulation the net linear impulse represents the influence of the external forces on the particle. The linear momentum of a particle changes in response to an external linear impulse. When the net external linear impulse vanish, the linear momentum will be conserved.

Angular momentum is the rotational analog of linear momentum and is also associated with a conservation principle. In this chapter, besides the law of conservation of the linear momentum, we shall also introduce the law of conservation of angular momentum for a particle.

Physicists take advantage of all conservation laws in several ways. In the process of particles interaction, physicists find the change in momentum and angular momentum to make predictions about some aspect of the motion of a particle when it is undesirable or impossible calculate the full details of motion. This is especially helpful in those cases where the force law is not known exactly. In the study of atoms, nucleons, and elementary particles, physicists use the conservation laws to decide what reactions are possible — if a hypothetical reaction violates a conservation law then it is impossible.

4.1 Linear Momentum, Linear Impulse, and Momentum Theorem

4.1.1 Linear momentum

Through Newton's second law the net force exerted on a particle determines the product of the particle's mass and acceleration. Through the law of kinetic energy the work done by the net force determines the change in the kinetic energy of the particle. Now we shall see that through the linear impulse momentum theorem, the linear impulse delivered by the net force determines the change in linear momentum of the particle.

Before we define linear momentum and derive the linear impulse — momentum theorem, Let's consider Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$. Using $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ for acceleration, we can write

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{4-1}$$

According to Eq. (4-1), the net force acting on a particle equals the change in the product $m\boldsymbol{v}$ per unit time. This product is called linear momentum, and labeled as $\boldsymbol{p}$. It is defined as

$$\boldsymbol{p}=m\boldsymbol{v} \tag{4-2}$$

The SI unit of linear momentum is kg m/s, the dimension of it is $[p]=\mathrm{MLT}^{-1}$.

4.1.2 The impulse-momentum theorem and Average force

Eq. (4-1) can be rewritten as

$$\boldsymbol{F}=\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \tag{4-3}$$

The two forms of Newton's second law — $\boldsymbol{F}=m\boldsymbol{a}$ and $\boldsymbol{F}=\mathrm{d}\boldsymbol{p}/\mathrm{d}t$ are equivalent if the mass is constant. In the case of variable mass, such as a rocket, then $\boldsymbol{F}=m\boldsymbol{a}$ can not be applied, and we turn to $\boldsymbol{F}=\mathrm{d}\boldsymbol{p}/\mathrm{d}t$. Rewrite Eq. (4-3) as

$$\mathrm{d}\boldsymbol{p}=\boldsymbol{F}\mathrm{d}t$$

Taking time integral of above equation from t_i to t_f we have

$$\boldsymbol{p}_f-\boldsymbol{p}_i=\int_{t_i}^{t_f}\boldsymbol{F}\mathrm{d}t \tag{4-4}$$

where $\boldsymbol{p}_i$ and $\boldsymbol{p}_f$ are the linear momentum at t_i and t_f respectively. The integration in the right side is defined as the linear impulse of the force during the time interval t_i to t_f, labeled as $\boldsymbol{I}$, thus

$$\boldsymbol{I}=\int_{t_i}^{t_f}\boldsymbol{F}\mathrm{d}t \tag{4-5}$$

Impulse is a vector. If $\boldsymbol{F}$ is a constant force, the direction of $\boldsymbol{I}$ is the same as that of force. If $\boldsymbol{F}$ is a variable one, the direction of $\boldsymbol{I}$ is determined by the vector integral of the right side in Eq. (4-5). The SI unit of linear impulse is Newton · second, or N · s ; the SI unit of linear momentum is kg m/s, they have the same dimension, that is $[I]=[p]=\mathrm{MLT}^{-1}$.

Eq. (4-4) can be expressed in the form

$$\boldsymbol{I}=\boldsymbol{p}_f-\boldsymbol{p}_i=\Delta\boldsymbol{p}$$

or

$$\boldsymbol{I}=m\boldsymbol{v}_f-m\boldsymbol{v}_i \tag{4-6}$$

Which we call the impulse-momentum theorem, or in words as

The change in linear momentum of a particle is equal to the linear impulse delivered to that particle by the net force.

Just as the change in the kinetic energy of a particle measures the net work done on the particle, the change in linear momentum of a particle measures the linear impulse delivered to the particle by the net force. We see that the vector law of impulse-momentum theorem Eq. (4-6) is analogous to the scalar law of kinetic energy theorem Eq. (3-12), both of them involve integration and connect the initial and final values of quantities without involving the values of these quantities at intermediate times.

In practical application, it is convenient to use the scalar forms of Eq. (4-6). Suppose that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are all in x-y plane, thus impulse is also in this plane, and the components of Eq. (4-6) are

$$I_x = \int_{t_i}^{t_f} F_x \mathrm{d}t = mv_{fx} - mv_{ix}$$
$$I_y = \int_{t_i}^{t_f} F_y \mathrm{d}t = mv_{fy} - mv_{iy} \qquad (4\text{-}7)$$

Like the law of kinetic energy, the impulse-momentum theorem is not a new and independent theorem but is a direct consequence of Newton's second law. Both theorem are special form of this law, useful for special purpose. Impulse momentum theorem is particularly useful for the problems of collision. For example, when a golf club strikes a golf ball, a large variable force of short duration is exerted on the ball by the club, observable changes in the motion of the golf ball occurs as a result of the force, which is usually called impact force.

In order to deal with sudden or abrupt change in the motion of particle, we can use Eq. (4-6) to calculate the linear impulse by means of measuring the change in linear momentum of the particle during the time interval of the collision.

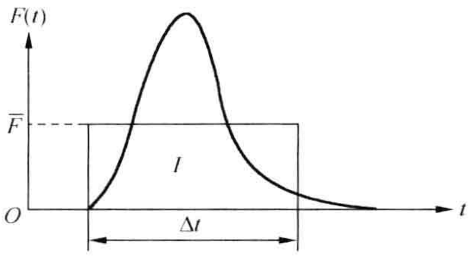


Fig. 4-1 The curve of an impact force

Determination of the impact force as a function of time is quit difficult and unnecessary, it is illustrated roughly in Fig. 4-1. When dealing with such a variable force, it is sometimes helpful to introduce an average force $\overline{F}$, defined by

$$\overline{\boldsymbol{F}} = \frac{1}{t_f - t_i} \int_{t_i}^{t_f} \boldsymbol{F} \mathrm{d}t \qquad (4\text{-}8)$$

which is represented by a straight line in Fig. 4-1.

So that Eq. (4-6) can be rewritten as

$$\boldsymbol{I} = \overline{\boldsymbol{F}}(t_f - t_i) = m\boldsymbol{v}_f - m\boldsymbol{v}_i \qquad (4\text{-}9)$$

which is represented by the area under the curve of the variable force in Fig. 4-1, its components are

$$I_x = \overline{F}_x(t_f - t_i) = mv_{fx} - mv_{fx}$$
$$I_y = \overline{F}_y(t_f - t_i) = mv_{fy} - mv_{iy} \qquad (4\text{-}10)$$

The time interval $\Delta t = t_f - t_i$ can be measured, so that, we can use impulse-momentum theorem to find the average impact force or its components.

Example 4-1 A baseball (m = 0. 1 45 kg) is moving with a velocity $\boldsymbol{v}_i = -31.8(\boldsymbol{i}+\boldsymbol{j})$ m/s, when it reaches the batter. After the baseball is hit by the batter it travels with a horizontal velocity $\boldsymbol{v}_f = +60\boldsymbol{j}$ m/s along y axis as shown in Fig. 4-2.

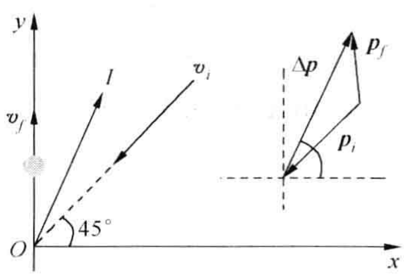


Fig. 4-2 For Example 4-1

(1) What linear impulse was delivered by the batter?

(2) The impact time, Δt for the baseball-bat collision is 1. 2 ms, a typical value. What average force acts on the baseball?

Solution (1) From the known we have

$$\boldsymbol{p}_i = -0.145 \times (31.8)(\boldsymbol{i}+\boldsymbol{j}) = -4.61(\boldsymbol{i}+\boldsymbol{j}) \text{ kg} \cdot \text{m/s}$$
$$\boldsymbol{p}_f = 0.145 \times 60\boldsymbol{j} = +8.7\boldsymbol{j} \text{ kg} \cdot \text{m/s}$$

Their directions shown in Fig. 4-2, and

$$\Delta \boldsymbol{p} = 8.7\boldsymbol{j} - [-4.61(\boldsymbol{i}+\boldsymbol{j})] = (4.61\boldsymbol{i} + 13.3\boldsymbol{j})\ \text{kg}\cdot\text{m/s}$$

Using Eq. (4-6) we have

$$\boldsymbol{I} = \Delta \boldsymbol{p} = (4.61\boldsymbol{i} + 13.3\boldsymbol{j})\ \text{N}\cdot\text{s}$$

for the net linear impulse delivered to the baseball. The magnitude of $\boldsymbol{I}$ obtained from its components, thus

$$I = \sqrt{I_x^2 + I_y^2} = \sqrt{4.61^2 + 13.3^2} = 14.1\ \text{N}\cdot\text{s}$$

The direction of $\boldsymbol{I}$ is given by

$$\tan\theta = \frac{I_y}{I_x} = \frac{13.3}{4.61} = 2.88 \quad \text{or} \quad \theta = 70.9°$$

That is, $\boldsymbol{I}$ makes an angle of 70.9° with the x axis as shown in Fig. 4-2. Using a vector triangle $\Delta \boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i$, we can get the same result.

(2) If $\Delta t = t_f - t_i = 1.2\ \text{ms} = 1.2\times 10^{-3}\ \text{s}$, thus, from Eq. (4-9) the magnitude of the average force is then $\bar{F} = \dfrac{I}{t_f - t_i} = \dfrac{14.1}{1.2\times 10^{-3}} = 11750\ \text{N}$, which is over a ton!

4.2 Conservation of Momentum

Much of our discussion so far has dealt with the motion of a single particle and with the laws governing the relationship between the particle and its surroundings. We have just seen how impulse of a force effects the motion of a particle, now let us generalize this theorem to a composite system of two particles being able to interact with each other, then lead to the law of conservation of momentum.

4.2.1 The momentum of a system of particles

Consider two particles that are free to move on a horizontal frictionless surface. Their masses are m_1 and m_2, and they each exert a force on the other while external forces are being exerted on them by the surrounding. $\boldsymbol{F}_1$ and $\boldsymbol{f}_{12}$ are the net external force and internal force exerted on m_1 by the surrounding and m_2 respectively, while $\boldsymbol{F}_2$ and $\boldsymbol{f}_{21}$ are the net external force and internal force exerted on m_2 by the surrounding and m_1 respectively as shown in Fig. 4-3. The pair of internal forces $\boldsymbol{f}_{12}$ and $\boldsymbol{f}_{21}$ include gravitational forces, electromagnetic forces, nuclei forces, and contact forces, such as tensions, compression, friction and so on.

Suppose the velocities of m_1 and m_2 are $\boldsymbol{v}_{i1}$ and $\boldsymbol{v}_{i2}$ at initial time, through interaction, they became $\boldsymbol{v}_{f1}$ and $\boldsymbol{v}_{f2}$ at final time. Using Eq. (4-4) for m_1 and m_2 respectively, yields

$$\int_{t_i}^{t_f} (\boldsymbol{F}_1 + \boldsymbol{f}_{12})\,\mathrm{d}t = m_1\boldsymbol{v}_{f1} - m_1\boldsymbol{v}_{i1} \quad (4\text{-}11\text{a})$$

$$\int_{t_i}^{t_f} (\boldsymbol{F}_2 + \boldsymbol{f}_{21})\,\mathrm{d}t = m_2\boldsymbol{v}_{f2} - m_2\boldsymbol{v}_{i1} \quad (4\text{-}11\text{b})$$

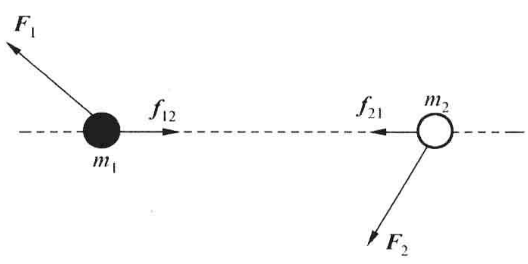


Fig. 4-3 A system of two particles

According to Newton's third law $\boldsymbol{f}_{12} = -\boldsymbol{f}_{21}$, add Eq. (4-11a) to Eq. (4-11b), and note $\boldsymbol{f}_{12} + \boldsymbol{f}_{21} = 0$, we have

$$\int_{t_i}^{t_f} (\boldsymbol{F}_1 + \boldsymbol{F}_2)\mathrm{d}t = (m_1 \boldsymbol{v}_{f1} + m_2 \boldsymbol{v}_{f2}) - (m_1 \boldsymbol{v}_{i1} + m_2 \boldsymbol{v}_{i2}) \tag{4-12}$$

The left side is the impulse delivered to the system by the resultant external force while the right side represents the change in total momentum of the system. Thus, Eq. (4-12) means that **the change in total linear momentum of a system is equal to the impulse delivered to that system by the resultant external force.** This result is called the linear impulse-momentum theorem for a system.

4.2.2 The conservation law of momentum

We are now ready to derive the conservation law of momentum: if the resultant external force is zero in Eq. (4-12), i. e.

If

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 = 0$$

then

$$m_1 \boldsymbol{v}_{f1} + m_2 \boldsymbol{v}_{f2} = m_1 \boldsymbol{v}_{i1} + m_2 \boldsymbol{v}_{i2} \tag{4-13}$$

This important result is called the law of conservation of momentum. It tells us that **if no external forces act on a system of particles, the total linear momentum of the system remains constant.**

In general case, the system may consist more than two particles, the impulse-momentum theorem and the law of conservation of momentum can be expressed as following:

$$\int_{t_i}^{t_f} \sum_j \boldsymbol{F}_j \mathrm{d}t = \sum_j m_j \boldsymbol{v}_{fj} - \sum_j m_j \boldsymbol{v}_{ij} \tag{4-14}$$

If $\sum_j \boldsymbol{F}_j = 0$, then

$$\sum_j m_j \boldsymbol{v}_j = \text{a constant vector} \tag{4-15}$$

the subscript i and f represent the initial state (time) and final state (time), j represents the jth particle, and the sum is over all particles.

Like the law of conservation of energy that we met in chapter 3, conservation of momentum is a law more general than Newtonian mechanics itself. It continues to hold in the subatomic realm, where Newton's laws fail. It holds for the highest particle speed, where Einstein's relativity theory prevails; it is only necessary to concern that the momentum is related to speed, rather than Eq. (4-2).

Eq. (4-15) is a vector equation and, as such, is equivalent to three scalar equations corresponding to momentum conservation in three mutually perpendicular directions. In the case of motion in x-y plane, the x and y components of Eq. (4-15) are given by

$$\begin{aligned} &\text{If} \quad \sum F_{jx} = 0 \quad \text{then} \quad \sum m_j v_{jx} = p_x = \text{a constant} \\ &\text{If} \quad \sum F_{jy} = 0 \quad \text{then} \quad \sum m_j v_{jy} = p_y = \text{a constant} \end{aligned} \tag{4-16}$$

Note that, it may happen, for example, there is an external force acting on the system but it acts, for example, in only the vertical direction, with no component in horizontal direction. In such a case, the horizontal components of the total momentum of the system remain constant, even though the vertical component does not.

Like all law derived from the Newton's second law, the impulse-momentum theorem and the law of conservation of momentum is valid only in inertial reference frames, that is, all velocity in above equations should be with respect to the inertial reference frames.

Example 4-2 as Fig. 4-4 shows, a cannon whose mass M is 5000kg, fires a 100kg ball with a muzzle speed $v = 300$ m/s in a direction of 30°to the horizontal. The cannon is mounted so that it can recoil freely.

(1) What is the velocity $\boldsymbol{v}_1$ of the recoiling cannon with respect to the earth (Neglect the friction between the wheel and the ground) ?

(2) What is the direction of the ball's velocity with respect to the earth?

(3) Suppose the recoil time interval $\Delta t = 2$ s, calculate the average acting on the buffer?

Solution (1) We choose the cannon plus the ball as our system. By doing so, the interacting forces between the cannon and the buffer caused by firing the ball are internal to the system. The external forces acting to the system have no components in the horizontal direction. Thus, the horizontal linear momentum of the system must remain unchanged as the cannon is fired, that is

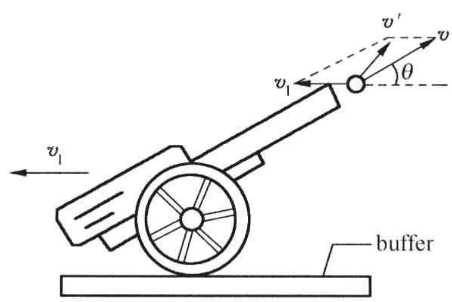


Fig. 4-4 For Example 4-2

$$\sum F_{jx} = 0, \quad \sum m_j v_{jx} = \text{a constant}$$

Choose a reference frame fixed with respect to the earth and assume that the recoiling velocity is $-\boldsymbol{v}_1$. Before the cannon is fired, the system has an initial momentum $p_i = 0$. After the cannon have fired, the ball has a horizontal velocity v with respect to the recoiling cannon, $\boldsymbol{v}$ is being the muzzle speed. In the reference frame of earth, however, the horizontal velocity of the ball is v'_x and

$$v'_x = v\cos\theta - v_1$$

where $\theta = 30°$, $\boldsymbol{v}_1$ is the cannon's recoiling speed with respect to the earth, and points to the left. Thus, the total linear momentum of the system after firing is

$$p_{fx} = m(v\cos\theta - v_1) - Mv_1$$

Where m is the mass of the ball, M is the mass of the cannon. Conservation of linear momentum in the horizontal direction requires

$$m(v\cos\theta - v_1) - Mv_1 = 0$$

Solving for v_1 yields

$$v_1 = \frac{m}{m+M} v\cos\theta = \frac{100}{100+5000} \times 300 \times \frac{\sqrt{3}}{2} = 5.09(\text{m/s})$$

(2) The velocity of the ball with respect to earth is v', which makes an angle α with horizontal direction, as Fig. 4-4 shows

$$v'_x = v\cos\theta - v_1, \quad v'_y = v\sin\theta$$

so

$$\alpha = \arctan\frac{v\sin\theta}{v\cos\theta - v_1} = \frac{300\times(1/2)}{300\times(\sqrt{3}/2)-5.09} = \arctan\frac{150}{254.7} = 30.5°$$

(3) Using the impulse-momentum theorem to the cannon on which the average force $\bar{\boldsymbol{F}}$ acts,

$$\bar{F}\Delta t = 0-(-Mv_1) = Mv_1$$

$$\bar{F} = \frac{Mv_1}{\Delta t} = \frac{5000\times 5.09}{2} = 12725(\text{N})$$

directing to the right. So that, the average force acting on the buffer is

$$\bar{F}' = -\bar{F} = -12725 \text{ N}$$

directing to the left in Fig. 4-4.

Example 4-3 Alpha particle scattering. An alpha particle collides with an oxygen atom at rest, scattering along the direction of $\theta = 72°$ to the incident direction while the oxygen atom recoil along the direction of $\beta = 41°$ to the alpha's incident direction. Find the ratio of the speeds of the alpha particle before and after the collision.

Solution Our system includes alpha particle and oxygen atom. Suppose the mass of alpha particle is m, its initial and final velocity are v_{1i} and v_{1f} respectively; the mass of oxygen atom is M, its final velocity is v_{2f}. In Fig. 4-5, $\theta = 72°$, $\beta = 41°$, the alpha's incident direction points to x axis. During the collision process, only internal forces act on the two particle, so that the momentum of the system remains constant. Using Eq. (4-15) we have two components equation in x direction:

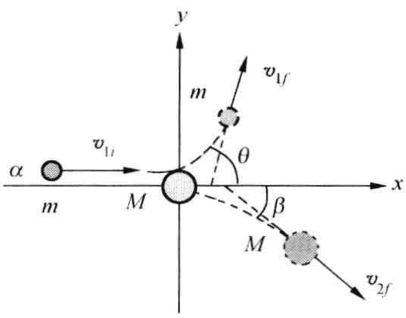


Fig. 4-5 For Example 4-3

$$mv_{1i} + 0 = mv_{1f}\cos\theta + Mv_{2f}\cos\beta \quad ①$$

in y direction:

$$0 = mv_{1f}\sin\theta - Mv_{2f}\sin\beta \quad ②$$

Eliminating Mv_{2f} by Eq. ①$\times\sin\beta$ plus Eq. ②$\times\cos\beta$, we have

$$mv_{1i}\sin\beta = mv_{1f}(\cos\theta\sin\beta + \sin\theta\cos\beta)$$

so that

$$\frac{v_{1f}}{v_{1i}} = \frac{\sin\beta}{\sin(\theta+\beta)} = \frac{\sin 41°}{\sin(72°+41°)} = 0.71$$

4.3 Collision

A collision is an isolated event in which a relatively strong force (a variable force) acts on each colliding particle for a relatively short time. An observable sudden or abrupt change in the motion of the colliding particles occurs as a result of the force, which is usually called impact force(Fig. 4-1). It must be possible to make a clean separation between times that are before the collision and those that are after the collision. In a true collision the impact forces acting on the particles are internal forces, while the external forces play no role.

In our daily experience, the examples of collision might be a hammer and a nail, a base ball and a bat, a stone dropped toward the earth…. In laboratories, many physicists spend their time playing what we can call"the collision games". A major goal of those "games" is to find out as much as possible about the forces that act during the collision, knowing the states of the articles both before and after the collision, which include all of our knowledge of the subatomic world—electrons, protons, neutrons, quarks, and so on. The objects that collide range from subatomic particles to the galaxies. No matter what is the nature of the objects that collide, the common rule of the collisions is that the momentum of the system is always conserved.

4.3.1 Elastic collision in one dimension

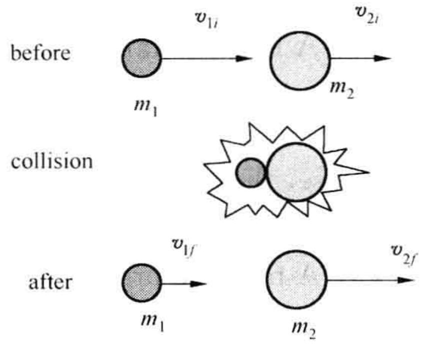


Fig. 4-6 Two particles collision

Fig. 4-6 shows the general situation of a collision in one dimension, which is usually called head-on collision. The conservation equation for linear momentum in this case can be written as

$$m_1 \boldsymbol{v}_{1i} + m_2 \boldsymbol{v}_{2i} = m_1 \boldsymbol{v}_{1f} + m_2 \boldsymbol{v}_{2f}$$

let the direction of the collision in x axis and the positive x pointing right, the x component of above equation is then

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{4-17}$$

in which v_{1i}, v_{2i}, v_{1f} and v_{2f} are all algebraic amounts of the corresponding velocities, if the direction of velocity is the same as the direction of positive x axis, then its magnitude is positive, otherwise negative.

If the total kinetic energy of two colliding particles is conserved, their collision termed complete elastic or briefly elastic, and the total kinetic energy before and after collision is

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \tag{4-18}$$

Because collision in which one particle, for example, m_1 is the projectile with initial speed $\boldsymbol{v}_{1i}$, another one, the target m_2 is at rest in the laboratory reference frame is quite common, so, we take m_2 to be at rest before collision, that is, setting $\boldsymbol{v}_{2i} = 0$ in Eq. (4-17) and Eq. (4-18), thus

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \tag{4-17$'$}$$

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \tag{4-18$'$}$$

Rewrite above equations as

$$m_2 v_{2f} = m_1 (v_{1i} - v_{1f}) \tag{4-19}$$

$$m_2 v_{2f}^2 = m_1 (v_{1i}^2 - v_{1f}^2) \tag{4-20}$$

Dividing Eq. (4-20) by Eq. (4-19), we have

$$v_{2f} = v_{1i} + v_{1f}$$

Substituting it into Eq. (4-17)$'$, we obtain

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \tag{4-21}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} \tag{4-22}$$

From above results, we note that v_{2f} is always positive (in Fig. 4-6, the target particle m_2 always moves along $+x$ direction); while v_{1f} may be of either sign, if $m_1 > m_2$, the projectile m_1 moves forward, if $m_1 < m_2$, it rebounds. Let us look at a few special situations:

(1) Equal masses. If $m_1 = m_2$, Eq. (4-21) and Eq. (4-22) reduce to

$$v_{1f} = 0, \quad v_{2f} = v_{1i} \tag{4-23}$$

which means that the particles of equal masses simply exchange velocities after collision, and this is true even though the target particle may not be initially at rest.

(2) A massive target. If $m_2 \gg m_1$, Eq. (4-21) and Eq. (4-22) reduce to

$$v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} \approx \left(\frac{2m_1}{m_2}\right)v_{1i}, \text{ or } \quad v_{2f} \approx 0 \tag{4-24}$$

In this case, the projectile m_1 simply bounces back in the same direction from which it came, its speed essentially unchanged. The target m_2 moves forward at a low speed, even becomes halt.

(3) A massive projectile. If $m_1 \gg m_2$, Eq. (4-21) and Eq. (4-22) reduce to

$$v_{1f} \approx v_{1i}, \quad v_{2f} \approx 2v_{1i} \tag{4-25}$$

In this case, the projectile m_1 simply keeps going, scarcely slow down by the collision, while the target m_2 leaps ahead, at twice the speed of m_1's initial speed.

One of the important application of above results is in a nuclear reactor. Some certain heavy nuclei capture neutrons and then undergo fission, emitting high-speed neutrons. These emitted neutrons go on to initiate additional fission reactions, and the process continues in a chain reaction. In thermal nuclear reactors low speed neutrons are used to initiate fission. Therefore, the high-speed neutrons must slow down or transfer most of their kinetic energy in collision before they can initiate another fission reaction. Thus elements of small mass, like hydrogen and helium, are usually chosen in the thermal reactor core to maximize the chance of a neutron (projectile) collision with a particle of nearly equal mass (target), so that the neutron becomes almost at rest, this makes the energy transfer as efficient as possible. While in another kind reactor called fast-breeder nuclear reactor high-speed neutrons are used to initiate fission. Designers strive to minimize the energy transfer by elements of large mass chosen in the reactor core, for example, liquid sodium, rather than water, is used as coolant in order to avoid having hydrogen (from water) in the reactor core.

4.3.2 Inelastic collision in one dimension

If the two colliding particles stick together after the collision — the collision is termed completely inelastic or briefly inelastic, as shown in Fig. 4-7. They may not lost all of their kinetic energy but they have lost as much of it as they can, so, the kinetic energy is not conserved at all, the "lost" kinetic always shows up in some other form — perhaps as thermal energy. In Fig. 4-7, after collision, the projectile and target becomes one particle of mass $(m_1 + m_2)$ moving along x axis, Eq. (4-17) of conservation of momentum becomes

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)v \tag{4-26}$$

so

$$v = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \tag{4-27}$$

If $\boldsymbol{v}_{2i} = 0$, thus we simply have

$$v = \frac{m_1}{m_1 + m_2} v_{1i} \tag{4-28}$$

Example 4-4 A ballistic pendulum is a device that was used to measure the speed of bullets. It consists of a large block of wood of mass M hanging from two long pairs of cords. A bullet of mass m is fired into the block, coming quickly to rest. The block with the bullet in it then swing upward. The change in vertical position of the block is measured as h before the pendulum comes momentarily to rest at the end of its arc, as shown in Fig. 4-8.

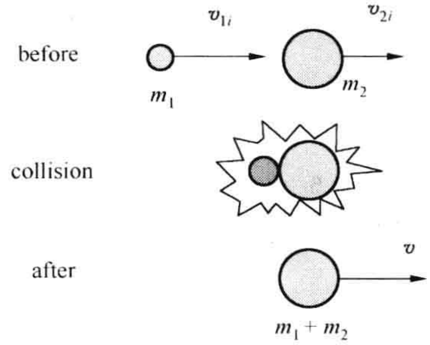


Fig. 4-7 An Inelastic collision

(1) If $M = 5.4$ kg, $m = 9.5$ g, $h = 6.3$ cm, what is the speed of the bullet?

(2) How much of the mechanical energy remains after the collision?

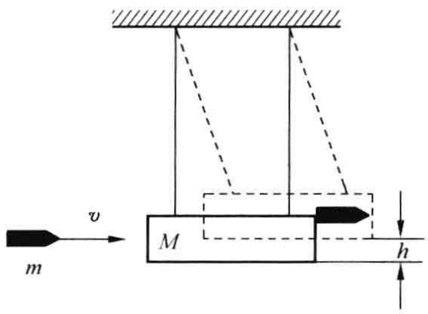


Fig. 4-8 For Example 4-4

Solution (1) The first stage of the motion is a inelastic collision, if the projecting speed of the bullet is v, the speed obtained by the block is v', from the conservation law of the momentum, we have

$$mv = (M+m)v' \quad ①$$

After collision, the second stage of the motion is the swing in which process the mechanical energy of the swinging pendulum remains constant, the kinetic energy of the system when the block is at the bottom (the reference position of zero potential energy) of its arc must then equal the potential energy of it when the block is at the top, thus

$$\frac{1}{2}(M+m)v'^2 = (M+m)gh \quad ②$$

Eliminating v' from Eq. ① and Eq. ②, leads to the projectile speed of the bullet

$$v = \frac{M+m}{m}\sqrt{2gh} = \left(\frac{5.4+0.0095}{0.0095}\right) \times \sqrt{2 \times 9.8 \times 0.063} = 633 \text{ (m/s)}$$

(2) The initial kinetic energy of the bullet is

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.0095 \times 630^2 = 1930(\text{J})$$

The mechanical energy of the system when the block is at top is

$$E = E_p = (M+m)gh = (5.4+0.0095) \times 9.8 \times 0.063 = 3.3(\text{J})$$

Thus, only 3.3/1903= 0.2% of the original kinetic energy of the bullet is transferred to mechanical energy of the pendulum. The rest is dissipated inside the block as thermal energy.

4.3.3 Elastic collision in two dimensions

Fig. 4-9 shows a typical situation of two dimensional collision in which a glancing collision (assumed elastic) occurs between a projectile particle m_1 and a resting target m_2 ($v_{2i} = 0$). Set the incident direction of m_1 as the x axis, thus $v_{1iy}=0$. According to the conservation of momentum, we can write down two components equations

x component: $m_1 v_{1i} = m_1 v_{1f}\cos\theta_1 + m_2 v_{2f}\cos\theta_2$ (4-29)

y component: $0 = m_1 v_{1f}\sin\theta_1 + m_2 v_{2f}\sin\theta_2$ (4-30)

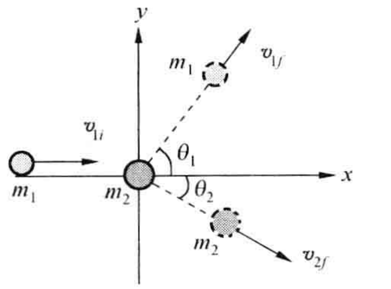


Fig. 4-9 A two dimensional collision

Where θ_1 and θ_2 represent the directions of v_{1f}, and v_{2f} respectively. From the conservation of kinetic energy, we have one more equation

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \tag{4-31}$$

These three equations contain seven variables, if we know any four of them, we can solve the three equations for the remaining three quantities. Usually, the known quantities are m_1, m_2 and one of the angles. The unknowns to be solved for are then the two final speeds and the remaining angle.

Example 4-5 Two particles of equal mass have an elastic collision (in two dimensions), the target particle being initially at rest. Prove that the two particles will always move off at right angles to each other after the collision.

Solution Let us solve the problem in a straightforward way rather than using Eq. (4-29) ~ Eq. (4-31). Fig. 4-9 shows the situation both before and after the collision. Because of conservation of momentum, $m\boldsymbol{v}_{1i} = m\boldsymbol{v}_{1f} + m\boldsymbol{v}_{2f}$, these three momentum vectors must form a closed triangle. Consider that the masses of the particles are equal, the corresponding velocities form a closed triangle also as Fig. 4-10 shows. Cancel the mass m, we have

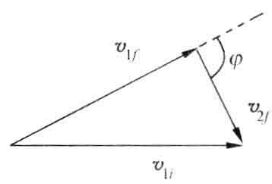


Fig. 4-10 For Example 4-5

$$\boldsymbol{v}_{1i} = \boldsymbol{v}_{1f} + \boldsymbol{v}_{2f} \quad ①$$

On the other hand, from the conservation of kinetic energy, Eq. (4-31) holds, with the mass canceled. It becomes

$$\boldsymbol{v}_{1f}^2 = \boldsymbol{v}_{1f}^2 + \boldsymbol{v}_{2f}^2 \quad ②$$

For Eq. ② holds, the triangle in Fig. 4-10 must be a right triangle and therefore the angle φ between v_{1f} and v_{2f} must be 90° i. e. $\varphi=90°$ which is what we set out to prove.

4.4 Conservation of Angular Momentum of a Particle

4.4.1 Angular momentum of a particle

Like all linear quantities, linear momentum has its angular counterpart. For a particle having linear momentum $\boldsymbol{p}$, located at point P with position vector $\boldsymbol{r}$ in the x-y plane (Fig. 4-11), the angular momentum $\boldsymbol{L}$ of this particle with respect to the origin O is defined by equation

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v} \tag{4-32}$$

The angular momentum vector $\boldsymbol{L}$ is perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{p}$ (in Fig. 4-11, this plane is coincident with x-y plane), and in the direction determined by the right-hand rule for the cross product, thus, $\boldsymbol{L}$ is directed along the positive z axis in Fig. 4-11.

The magnitude of angular momentum is given by

$$L = rmv\sin\varphi \tag{4-33}$$

where φ is the angle between the linear momentum and the direction of the position vector. If the particle is moving directly away from the origin ($\varphi=0°$), or directly toward it ($\varphi=180°$), thus the particle has no angular momentum about that origin. The SI unit of angular momentum is $kg \cdot m^2/s$, equivalent to $J \cdot s$. Apply the definition Eq. (4-32) of angular momentum to the particle undergoing a circular motion, as shown in Fig. 4-12, the angular momentum of the particle with respect to the center O of the circle is given by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}$$

which is directed upward, perpendicular to the plane of circle. Its magnitude is

$$L = mrv \sin 90° = mrv = mr^2\omega$$

in which r is the radius of the circle, ω is the angular speed of the particle around the center.

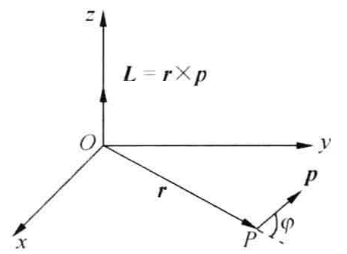


Fig. 4-11 Define the angular momentum of a particle

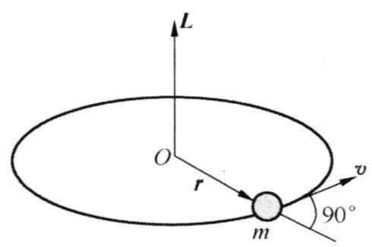


Fig. 4-12 Angular momentum of a particle in circular motion

4.4.2 Newton's second law in angular form and Torque

We start with the defining equation for angular momentum of a particle, Eq. (4-32) $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, by differentiating each side of it with respect to time t, yield

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} + \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} \tag{4-34}$$

Because $\boldsymbol{v} = d\boldsymbol{r}/dt$, $\boldsymbol{p} = m\boldsymbol{v}$, we can write the second term as

$$\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} = \boldsymbol{v} \times (m\boldsymbol{v}) = m\boldsymbol{v} \times \boldsymbol{v} = 0$$

so that Eq. (4-34) becomes

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$$

Using Newton's second law $\boldsymbol{F} = d\boldsymbol{p}/dt$, we have

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} \tag{4-35}$$

Where $\boldsymbol{F}$ is the net external force acting on this particle, $\boldsymbol{r}$ is the position vector of the particle with respect to the origin O. Here we introduce a quantity torque exerted on the particle by this force with respect to the origin point O, labeled as M. It is defined as

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{4-36}$$

in Fig. 4-13. Where $\boldsymbol{M}$ is perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$ and points to the direction determined by the right-hand rule for the cross product.

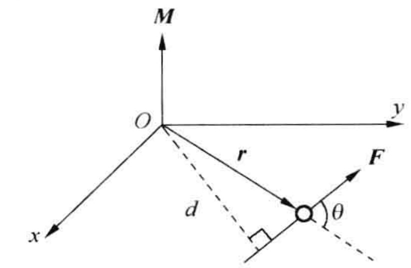


Fig. 4-13 Definition of the Torque $\boldsymbol{M}$ exerted on a particle by the net force $\boldsymbol{F}$

The magnitude of torque $\boldsymbol{M}$ is given by

$$M = rF\sin\theta = Fd \qquad (4\text{-}37)$$

where θ is formed by $\boldsymbol{F}$ and the positive direction of $\boldsymbol{r}$, d is the perpendicular distance to the line of action of the force from the origin O, and is called the moment arm of the force. Therefore Eq. (4-35) becomes

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \qquad (4\text{-}38)$$

Because $\boldsymbol{F}$ is the net external force, so that $\boldsymbol{M}$ is represents the net torque produced by external forces. Eq. (4-38) is indeed the angular form of Newton's second law:

The net torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

4.4.3 Conservation of angular momentum of a particle

When the net torque equals zero, $\mathrm{d}\boldsymbol{L}/\mathrm{d}t=0$, the angular momentum of the particle does not change—it remains constant, and is thus conserved, we can write

$$\text{If } \boldsymbol{M} = 0, \text{ then } \boldsymbol{L}_i = \boldsymbol{L}_f, \text{or } \boldsymbol{L} = \text{a constant vector} \qquad (4\text{-}39)$$

this is the law of conservation of angular momentum for a particle: **If the net torque is equal to zero, the angular momentum of a particle remains constant.**

The case where the net force is a central force merits special consideration. A central force is the one whose direction is always toward or away from the force center then $\boldsymbol{F}$ is along $\boldsymbol{r}$. Electrical and gravitational forces provide examples of central forces. When $\boldsymbol{F}$ and $\boldsymbol{r}$ are parallel or anti-parallel, their cross product is zero. Therefore. for a central force, $\boldsymbol{r} \times \boldsymbol{F} = 0$, $\boldsymbol{M}=0$. **The angular momentum of a particle moving under the influence of a central force is conserved.**

Example 4-6 Halley's comet moves about the sun in an elliptic orbit (Fig. 4-14) with a perihelion distance $R_P = 0.885\times10^{11}$ m and an aphelion distance $R_A=52.5\times10^{11}$ m. We know the perihelion speed is $v_P = 5.4\times10^4$ m/s. Find the aphelion speed $v_A=?$

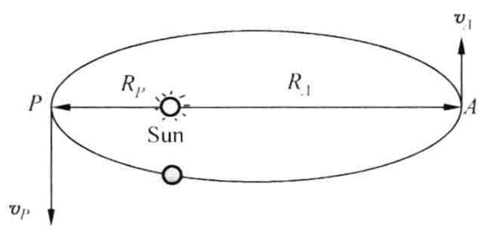


Fig. 4-14 For Example 4-6

Solution From Fig. 4-14 we see that the sun's force of gravity acts along the line joining the sun and the comet, i. e. it is a central force ($\boldsymbol{M}=0$), so that the angular momentum of the comet with respect to the center of the sun is conserved, thus

$$L_P = L_A$$

or

$$mv_PR_P = mv_AR_A$$

therefore

$$v_A = v_P\left(\frac{R_P}{R_A}\right) = 5.4\times10^4\times\frac{0.885\times10^{11}}{52.5\times10^{11}} = 9.1\times10^2\,(\mathrm{m/s})$$

Example 4-7 Planets move in elliptic orbits with the sun located at one focus. Prove the Kepler's second law of planetary motion: in equal time intervals, the radius vector from the sun to a planet sweeps out equal areas.

Solution Because the sun's gravitational force acts along the line joining the sun and the planet, the angular momentum of the planet relative to the center of sun is conserved, that is

$$\boldsymbol{L} = \text{a constant vector}$$

which means:

(1) The direction of $\boldsymbol{L}$ is remaining unchanged, as shown in Fig. 4-15, so that the orientation of the planet determined by $\boldsymbol{r}$ and $\boldsymbol{p}$ is pointing to a constant direction, in the other words, the planet moves always in a same plane in which the orbit of the planet locates.

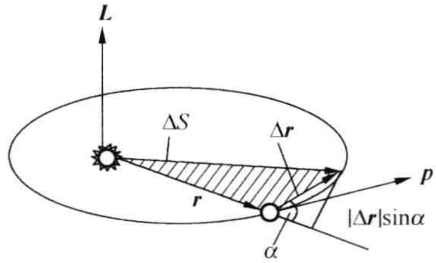


Fig. 4-15 For Example 4-7

(2) The magnitude of the angular momentum of the planet relative to the sun is given by

$$L = mrv\sin\alpha = mr\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|\sin\alpha = m\lim_{\Delta t\to 0}\frac{r|\Delta\boldsymbol{r}|\sin\alpha}{\Delta t}$$

Here $r|\Delta\boldsymbol{r}|\sin\alpha \approx$ twice of the shadowed area ΔS in Fig. 4-15, i. e.

$$r|\Delta\boldsymbol{r}|\sin\alpha \approx 2\Delta S$$

so that

$$L = 2m\lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t} = 2m\frac{\mathrm{d}S}{\mathrm{d}t}$$

in which $\mathrm{d}S/\mathrm{d}t$ equals the area swiped by the radius vector $\boldsymbol{r}$ in an unit time interval, called area speed. Because that the magnitude of L is a constant so that the area speed

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{L}{2m} = \text{a constant}$$

It is the Kepler's second law of planetary motion that we set out to prove.

4. 5 The Center of Mass

4. 5. 1 The center of mass

We can simplify the description of the motion of a system that consists of many particles by introducing a new concept of the center of mass. Suppose there are several particles, with mass $m_1, m_2, \ldots, m_i$, and so on, shown in Fig. 4-16. Let the coordinates of m_1 be (x_1, y_1), those of m_2 be (x_2, y_2), of m_i be (x_i, y_i) and so on. The definition of the center of mass of the system is the position having the coordinates (x_c, y_c) given by

$$x_c = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_ix_i}{\sum_i m_i} = \frac{1}{m}\sum_i m_ix_{i_i} \tag{4-40}$$

$$y_c = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_iy_i}{\sum_i m_i} = \frac{1}{m}\sum_i m_iy_{i_i} \tag{4-41}$$

In which $m=\sum_i m_i$ is the total mass of the system. The position vector $\boldsymbol{r}_c$ of the center of mass can be expressed in terms of the position vector $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ⋯of the particles as

$$\boldsymbol{r}_c=\frac{m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2+m_3\boldsymbol{r}_3+\cdots}{m_1+m_2+m_3+\cdots}=\frac{\sum_i m_i\boldsymbol{r}_i}{\sum_i m_i}=\frac{1}{m}\sum_i m_i\boldsymbol{r}_i \tag{4-42}$$

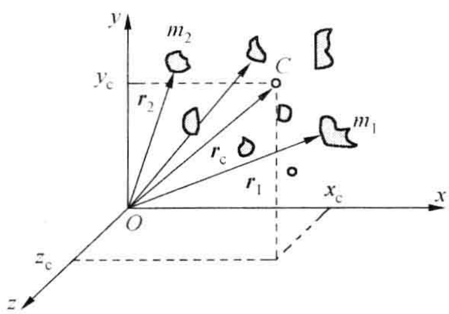


Fig. 4-16 The center of mass

In Eq. (4-42) it is necessary to point out that the position of the center of mass is not related to the origin of the coordinate system chosen.

The most simple particle system is that consists of two particles with equal mass, obviously its center of mass is located right at the middle of the line connecting to one another. For the two particle system with different mass, the position of the center of mass is on the connecting line but closer to the heaver one.

For the case that the mass of a body is continuously distributed, we must divide the body into a number of small volume element dV with mass dm and replace the sum in the equations above by an integral then the expression of the definition of the center of mass becomes

$$x_c=\frac{\int x\mathrm{d}m}{\int \mathrm{d}m}=\frac{\int x\mathrm{d}m}{m},\quad y_c=\frac{\int y\mathrm{d}m}{\int \mathrm{d}m}=\frac{\int y\mathrm{d}m}{m},\quad \boldsymbol{r}_c=\frac{\int \boldsymbol{r}\mathrm{d}m}{m} \tag{4-43}$$

In which x , y are the coordinates of mass element dm, d$m=\rho$dV, ρ is the mass per unit volume, m is the total mass of the body. There are some general points about such problems should be mentioned. First, whenever a homogeneous body has a geometric center, such as a solid sphere, a disk, a thin rod, or a solid cube, the center of mass always coincides with the geometric center. Second, whenever a body has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, the center of mass is not always located within the body. For example, the center of mass of a thin hoop is right at the middle of the hole.

Example 4-8 Find the center of mass of the earth-sun system. The mass of earth and that of sun are known as 5.98×10^{24} kg and 1.99×10^{30} kg, the distance from earth to sun is 1.5×10^{11} m.

Solution This is a two particle system, take the center of the sun as the origin of x axis, then $x_s=0$, $x_e=1.5\times10^{11}$ m, according to Eq. (4-40), the center of mass locates at

$$x_c=\frac{m_s x_s+m_e x_e}{m_s+m_e}=\frac{5.98\times10^{24}\ \text{kg}\times1.5\times10^{11}\ \text{m}}{1.99\times10^{30}\ \text{kg}+5.98\times10^{24}\ \text{kg}}=4.49\times10^{5}\ \text{m}$$

from the center of the sun, note that the radius of sun is 6.96×10^{8} m, the result means that the center of mass of the earth-sun system locates within the sun on the line connecting them and is about $0.645\times10^{-3}R_s$.

Example 4-9 Calculate the center of mass of a thin semicircle of radius R.

Solution Take the coordinate as show in Fig. 4-17. Suppose the mass per unit length is λ, the mass of a element arc dl is $dm = \lambda dl = \lambda R d\theta$, the coordinates are

$$x = R\cos\theta, \quad y = R\sin\theta$$

Because the **symmetry** of the semicircle is the y axis, the center of mass definitely lies on it, that is $x_c = 0$, from Eq. (4-43), we have

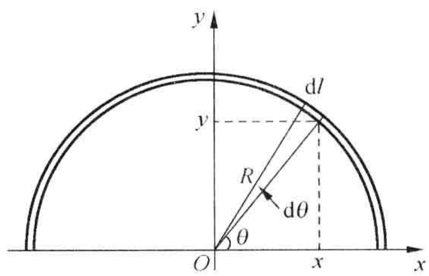


Fig. 4-17 For Example 4-9

$$y_c = \frac{\int y dm}{m} = \frac{\int_0^{\pi} R\sin\theta \cdot \lambda R d\theta}{\pi R\lambda} = \frac{2}{\pi}R$$

which indicates that the center of mass locates at the point $2R/\pi$ from the origin on the y axis.

4.5.2 Motion of the center of mass

To find one more useful way to describe the motion of a system of particles, rewrite Eq. (4-42) as $m\boldsymbol{r}_c = \sum_i m_i \boldsymbol{r}_i$ and taking the time derivative of it, we obtain $m\frac{d\boldsymbol{r}_c}{dt} = \sum_i m_i \frac{d\boldsymbol{r}_i}{dt} = \sum_i m_i \boldsymbol{v}_i$, let $\boldsymbol{v}_c = \frac{d\boldsymbol{r}_c}{dt}$ represent the velocity of the center of mass, the above equation can be expressed as

$$m\boldsymbol{v}_c = \sum_i m_i \boldsymbol{v}_i = \sum_i \boldsymbol{p}_i \tag{4-44}$$

In which $\sum_i \boldsymbol{p}_i = \boldsymbol{p}$ is the total momentum of the system, thus we have proved that the total momentum of the system equals to the product of the total mass and the velocity of the center of mass. The change rate of the total momentum with time is given by

$$\frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}_c}{dt} = m\boldsymbol{a}_c \tag{4-45}$$

where $\boldsymbol{a}_c$ is the acceleration of the center of mass. On the other hand, from Eq. (4-3), the change rate of the momentum of a particle with time equals to the total external force exerted on it, furthermore, due to Newton's third law, the total internal force of a system is zero, and the total external force equals to the vector sum of the external forces exerted on all particles of the system. So that, the total external force $\boldsymbol{F}$ equals to the change rate of the total momentum with time, Eq. (4-45) then can be rewritten as

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = m\boldsymbol{a}_c \tag{4-46}$$

Eq. (4-46) indicates that **the total external force on the system equals to the product of the total mass and the acceleration of the center of mass of the system** which is called as **the theorem of motion of the center of mass.** The form of this law is the same as that of Newton's second law, which indicates that the center of mass moves just as though all the mass concentrated at that point and it were acted on a net force equal to the sum of the external forces on the system.

The result above is very important when we analyze the motion of a system of particles, a rigid body or an extended; for example, we describe the motion of a diver as a combination of translational motion of the center of mass along a parabolic path and rotational motion about an axis through the center of mass.

Finally, note that if the net external force is zero, Eq. (4-46) shows that the acceleration $\boldsymbol{a}_c$ of the center of mass is zero. So the velocity $\boldsymbol{v}_c$ of the center of mass is constant, the total momentum is constant as well. This reaffirms the principle in section 4-2 of conservation of momentum.

Example 4-10 a cannon shell traveling in a parabolic trajectory (neglecting the air resistance) with initial velocity 500 m/s at the 60° with the horizontal explodes at the point of the maximum height, splitting into two fragments with equal mass as shown in Fig. 4-18, one fragment falls downward. Find the position of the other piece.

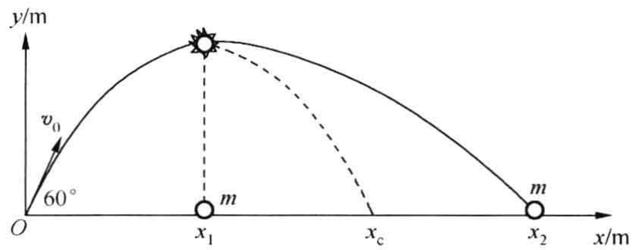


Fig. 4-18 For Example 4-10

Solution Set the coordinate system as shown in Fig. 4-18. Because that before and after the explosion there is no external force in horizontal direction, according to Eq. (4-46), the x component of motion of the center of mass is unchanged, the position of the center of mass is the range of the parabolic trajectory

$$x_c = R = \frac{v_0^2 \sin\theta\cos\theta}{g} = \frac{2 \times 500^2 \times 0.866 \times 0.5}{9.8} \approx 21.6 \times 10^3 \text{ m}, \quad y_c = 0$$

The position of the fragment falling down is $x_1 = 0.5R = 10.8 \times 10^3$ m, $y_1 = 0$, from Eq. (4-21) and $m_1 = m_2 = m$, we have

$$x_c = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m \times 10.8 \times 10^3 + m x_2}{2m} = 21.6 \times 10^3 \text{(m)}$$

by solving this equation, the position of the other piece is obtained as

$$x_2 = 32.4 \times 10^3 \text{ m}$$

the position of the second fragment in y axis $y_2 = 0$ too, which indicates that both fragments fall in x-y plane.

Questions

4-1 If you kick out a ball that originally was at rest on the deck of a moving ship with constant velocity, whether or not the increment of the momentum of the ball with respect to the ship and to the ground are equal to each other? How about the answer to the same question about the kinetic energy of the ball?

4-2 (1) When the nozzle of a fire hose discharges a large amount of water at high speed, several strong firemen are needed to hold the nozzle steady. Explain the reason.

(2) When firing a shotgun, a hunter always presses it tightly against his shoulder. Why?

4-3 Many features of cars, such as collapsible steering wheels and padded dashboards, are meant to change more safely the momentum of passengers during accidents. Explain their usefulness, using the impulse concept.

4-4 To solve problem 4-7, two students give two equations as following,

(1) $Mv = (M-3m)v' + mu - 2mu$;

(2) $Mv = (M-3m)v' + m(v+u) + 2m(v-u)$ what is wrong with the solutions?

4-5 Can a spacecraft or rocket get propulsion in empty space there is nothing in the environment to push against? What will happen when a rocket engine produces a large quantity of hot high pressure gas from the combustion of fuel in combustion chamber and then ejects this gas at high speed at the tail end of the rocket while pushes on the gas. Compare this with that a gun recoils when it ejects a projectile (say, a bullet), explain the mechanism of a rocket's propulsion.

4-6 Make schemes in two dimensions about the following example, specify the angular momentum of the particle with respect to a reference point:

(1) a satellite in a circular orbit;

(2) a satellite in an elliptical orbit, for both, the center of earth is the reference point;

(3) an airplane flying straight possesses angular momentum with respect to an arbitrary point on the earth;

(4) for a rotating wheel, each bit of mass contributes some angular momentum to the total with respect to the axis.

In which ones among above examples the conservation of angular momentum hold?

4-7 A man walks back and forth from the stem to the stern on a boat which is at rest on a still pond; can the boat move forward and reach the bank? Give the reason?

4-8 A man and a boy are standing some distance apart on the frictionless surface of a frozen pond and pull on the ends of a light rope that stretched between them, midway between them there is a bag on the ice. Then the man pulls the rope and moves toward the bag, if the mass of the man is 1.5 times the mass of the boy, who will reach the bag first? Why?

4-9 What path does the diver's center of mass follow when the diver is doing a forward one-and a-half-somersault dive in the air after leaving the spring board ?

Problems

4-1 In a high jump, a 60.0 kg man jumps over the 2.0 m height bar, and vertically falls to the bumping cushion. Suppose that it takes 2.0 s for the man to reduce his velocity to zero.

(1) What average force acts on the jumper?

(2) If the jumper directly fell to the ground from that height and it takes 0.20s to get rest, what average force would act on the man?

4-2 A 140 g baseball, in horizontal flight with a speed v_i of 40 m/s, is struck by a batter. After leaving the bat, the ball travels in the opposite direction with a speed v_f, 40 m/s also.

(1) What impulse acted on the ball while it was in contact with the bat?

(2) Suppose that the impact time Δt for the baseball-bat collision is 1.2 ms, a typical value. What average force acts on the baseball?

(3) What is the average acceleration of the baseball?

4-3 A 2.5 g ping-pong ball bounces off a bat, the magnitude of the incident and the leaving velocity are $v_1 = 10$ m/s and $v_2 = 20$ m/s respectively, they are in the same plane, make 45°and 30°angle with the normal of the bat as shown in Fig. 4-19.

(1) Find the impulse acting on the ball during the collision.

(2) Suppose that the impact time takes 0.01 s, find the average impact force exerted on the ball by the bat (magnitude and direction).

4-4 Coal drops at the rate of 200 kg per second from a hopper onto a horizontal moving belt which transports it to the washing plant (Fig. 4-20). If the belt travels at the speed of 3. 0 m/s, what is the power of the motor driving the belt (ignore friction loss)?

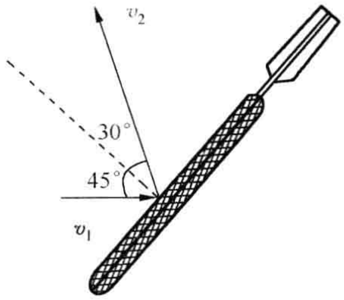


Fig. 4-19 For problem 4-3

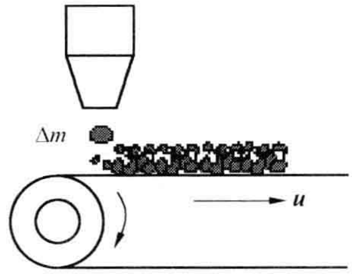


Fig. 4-20 For problem 4-4

4-5 A cannon whose mass M is 1500 kg fires a 70 kg ball in the horizontal direction with a muzzle speed v_0 of 50 m/s. The cannon is mounted so that it can recoil freely.

(1) What is the velocity v of the recoiling cannon with respect to the earth?

(2) What is the initial velocity v_i of the ball with respect to the earth (refer to Example 4-2)?

4-6 A workman throws a package of 50kg mass with a speed of 1. 0 m/s in $+x$ direction into a 200 kg flatcar, what is the velocity that the flatcar gains?

(1) The car is at rest in the beginning;

(2) It moves with initial speed of 2. 0 m/s in $+x$ direction;

(3) It moves with the same initial speed, hut in $-x$ direction. Neglect the friction between the flatcar and the ground.

4-7 A loaded boat of total mass M sails on a still pond with a velocity $\boldsymbol{v}$ as shown in Fig. 4. 21(a), then a man throws a body of mass m from the stem and another man throws a body of mass $2m$ from the stern simultaneously, with the same speed $\boldsymbol{u}$ but in opposite direction horizontally relative to the boat as Fig. 4-21 (b) shows. Find the velocity of the boat just after the two bodies are thrown.

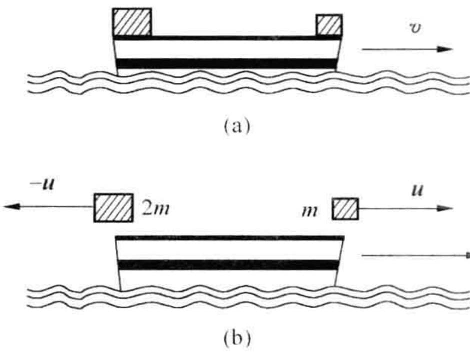


Fig. 4-21 For problem 4-7

4-8 A man of 60 kg stands on a 100 kg canoe at rest in still water and the length of the boat is 5. 0 m. What is the distance that the canoe moves when the man walks from the stem to the stern? Neglect the resistance of water.

4-9 A sportsman of mass M with a basketball of mass m in his hand jumps up forward with the velocity $\boldsymbol{v}_0$ that forms an angle θ to the horizontal, he throws the ball with velocity $\boldsymbol{u}$ relative to himself backward horizontally just as he jumps to the highest position. What is the increase in the distance that the sportsman passes forward compared with the situation that if he jumps but does not throw the ball.

4-10 In Fig. 4-22 a boy standing at one end of a light flatcar puts the shot along the length of the car twice. First, with the car locked in position, then with the car free to roll on horizontal rails with negligible friction. On both trials the shot, just after it leaves the hand of the boy, has the same velocity(both magnitude and direction) relative to him. On which trial, if either, does the shot cover the greater distance measured relative to the ground? Suppose that the mass of the flatcar with the boy is M while the mass of the shot is m . What is the ratio of the two distances d_2/d_1?

4-11 Fig. 4-23 shows an elastic collision of a deuteron(the nucleus of an isotope of Hydrogen with a mass of 2. 0 u, u is the atomic mass unit) and a proton (with a mass of 1. 0 u). This collision took place within the emulsion of a photographic film; in this medium the particles produce visible tracks because their passage exposes the film. The deuteron has an initial speed of 2.7×10^7 m/s and the final speed of 2.2×10^7 m/s. The proton is initially at rest. Calculate the final speed of the proton, and the final directions of motions of the deuteron and the proton.

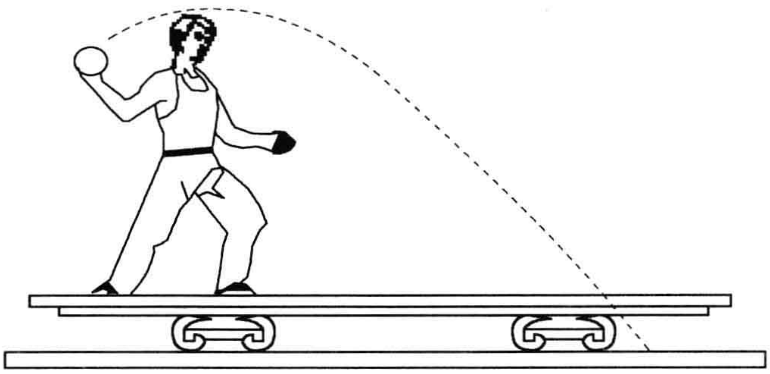


Fig. 4-22 For problem 4-10

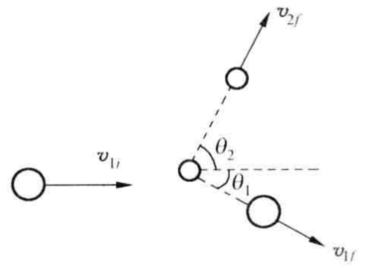


Fig. 4-23 For problem 4-11

4-12 A bullet weighting 4.0 g is fired at a speed of 600 m/s into a ballistic pendulum of weight 1.0 kg and thickness 25 cm. The bullet passes through the pendulum and emerges with a speed of 100 m/s. Calculate the average retarding force acting on the bullet in its passage through the block, and the height to which the pendulum rises (refer to Fig. 4-7 in Example 4-4).

4-13 Another way to measure the speed of the bullet is that to fire the bullet into (and embedded in) a block connected to a spring with one end fixed, then to measure the compression of the spring. Suppose that the mass of the bullet is 20 g, the block is 8.98 kg, the spring constant is 100 N/m, the compression of the spring is 10cm and the coefficient of kinetic friction between the block and the horizontal surface is 0.2. Make a scheme; calculate the initial speed of the bullet.

4-14 Fig. 4-24 shows that a crate of mass m released from the top of an incline slides down along its frictionless surface into a small cart of mass M placed at a horizontal plane, and moves with the cart over a distance $S=0.25$ m, then stops. Suppose that $m=1$ kg, $M=4.0$ kg, $h=5.0$ m, and $\theta=60^\circ$. Find the coefficient of friction between the cart and the ground.

4-15 A 5.0 g bullet is fired with initial speed $v=400$ m/s horizontally at a block of mass 1.0kg resting on a smooth tabletop. The bullet gets into the block and embeds itself in it while the block moves a distance as shown in Fig. 4-25. Find

(1) the work done by the force exerted on the bullet by the block;

(2) the work done by the force exerted on the block by the bullet;

(3) the mechanical energy dissipated.

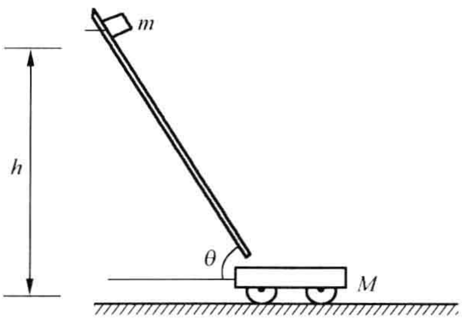


Fig. 4-24 For problem 4-14

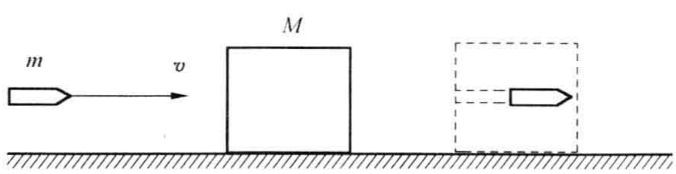


Fig. 4-25 For problem 4-15

4-16 In Fig. 4-26, a disk of mass M is connected to a light spring of force constant k with another end fixed. A small body of mass m is released from the height h above the disk and falls into it, then moves together with it. Find the maximum extension of the spring.

4-17 A particle of mass m is released from rest at point P in Fig. 4-27.

(1) what torque does the gravitational force acting on the particle exert about the origin O?

(2) What is the angular momentum of falling particle about the origin? Show that this result satisfy the relation $\boldsymbol{M}=\mathrm{d}\boldsymbol{L}/\mathrm{d}t$.

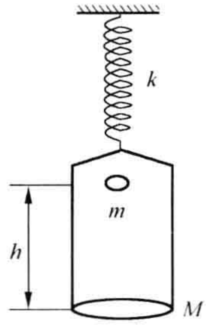


Fig. 4-26 For problem 4-16

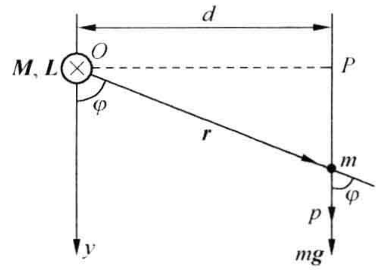


Fig. 4-27 For problem 4-17

4-18 Show that a particle of mass m moving in a circular path of radius R at constant speed v will have constant angular momentum relative to an origin at the center of the circle.

4-19 Fig. 4-28 shows the relaxed state of a string of length l with one end fixed at point O and another end connected a ball at position A, $OA=h$. Let the ball move along AB which is perpendicular to OA, when the distance OB is equal to l so that the string is then stretched to cause the ball moving around point O with radius l. Find the ratio E_k/E_{ki} (E_{ki} is the ball's initial kinetic energy while E_k is its kinetic energy when moving in a circle).

4-20 A child whirls a ball of mass m in a circle around her head, the initial radius $r_i=130$ cm with an initial angular speed ω_i of 35 r/min. Then the child pulls in the cord, shortening the radius to $r_f=85$ cm. What is the angular speed ω_f of the ball in this new orbit (Fig. 4-29)?

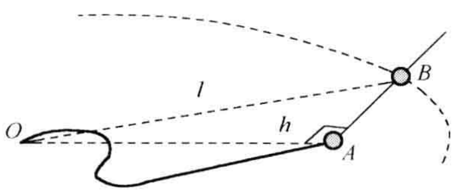


Fig. 4-28 For problem 4-19

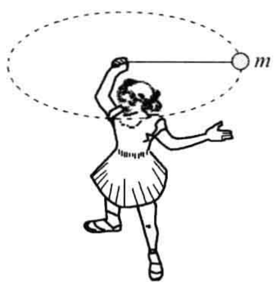


Fig. 4-29 For problem 4-20

4-21 In the Bohr-atom model an electron of mass 9.11×10^{-31} kg revolves in a circular orbit about the nucleus. It completes an orbit of radius 0.53×10^{-10} m in 1.51×10^{-16} s. What is the angular momentum L of the electron in this orbit?

4-22 Find the position of the center of mass for a water molecule which consists of one oxygen atom and two hydrogen atoms, $m_O=16m_H$, $\theta=37.7°$ formed by x axis and the line connecting oxygen atom and hydrogen atom, the length of each line is $d=1.0\times10^{-10}$ m. Use the coordinate system shown in Fig. 4-30.

4-23 Find the position of the center of mass for an uniform semicircular thin sheet with the surface mass density σ and radium R. Use the same coordinate system as in Example 4-9.

4-24 Another situation of Example 4-10, if $m_2=2m_1$, the first fragments is found at $x_1=16\times10^3$ m $y_1=$ 60 m after explosion, calculate the coordinates of the other piece, suppose the initial parabolic trajectory is in x-z plane, θ and v_0 are the same as in Example 4-10.

4-25 Using the theorem of motion of the center of mass and Eq. (4-40) solve problem 4-8 again.

4-26 A soft rope of length l and mass m winded on a horizontal table surface, a person lift one end of it with his hand at a uniform vertical speed $\boldsymbol{v}_0$, try to find the lifting force when the distant from the up end of the rope to the table surface is y (Fig. 4-31). Hint: use Eq. (4-41) and Eq. (4-46), the answer is $F=\frac{m}{l}(v_0^2+yg)$.

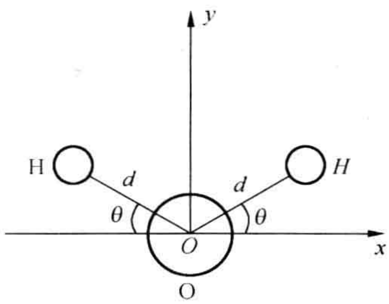


Fig. 4-30 For problem 4-22

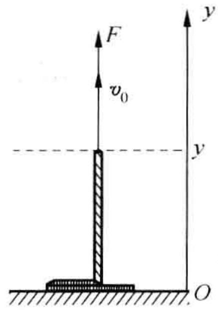


Fig. 4-31 For problem 4-26

※4-27 If the initial condition of problem 4-26 is the end of the soft rope just leave the table surface at rest, then release it, try to find the normal force exerted on the rope when the left length is y from the table surface (hint: find $y_c(y)$ first, for the particle system of the rope, weight should be mg rather than $(y/l)\,mg$, the answer is $F=3mg(1-y/l)$).

Chapter 5

Rotation of a Rigid Body

Our discussion of the principles of mechanics so far has been concerned only with translational motion of a particle—an ideal model whose size and shape can be neglected in the problems involved. But in our everyday life experience, in factories, laboratories and in nature phenomena, there are many problems in which this model and its translational motion are inadequate, instead of this, we need to study another important class of motion—rotation. Rotation is the motion of wheels, gears, motors, handles, clocks, helicopter blades as well as the motion of spinning electrons and atoms, spinning planets and so on. It is the motion of acrobats, of high divers, or of ballet dancers etc.. Rotation motion is all around us.

To discuss the principles governing the motion of rotation, we need to introduce you another important ideal model—rigid body. A body is rigid if the particles in the body do not move relative to one another under the action of an external force. Thus, the body has a perfectly definite and unchanged shape and size and all its parts have a fixed position relative to one another, in the other words, all possible deformations can be neglected so that the assumption that rigidity is a good approximation.

In this chapter, we will focus on the subject of the rotation of a rigid body around a fixed axis, in which the concepts, principles and laws are the bases of study more complicated rotation problems.

5.1 Motion of a Rigid Body

5.1.1 Translation and rotation of a rigid body

A rigid body can undergo two types of motion, translation and rotation. The motion is a translation when all particles describe parallel paths so that the lines joining any two points in the body always remain parallel to its initial position, all points in the body have same instantaneous velocity and acceleration, therefore, the motion of any point of the body can represent the translational motion of entire rigid body, usually we choose the center of mass as the representative point whose mass is equal to the mass of the rigid body and that is acted on by a force equal to the sum of all external forces applied to the body. The principle of translational motion of a particle given by Newton's second law which we studied before, is applied to the translation motion of the representative point of the rigid body.

The motion is a rotation of a rigid body around an axis when all the particles describe circular paths around a line called the axis of rotation. The axis may be fixed or may be changing its direction relative to the body during the motion.

A rigid body can simultaneously have two kinds of motion, the most general motion of a rigid body can always be considered as a combination of a rotation and a translation. For example, the rolling of the wheels of a bicycle in Fig. 5-1, which can be regarded as a translation represented by the axis with respect to the earth and a rotation around this axis. The displacement $\Delta \boldsymbol{r}$ of point A on the rim relative to the reference frame fixed on the earth, is the vector addition of the displacement $\Delta \boldsymbol{r}_0$ of the origin O on the axis and the displacement $\Delta \boldsymbol{r}'$ of the point A relative to the origin O shown in the figure.

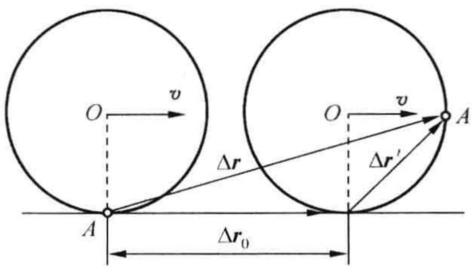


Fig. 5-1 The combination of a translation motion and a rotation of the wheel on a bicycle

5.1.2 Rotation of a rigid body around a fixed axis

The simplest and most common case of rotation is the rotation of a rigid body around a fixed axis, there are mane examples relevant to it in daily experience, such as a pulley rotates about a fixed horizontal axis, and a disc about its central vertical axis, a door about its hinge, a grindstone wheel; a motor as well as a gear etc..

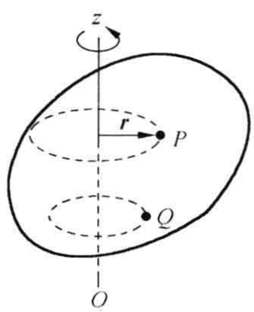


Fig. 5-2 A rigid body rotates about z axis. Each particle moves in a circle with center lying on z axis

Fig. 5-2 shows a rigid body rotating about a fixed axis which coincides with the z axis, every point of the body moves in a circle whose center lies on the axis of rotation and every point experiences the same angular displacement during a particular time interval. To describe the orientation of the body at any instant, we select one arbitrary particle P in the body and use it as a reference point, the circular motion of this particle is then representative of the rotational motion of the entire rigid body, that is, the angular position, angular speed and angular acceleration of the entire rigid body are all represented by those of particle P.

So that, the kinematics equations of the circular motion of a particle discussed in section 1-4 are all valid in the motion of the rotation of a rigid body about a fixed axis. For the sake of convenient, let us rewrite them in the followings:

angular speed

$$\omega = \frac{d\theta}{dt} \tag{5-1}$$

angular acceleration

$$\beta = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{5-2}$$

where θ is the angular position of the body with respect to a reference line. Moreover, equations

$$\omega = \omega_0 + \beta(t - t_0) \tag{5-3}$$

$$\theta = \theta_0 + \omega_0(t - t_0) + \frac{1}{2}\beta(t - t_0)^2 \tag{5-4}$$

$$\omega^2 = \omega_0^2 + 2\beta(\theta - \theta_0) \tag{5-5}$$

are specially useful for the motion with constant angular acceleration.

However, as you may already realize from the motion of points P and Q in Fig. 5-2 that the linear speed of different point in the body may be different, depending on the perpendicular distance from the particle to the axis, that is the radius of the circle the particle moving along. Equations

$$s = \theta r \tag{5-6}$$

$$v = \omega r \tag{5-7}$$

$$a_t = \beta r \tag{5-8a}$$

$$a_n = \omega^2 r \tag{5-8b}$$

hold for the relations between the angular quantities of a rigid body rotating about a fixed axis and the linear quantities of the points in it.

It is important when we study angular motion by means of analogy between linear and angular motion to find the corresponding quantities and the governing laws of rotation, which will head us well-understand and in easier way to solve problems met in rotation.

5.2 Torque, the Law of Rotation, and Rotational Inertia

5.2.1 Torque

In this section we shall deal with the dynamic problem for the rigid body rotating about a fixed axis. If you want open a door or a window, you must certainly apply a force; that alone, however, is not enough, where you apply that force and in what direction you push are also important. The physics concept hidden in this experience is torque of the force applied on the body. The first place where we learned this word is in section 4-4 where Eq. (4-36) $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$ defined the torque acting on a particle with respect to the fixed point around which the body rotates.

Suppose that the force $\boldsymbol{F}$ acting on the body is located in the plane perpendicular to the axis OO' about which the body rotates as shown in Fig. 5-3(a). The force is applied at point P whose position is defined by the vector $\boldsymbol{r} = \overrightarrow{OP}$. The directions of $\boldsymbol{r}$ and $\boldsymbol{F}$ make an angle θ with each other. The torque produced by $\boldsymbol{F}$ with respect to the axis is defined as

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{5-9a}$$

The magnitude of $\boldsymbol{M}$ is given by

$$M = rF\sin\theta \tag{5-9b}$$

or

$$M = rF_{\perp} \tag{5-10a}$$

which means that only the tangential component of $\boldsymbol{F}$ ($F_t = F\sin\theta$) can have any turning effect on the body. The radial component ($F_n = F\cos\theta$) passes through the axis and can not cause the body to rotate. Torque plays the same role in generating angular acceleration that force does in generating linear acceleration. Eq. 5-9(a) can also be written as

$$M = r_{\perp} F \tag{5-10b}$$

Same as d in Eq. (4-37), $r_{\perp}$ called the moment arm of the force is the perpendicular distance from the axis of rotation to the line of action of the force, as Fig. 5-3(a) shows. The SI unit of torque is the Newton • meter (abbr. N • m), thus the dimension of torque is $[M] = ML^2T^{-2}$.

The direction of the torque in Eq. (5-9) is determined by the right-hand-rule for a vector product, that is to sweep the vector $\boldsymbol{r}$ into the vector $\boldsymbol{F}$ with your right hand through angle $\theta(\theta<180°)$, thus, your outstretched thumb then points in the direction of vector $\boldsymbol{r}\times\boldsymbol{F}$. So that torque is at right angle to the plane formed by $\boldsymbol{r}\times\boldsymbol{F}$, or parallel to the axis OO', and it has only two possible direction (orientation), if it causes the body to rotate counterclockwise, it is defined as a positive torque, otherwise, negative. Fig. 5-3(b) shows a general case that the force $\boldsymbol{F}$ is not placed in the plane perpendicular to the axis. We can resolve the force into two components, one, $\boldsymbol{F}_1$ is in that plane, the other, $\boldsymbol{F}_2$ is at right angle to that plane, and the later has no effect at all to rotate the body about the axis. So, the force in Eq. (5-9) refers only to the force component in the plane perpendicular to the axis.

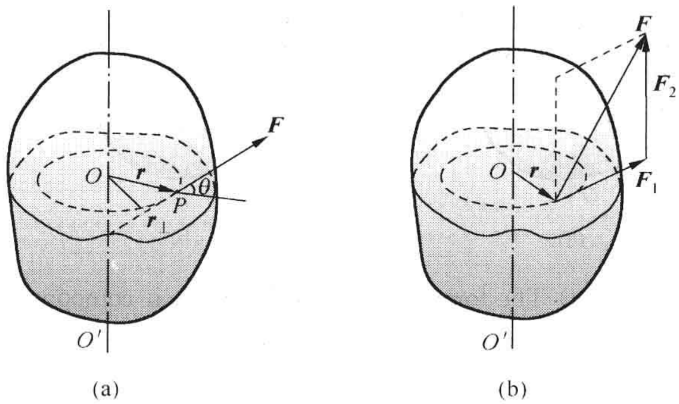


Fig. 5-3 (a) The definition of the torque with respect to axis OO' is given by $\boldsymbol{r}\times\boldsymbol{F}$.
(b) Resolve a force that is not in the plane perpendicular to the axis

5.2.2 The law of rotation (Newton's second law for rotation)

We are now ready to consider the dynamic equation of rotation about a fixed axis. Our method is treating the body as a collection of particles, all with same angular acceleration. In Fig. 5-4(a), point P_i is one of the particles of the body with mass Δm_i. The particle is acted on by an external force $\boldsymbol{F}_i$, and by an internal force $\boldsymbol{f}_i$, the resultant of the forces exerted by all other particles in the body. Applying Newton's second law to particle P_i, gives

$$\boldsymbol{F}_i + \boldsymbol{f}_i = \Delta m_i \boldsymbol{a}_i$$

whose radial and tangential components are

$$F_{in} + f_{in} = \Delta m_i a_{in} = \Delta m_i r_i \omega^2$$

$$F_{it} + f_{it} = \Delta m_i a_{it} = \Delta m_i r_i \beta$$

The first equation does not concern us further. Let us multiply both sides of the second equation by the distance r_i of the particle from the axis, we obtain

$$F_{it} r_i + f_{it} r_i = \Delta m_i r_i^2 \beta \tag{5-11}$$

Comparing it with Eq. (5-10a), we find that the first term on the left is the torque of the external force about the axis, and the second is the torque of the internal force. We can write the equations corresponding to Eq. (5-11) for all particles of the body and add up all of them, then obtain

$$\sum_i F_{it} r_i + \sum_i f_{it} r_i = \sum_i (\Delta m_i r_i^2) \beta \tag{5-12}$$

According to Newton's third law, consider a pair of internal action—reaction forces, $\boldsymbol{f}_{ij}$ and $\boldsymbol{f}_{ji}$ as Fig. 5-4(b) shows. Because they act along the same line, their moment arm about the axis is the same, and also they are equal to each other but in opposite direction, so the resultant torque of them about the axis is zero. Since all internal forces exist in pair, so that

$$\sum_i f_{it} r_i = 0$$

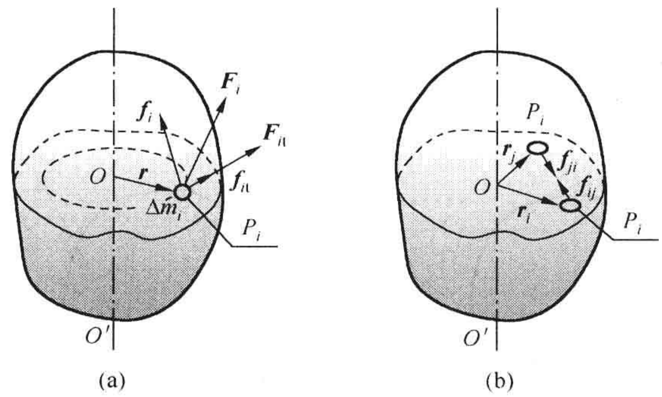


Fig. 5-4 (a) The forces acting on P_i and their components.
(b) The resultant torque of a pair of internal forces is zero

The sum of the left side of Eq. (5-12) is then simply the resultant torque of the external forces about the axis, or

$$M = \sum_i F_{it} r_i$$

Eq. (5-12) becomes

$$M = (\sum_i \Delta m_i r_i^2) \beta$$

The sum on the right side has a certain amount, it is defined as the rotational inertia of the body about the axis, and is represented by I, that is

$$I = \sum_i \Delta m_i r_i^2 \tag{5-13}$$

Eq. (5-12) finally becomes

$$M = I \beta \tag{5-14}$$

which means that **the angular acceleration of a rigid body is proportional to the resultant external torque and inversely proportional to the rotational inertia of the body. This is called the Law of Rotation.**

It is obvious that the angular acceleration of a rigid body about a fixed axis is given by an equation having exactly the same form as that for the linear acceleration of a particle,

$$\boldsymbol{F} = m\boldsymbol{a}$$

The resultant torque M, the angular acceleration β, and the rotational inertia I of a rigid body about a fixed axis correspond to the resultant force $\boldsymbol{F}$, the linear acceleration $\boldsymbol{a}$, and the mass m respectively. That is why Eq. (5-14) is also called as the Newton's second law for rotation.

5.2.3 Calculation of rotational inertia

If a rigid body is made up of discrete particles, we can calculate its rotational inertia by using Eq. (5-13). If the mass of the body is continuous distributed, we must replace the sum in Eq. (5-13) by an integral and the definition of the rotational inertia becomes

$$I = \int r^2 \mathrm{d}m = \int r^2 \rho \,\mathrm{d}V \tag{5-15}$$

where ρ is the mass per unit volume, dV is the volume element, and r is the distance from dm to the axis. In the case of the mass distribution over a surface, Eq. (5-15) becomes

$$I = \int r^2 \mathrm{d}m = \int r^2 \sigma \mathrm{d}s$$

where σ is the mass per unit area, ds is the area element. In the case of the mass distribution over a line, Eq. (5-15) becomes

$$I = \int r^2 \mathrm{d}m = \int r^2 \lambda \mathrm{d}l$$

Where λ is the mass per unit length, and dl is the line element.

The SI unit of rotational inertia is kg · m^2, and its dimension is $[I] = \mathrm{ML}^2$.

Example 5-1 Fig. 5-5(a) shows a uniform rod of mass m and length l, calculate its rotational inertia

(1) about an axis at right angles to the rod, through its middle point;

(2) about a perpendicular axis through its end point.

Solution (1) We choose dx as a mass element located at position x. The mass per unit length of the rod is $\lambda = m/l$, so that the mass dm of the element is d$m = (m/l)\,$dx. From Eq. (5-15) we have

$$I_1 = \int x^2 \mathrm{d}m = \int_{-l/2}^{l/2} x^2 \left(\frac{m}{l}\right) \mathrm{d}x = \frac{1}{12} m l^2$$

(2) We change the origin from meddle point to the end point as Fig. 5-5(b) shows, the integral becomes

$$I_2 = \int x^2 \mathrm{d}m = \int_0^l x^2 \left(\frac{m}{l}\right) \mathrm{d}x = \frac{1}{3} m l^2$$

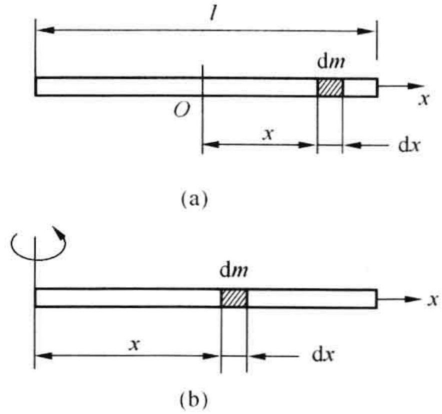


Fig. 5-5 For Example 5-1

Example 5-2 Calculate the rotational inertia about the axis of symmetry in the two cases:

(1) The mass m is distributed uniformly in such a way that forms a thin hoop of radius R (Fig. 5-6(a)).

(2) The mass m is distributed uniformly in such a way that forms a thin disk of radius R (Fig. 5-6(b)).

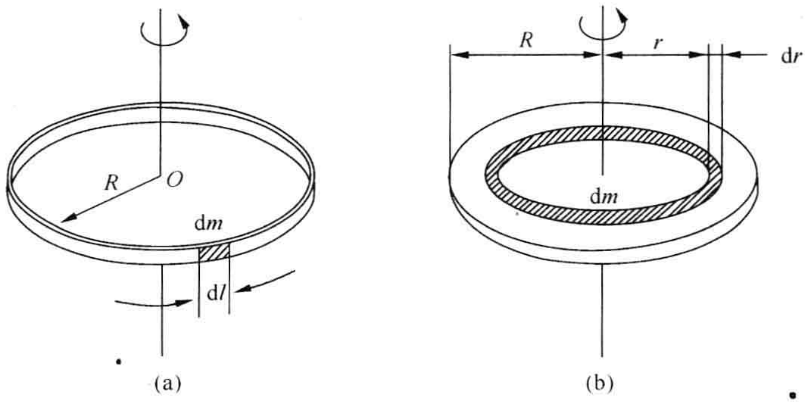


Fig. 5-6 For Example 5-2

Solution (1) In this case, mass per unit length $\lambda = \dfrac{m}{2\pi R}$, we choose an arc element dl, $dm = \lambda dl$. From Eq. (5-15), we have

$$I_1 = \int R^2 \mathrm{d}m = \int_0^{2\pi R} R^2 \frac{m}{2\pi R} \mathrm{d}l = mR^2$$

(2) In this case, the disk can be regarded as made up of a large number of thin concentric rings fitting one around another. We choose such a ring of radius r, width dr, located at r, as mass element. The mass per unit area is $\sigma = \dfrac{m}{\pi R^2}$, and the area element is $ds = 2\pi r dr$, the mass element is $dm = \sigma \cdot ds$. From Eq. (5-15) we have

$$I_2 = \int r^2 \mathrm{d}m = \int r^2 \cdot \frac{m}{\pi R^2} \cdot 2\pi r \mathrm{d}r = \int_0^R \frac{2m}{R^2} r^3 \mathrm{d}r = \frac{1}{2} mR^2$$

From above examples we note that the rotational inertia of a rigid body depends on

(1) the total mass of the body;

(2) the distribution of the mass;

(3) the position of the axis relative to the body. Table 5-1 summarizes the rotational inertia of several common bodies, about various axes.

Table 5-1 Rotational inertia of several common rigid bodies

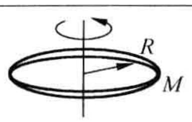
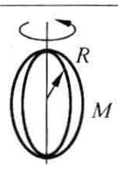
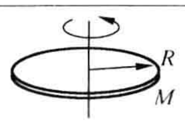
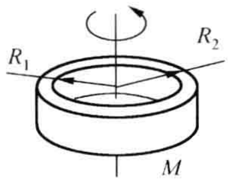
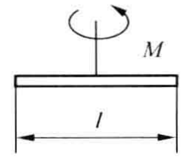
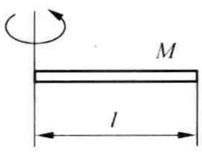
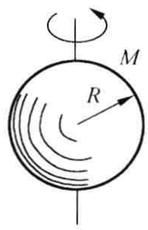
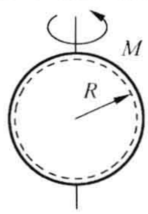
	Body and axis	Rotational Inertia
	Thin hoop about symmetry axis	MR^2
	Thin hoop about diameter	$\frac{1}{2} MR^2$
	Disk or cylinder about symmetry axis	$\frac{1}{2} MR^2$

table continued

	Body and axis	Rotational Inertia
	Hollow cylinder about symmetry axis	$\frac{M}{2}(R_1^2+R_2^2)$
	Thin rod about perpendicular axis through center	$\frac{1}{12}Ml^2$
	Thin rod about perpendicular axis through the end	$\frac{1}{3}Ml^2$
	Solid sphere about any diameter	$\frac{2}{5}MR^2$
	Thin spherical shell about diameter	$\frac{2}{3}MR^2$

5.3 Applying the Law of Rotation

Since the law of rotation $M = I\beta$ is the Newton' s second law for rotation, making an analogy between rotational problem and translational problem would help you to understand and solve the rotational problem, keeping the following points in mind, would be also helpful:

(1) Be sure to include all the forces and their torques acting on the rigid body involved, and write them down.

(2) Choose a positive direction as a reference to determine the signs of the torques, angular displacement, velocity, and angular acceleration.

(3) Write the equation of law of rotation to rigid body, and to find the unknown quantities.

(4) For the particles, if any, connected to the rigid body, apply Newton's second law to it, write the equation of relation between the rotational quantities and the linear quantities.

Example 5-3 A pulley fixed on a horizontal axis as Fig. 5-7 shows, two blocks m_1 and m_2 suspended from a light string which runs over the disk of the pulley without slipping. The pulley has radius R and a rotational inertia I about its axis, and rotates without friction. Find

(1) the angular acceleration of the pulley and the tensions in the two parts of the string;

(2) suppose that the blocks move from rest, what is the time required when the tangential acceleration equals to the normal acceleration for the points at the rim of the disk.

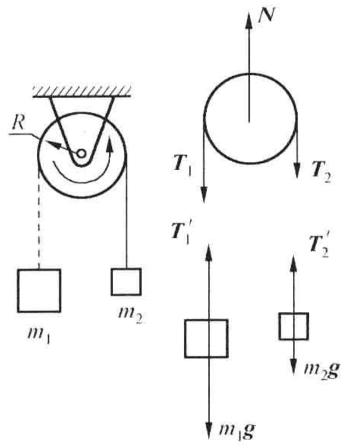


Fig. 5-7 For Example 5-3

Solution (1) Suppose that the tensions in the two parts of the string attached to the masses are T_1 and T_2. If the rotational inertia of the pulley were zero as it used to be assumed in particle dynamics (chapter 2), then these tensions would be equal. But if the rotational inertia is not zero, then a difference between T_1 and T_2 is required for the acceleration of the pulley.

The upward supporting force at the axis generates no torque; T_1 and T_2 generate torques T_1R and T_2R about the axis, the torque being reckoned as positive if it tends to produce counterclockwise acceleration. The equation of rotational motion of the pulley is then

$$T_1R - T_2R = I\beta \quad ①$$

For the attached blocks, we use Newton's second law. Because the mass of the string is neglected, so $T_1 = T'_1$, $T_2 = T'_2$,

$$m_1g - T_1 = m_1a \quad ②$$

$$T_2 - m_2g = m_2a \quad ③$$

Note that the angular and linear acceleration are related by the equation

$$a_t = \beta R \quad ④$$

and we have one more relation

$$a_t = a \quad ⑤$$

here a_t is the tangential acceleration of the points at the rim of the disk. Because there is no slipping between the rim and the string, so that it also represents the acceleration a of the blocks. Solving above equations, we obtain

$$a_t = a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2}, \quad \beta = \frac{(m_1 - m_2)g/R}{m_1 + m_2 + I/R^2}$$

$$T_1 = m_1(g - a) = m_1g \cdot \frac{2m_2 + I/R^2}{m_1 + m_2 + I/R^2}, \quad T_2 = m_2(g + a) = m_2g \cdot \frac{2m_1 + I/R^2}{m_1 + m_2 + I/R^2}$$

(2) For the points at the rim of the disk, normal acceleration $a_n = \omega^2 R$, because $\omega = 0$ at $t = 0$ and β is a constant so that

$$\omega = \beta t, \quad a_n = \beta^2 t^2 R$$

When $a_t = a_n$, that is $\beta R = \beta^2 t^2 R$, then

$$t^2 = \frac{1}{\beta}, \quad t = \sqrt{\frac{m_1 + m_2 + I/R^2}{(m_1 - m_2)g/R}}$$

From the expressions of β, a_t and a_n, we see that the disk rotates with a constant angular acceleration, therefore the points on it rotate with a constant tangential acceleration, but the total linear acceleration changes with time.

Example 5-4 A uniform rod of length l, mass m pivots about a horizontal axis O, and can freely rotate in vertical plane. In the beginning, the rod is at rest in horizontal position as the dash line shows in Fig. 5-8. Find the angular acceleration and angular speed as it wings over an angle θ.

Solution In Example 2-4, we regarded the bob of a pendulum as a particle, but in this case, the motions of the different point in the rod are not all the same, that is, the shape and sides of the body can not be neglected at all. We must use the law of rotation to treat it.

Obviously, it is only the weight of the rod that exerts a torque on it to cause its rotation. Because the mass is uniformly distributed over the rod, we divide it into a numbers of small elements, and take one element dm of width dr whose position is r related to the axis, and weight is dm**g** , the torque element of it about O, is then

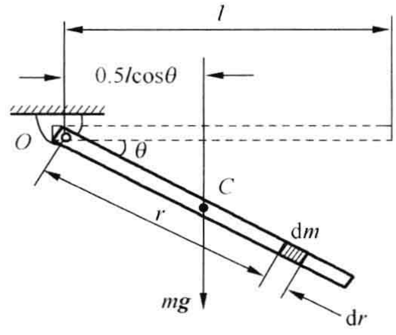


Fig. 5-8 For Example 5-4

$$dM = r\cos\theta dmg, \quad dm = \frac{m}{l}dr$$

The total torque is

$$M = \int dM = \frac{mg}{l}\cos\theta\int_0^l r dr = \frac{l}{2}\cos\theta \cdot mg \qquad ①$$

which is equal to the torque as if the total mass were concentrated at point C (the mass center), then, its torque would be.

Substituting Eq. ① into equation $M = I\beta$, where $I = \frac{1}{3}ml^2$, then we obtain the angular acceleration

$$\beta = \frac{3g}{2l}\cos\theta \qquad ②$$

To find angular speed, rewrite $M = I\beta$ as $M = Id\omega/dt$

or

$$Mdt = Id\omega \qquad ③$$

replace M by equation ①, then multiply both sides of equation ③ with ω, that is

$$\frac{1}{2}mgl \cdot \cos\theta \cdot \omega dt = \frac{1}{3}ml^2\omega d\omega$$

$$\frac{1}{2}g\int_0^\theta \cos\theta \cdot d\theta = \frac{1}{3}l\int_0^\omega \omega d\omega$$

we have

$$g\sin\theta = \frac{1}{3}l\omega^2$$

or

$$\omega = \sqrt{3g\sin\theta/l}$$

5.4 Kinetic Energy and Work in Rotational Motion

5.4.1 Kinetic energy of rotation

The rapidly rotating flywheel of a machine, the rotating rotor blades of a helicopter and all rotating bodies certainly have kinetic energy. In order to get the expression of kinetic ener-

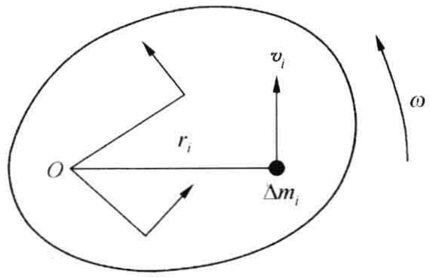


Fig. 5-9 Different particles have different linear speed

gy of rotation, similarly to what we have done in section 5-2, we treat the rigid body rotating with angular speed ω as a collection of a number of particles, all with different tangential speed as shown in Fig. 5-9. Take the ith particle of mass Δm_i, perpendicular distance r_i from the axis, we can write the particle's kinetic energy as

$$\Delta E_k = \frac{1}{2}\Delta m_i v_i^2$$

The kinetic energy of rotation of the body should be the sum of kinetic energies taken over all the particles that make up the body:

$$E_{k2} = \frac{1}{2}\Delta m_1 v_1^2 + \frac{1}{2}\Delta m_2 v_2^2 + \cdots + \frac{1}{2}\Delta m_i v_i^2 + \cdots = \frac{1}{2}\sum_i \Delta m_i v_i^2$$

The problem with above equation is that v_i is not the same for all particles. We get around this by using the relation $v_i = \omega r_i$, so that we have

$$E_k = \sum_i \frac{1}{2}\Delta m_i v_i^2 = \frac{1}{2}(\sum_i \Delta m_i r_i^2)\omega^2$$

in which ω is the same for all particles and $\sum \Delta m_i r_i^{\,2} = I$ is the rotational inertia of the body about the axis, thus E_k becomes

$$E_k = \frac{1}{2}I\omega^2 \qquad (5\text{-}16)$$

which is the expression of kinetic energy of a body in pure rotation we seek, and is the angular equivalent of the formula $E_k = \frac{1}{2}mv^2$ in pure translational motion. They are both kinetic energy, expressed in the ways that are appropriate to the problem at hand. Therefore, they have same unit and dimension.

5.4.2 Work done by torque, kinetic energy theorem of rotation

In Fig. 5-10, suppose that the rigid body acted by a force $\boldsymbol{F}$ rotates through an element angular displacement $d\theta$, the work done by the force is

$$dW = \boldsymbol{F}\cdot d\boldsymbol{r} = F\cos\varphi \cdot ds = F_t(rd\theta)$$

where ds is the length of arc passed by the acting point of the force. From Eq. (5-10a), we see $F_t r$ is the torque, so that

$$dW = Md\theta \qquad (5\text{-}17)$$

The work done during a finite angular displacement θ, is then

$$W = \int_0^\theta Md\theta \qquad (5\text{-}18)$$

Which holds for all rigid bodies rotating about a fixed axis, and is the rotational equivalent of equation

$$W = \int Fdx$$

in chapter 3 for one dimensional translational motion.

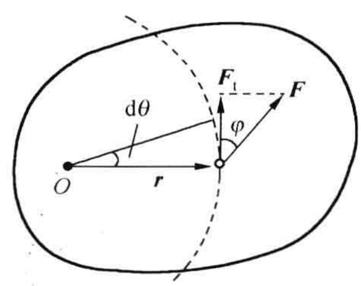


Fig. 5-10 For the work done by a torque

In the special case of constant torque, Eq. (5-18) becomes

$$W = \int_0^\theta M\mathrm{d}\theta = M\int_0^\theta \mathrm{d}\theta = M\theta \tag{5-19}$$

From Eq. (5-17), we can get the power for the rotational motion. Assume M is not change with time, then

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = M\frac{\mathrm{d}\theta}{\mathrm{d}t} = M\omega \tag{5-20}$$

Which is the angular analogy of $P = Fv$. Both formulas can be used to calculate the instantaneous power.

You may expect that there should be an angular equivalent of the kinetic energy theorem. To derive it, we start from the law of rotation, substituting β by $\mathrm{d}\omega/\mathrm{d}t$, thus

$$M = I\beta = I\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Multiplying both sides of above equation by $\mathrm{d}\theta$, we get the element work done by torque

$$\mathrm{d}W = M\mathrm{d}\theta = I\frac{\mathrm{d}\omega}{\mathrm{d}t}\mathrm{d}\theta = I\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\mathrm{d}\omega = I\omega \cdot \mathrm{d}\omega$$

When the angular speed changes from ω_i to ω_f, the work done by the torque is therefore

$$W = \int_{\omega_i}^{\omega_f} I\omega \cdot \mathrm{d}\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = E_{kf} - E_{ki} \tag{5-21}$$

Eq. (5-21) tells us that **the work done by the resultant torque acting on a rotating rigid body is equals to the change in rotational kinetic energy of that body.** This is the kinetic energy theorem in rotational motion. Like in translational motion, application of Eq. (5-21), usually can simplify the problem we solve.

Example 5-5 Calculate the work done by torque and use the kinetic energy theorem to solve the problem in Example 5-4.

Solution Let us return to Fig. 5-8, and use the result in Example 5-4, the torque produced by weight is

$$M = \frac{1}{2}l\cos\theta \cdot mg$$

When the rod turns through an angular displacement $\mathrm{d}\theta$, the work element is $\mathrm{d}W = M\mathrm{d}\theta = \frac{1}{2}mgl\cos\theta \cdot \mathrm{d}\theta$ during the finite angular displacement θ, the total work done by the torque is

$$W = \int \mathrm{d}W = \int_0^\theta \frac{1}{2}mgl\cos\theta \cdot \mathrm{d}\theta = \frac{1}{2}mgl\sin\theta$$

Substituting it to Eq. (5-21), we have

$$W = \frac{1}{2}mgl \cdot \sin\theta = \frac{1}{2}I\omega^2 - 0$$

where $I = \frac{1}{3}Ml^2$, so that

$$\omega = \sqrt{\frac{3g\sin\theta}{l}}, \quad \beta = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \frac{3g}{2l}\cos\theta$$

which is the same as we obtained in Example 5-4.

Discussion note that the part of $\frac{1}{2}l\sin\theta$ in equation $W = \frac{1}{2}mgl\sin\theta$, is the change in vertical position of the center of mass C, if the total mass of the rod were concentrates at this point, and if the initial position (horizontal) were chosen as the reference of zero gravitational potential, $\frac{1}{2}mgl\sin\theta$ would be the change in potential energy of the body during the motion (by taking integration of element potential energy we can get the same result). In this case, only gravitational force (weight) acts on the body. Use the conservation law of mechanical energy for the system of the body and the earth, we have

$$0 = \frac{1}{2}I\omega^2 - \frac{1}{2}mgl\sin\theta$$

from which the same result $\omega = \sqrt{3g\sin\theta/l}$ is obtained. From above example, we can make a conclusion that, if the rigid body is acted by a conservative force, we can introduce the corresponding potential energy; if the conservative force is the only force doing work, the total mechanical energy of the system will be also conserved.

5.5 Angular Momentum of a Rigid Body and Conservation of Angular Momentum

5.5.1 Angular momentum of a rigid body

We have already defined the angular momentum of a particle undergoing a circular motion, with respect to the origin as $\boldsymbol{L} = m\boldsymbol{r} \times \boldsymbol{v}$ in section 4-4. Now consider a rigid body rotating about a fixed axis z as shown in Fig. 5-11, every particle that make up the body moves around this axis in a circular path with the origin O_i on the axis. Again, we take the ith particle of mass Δm_i, path radius r_i, linear speed v_i in tangential direction, the magnitude of its angular momentum is $\Delta m_i v_i r_i$. By the right hand role we see that the direction of the angular momentum is along axis z, and it is true for all the mass elements in the body, therefore the angular momentum of this rigid body with respect to the axis is defined as

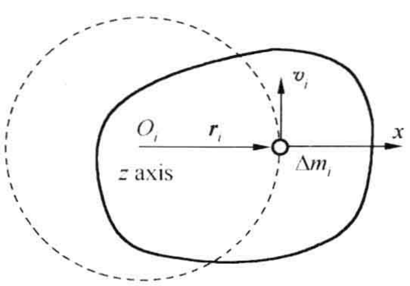


Fig. 5-11 Angular momentum of a rigid body

$$L = \sum_i (\Delta m_i v_i r_i) = (\sum_i \Delta m_i r_i^2)\omega = I\omega$$

that is

$$L = I\omega \tag{5-22}$$

If the body rotates anticlockwise then L points up along z axis, otherwise down. Note that the inertial of rotation I in the right side of Eq. (5-22) must be about the same axis as it in the left. Obviously, Eq. (5-22) is the angular equivalent of $\boldsymbol{p} = m\boldsymbol{v}$ in translational motion.

5.5.2 Conservation of angular momentum

In section 4-4, we have known about the conservation law of angular momentum for a particle. We are now ready to answer the question about how this law is for a rigid body rotation. Let us begin from Eq. (5-22), and take time derivative of it. Because that the rotational inertia about a fixed axis is a constant, therefore

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\beta$$

Compare this equation with the law of rotation $M = I\beta$, we have

$$M = \frac{dL}{dt} \tag{5-23}$$

Eq. (5-23) is another form of the law of rotation, which means the net torque acting on a rigid body is equal to the time rate of change of the body's angular momentum.

It can be proved that Eq. (5-23) hold for any body or a system rotating about a fixed axis. In the later case, M refers to the net external torque; L refers to the resultant angular momentum of the system about the same axis. If no external torque acts on the body or the system (so called isolated system), that is if

$$M = 0$$

then

$$\frac{dL}{dt} = 0 \quad \text{or} \quad L = I\omega = \text{a constant} \tag{5-24}$$

this is the law of conservation of angular momentum.

If no external torque acts on a system, the angular momentum of that system remains constant. The system involves almost all possible cases: a single body, a system of particles, a couple of rigid body, rigid bodies and particles. In the latter cases, the convenient form of Eq. (5-24) is

$$\sum_i L_i = \sum_i I_i\omega_i = \text{constant} \tag{5-25}$$

where i refers to the ith body or particle. Eq. (5-25) is the angular equivalent of Eq. (4-15), in translational motion.

Like the other two conservation laws that we have studied, this law holds beyond the limitation of Newtonian mechanics. It holds for particles whose speeds approach that of light and it remains true in the world of subatomic particles. No exception to it has ever been found.

Let us look at some interesting examples to see how this law governs those phenomena.

The spin of the earth: For a single body, if the situation meets Eq. (5-24), and the rotational inertia I is a constant, then the angular speed about the fixed axis will remains constant, that means, once the body starts to rotate, it will continue forever. The most important example of this case is the spin of the earth around its own axis, otherwise, the length of day would be all various!

On the other hand, if the rotational inertial varies while the body rotates, the angular speed ω will be then changed consequently.

The spinning demonstrations: Fig. 5-12(a) shows a man with two dumb bells in his outstretched hands, seated on a stool that has been set into rotation at an initial speed ω_i about the vertical axis. Then, as shown in Fig. 5-12(b), he pulls in his arms, thus reducing his rotational inertia from its initial value I_i, to a smaller value I_f. Consequent upon this, his angular speed increases remarkably from ω_i to ω_f. The same phenomenon is demonstrated by ballet dancers and by figure skaters. The man, the stool, and the dumbbell form an isolated system, on which no external torque acts. No matter how the man changes the distribution of the mass, therefore changes the rotational inertia about the axis, the angular momentum of that system must remain constant, that is

$$I_i\omega_i = I_f\omega_f$$

because $I_i > I_f$, so that $\omega_i < \omega_f$.

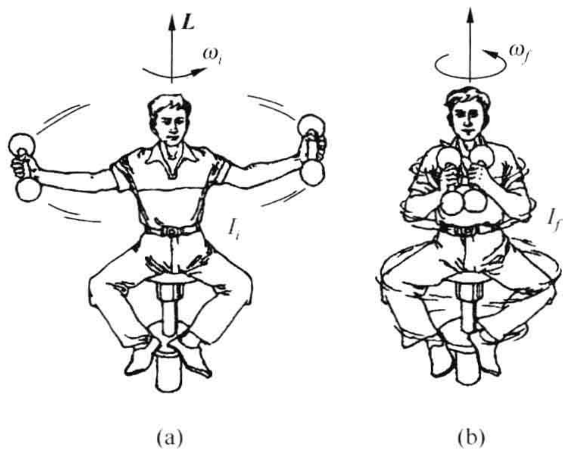


Fig. 5-12 A spinning demonstration for conservation of angular momentum

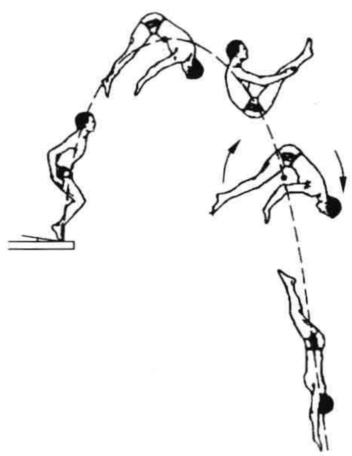


Fig. 5-13 The angular momentum of the diver is conserved

The diver's demonstration In Fig. 5-13, the motion of the diver is a combination of a translation (represented by his center of mass) and a rotation (about the axis through his center). The linear motion follows a parabolic path. The diver is an isolated system and his angular momentum can not be changed. By pulling his arms and legs into the tuck position, he can considerably reduce his rotational inertia and thus increase his angular speed. Pulling out of the tuck position at the end of the dive increases his rotational inertia and thus slows his rotation rate as he enters the water.

Re-orientating spacecraft In order to make the spacecraft orientation controlled by means of the internal action-reaction, engineers design a flywheel rigidly mounted on the spacecraft, as shown in Fig. 5-14(a). The system of spacecraft and flywheel has zero total angular momentum. To change the orientation of the spacecraft, start up the flywheel, as long as the flywheels rotates clockwise, to remain the total angular momentum equal to zero, the spacecraft will rotates counter-clockwise. Once the flywheel is braked to rest, the spacecraft will consequently stop with a changing in orientation by angle $\Delta\theta$, as shown in Fig. 5-14(b).

Conservation of angular momentum plays a vital role not only in sports, in daily experiences around us but also in atoms, nucleus, and primary-particles world and in all sorts of astrophysical phenomena. It is a universal law.

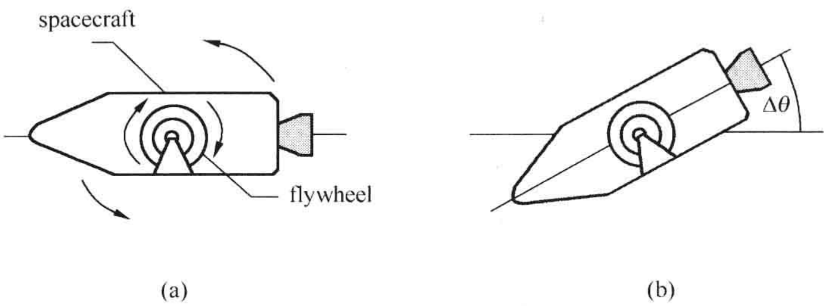


Fig. 5-14 Re-orientation of a spacecraft

Example 5-6 When a gymnast jumps off from a balance beam, initially she outstretches her body erect while rotating about her center of mass at a rate of 1 r/s, then she contracts her body to a fetal position.

(1) What is her new rate of rotation?

(2) What is the change in rotational kinetic energy? Assume that the rotational inertia is 10 kg • m^2 in the erect position and 4 kg • m^2 in the fetal position (regard this process as a pure rotation, neglect the change of the position of center of mass).

Solution Since there is no external torque, the gymnast's angular momentum is conserved.

(1) Let $I_1\omega_1$ and $I_2\omega_2$ represent the initial and the final angular momentum of the gymnast, from Eq. (5-24) we have

$$I_1\omega_1 = I_2\omega_2$$

and the rate of rotation $n = \omega/(2\pi)$, so that

$$I_1 n_1 = I_2 n_2$$

$$n_2 = n_1 I_1/I_2 = 1\times 10/4 = 2.5\ (\mathrm{r/s})$$

(2) The change of kinetic energy of rotation is

$$E_{k2} - E_{k1} = \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}I_1\omega_1^2 = \frac{1}{2}I_2\,(2\pi\cdot n_2)^2 - \frac{1}{2}I_1\,(2\pi\cdot n_1)^2$$

$$= \frac{1}{2}\times 4\times(2\pi\times 2.5)^2 - \frac{1}{2}\times 10\times(2\pi\times 1)^2 = 493.5 - 197.4 = 296.1(\mathrm{J})$$

The increase of kinetic energy comes from the work done by the gymnast when she contracts her body.

Example 5-7 A men of m_1=80 kg is standing on the rim of a stationary uniform circular platform of mass m_2=140 kg and radium r = 4 m, which can freely to rotate about its central vertical axis. The man throws a package of mass m=1 kg out in a direction tangential to the rim at a speed u = 20 m/s relative to the platform. What angular velocity of the man and platform is gained in consequence?

Solution For the system of the platform, the man and the package, which is at rest in the beginning, the total angular momentum L_0 is zero. At the moment when the package (a particle) is thrown with a speed u relative to the platform while the platform and the man start to rotate with an angular speed ω about the center axis of the platform relative to the ground. Since there is no external torque acting on the man-package-platform system, so that, the angular momentum of the system is conserved, that is $L = L_0 = 0$. To find the final angular momentum, we need to find the speed v of the package relative to the ground. Let v_r represent the speed at the rim of the platform in the opposite direction relative to the ground, we have

$$v = u - v_r$$

Therefore the angular momentum of the package is given by

$$mvr = m(u - v_r)r$$

The total angular momentum of the system is then

$$0 = m(u - v_r)r - \left(\frac{1}{2}m_2 r^2\right)\omega - m_1 v_r r$$

And the speed at the rim

$$v_r = \omega r$$

so that the angular speed of the platform is

$$\omega = \frac{mu}{\frac{1}{2}m_2 r + m_1 r + mr} = \frac{1 \times 20}{0.5 \times 140 \times 4 + 80 \times 4 + 1 \times 4} = 0.033(\text{rad/s})$$

This example reminds us that the speeds in the conservation law of angular momentum are with respect to the inertial reference frame, and the simultaneity of the law.

Example 5-8 Fig. 5-15 shows that a uniform vertical rod of mass M and length l pivots about an axis O and can freely rotate. Suppose a particle of mass m flies to the end of the rod at a speed v_0 in horizontal and sticks to the rod . What is the angular speed of the rod?

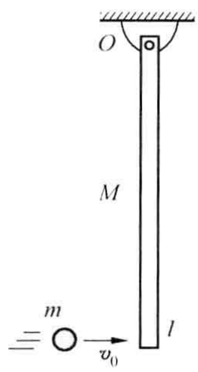


Fig. 5-15 For Example 5-8

Solution For the system of the rod and the particle, because that in the moment of collision, no external torque is acting on, the total angular momentum is conserved, and note that the collision is inelastic, the particle sticks to the end of the rod, becoming a part of it, that is

$$L_0 = L$$

$$mv_0 l = \frac{M}{3}l^2\omega + mvl \quad ①$$

and

$$v = \omega l \quad ②$$

so,

$$\omega = \frac{3m}{3m + M} \cdot \frac{v_0}{l}, \quad v = \frac{3m}{3m + M} \cdot v_0$$

obtained, the linear speed v of the particle after collision is the same as that of the end of the rod.

From the above examples we can make the conclusion that the angular momentum of the individual body may be changed during the interaction (separation in Example 5-7, collision in Example 5-8 etc.), but the total amount of the angular momentum will remains constant — the angular momentum may exchange among the bodies in the system. The interaction forces may have torque about the axis, since they belong to the internal torque, so, have no effect to the total angular momentum of the isolated system.

Making an analogy between the examples above and the corresponding problems in chapter 4, say, Example 5-7 and Example 4-2; Example 5-8 and Example 4-4, we can conclude once again that the way of understanding and solving the rotational problems is all equivalent to that in translational motion.

Questions

5-1 Consider a point on the rim of a wheel which rotates about its axis.

(1) If the wheel's angular velocity is a constant, does the point have a centripetal (radial) acceleration? a tangential acceleration?

(2) If the wheel rotates with a constant angular acceleration, what are the answers of the same questions? Do the magnitudes of these accelerations change with time?

5-2 (1) When we say that a point on the equator has an angular speed of 2π rad/day, what reference frame do we have in mind? Answer the same question about a rotating stool in your class room?

(2) What is the reference frame in which the law of $M=I\beta$ holds.

5-3 About what axis is the rotational inertia of a man the least? About what axis through his center of mass is his rotational inertia the greatest?

5-4 The net external force acting on a rigid body is $\boldsymbol{F}$ while the net external torque is $\boldsymbol{M}$, give some examples and explain in which case the following situations will happen

(1) $\boldsymbol{F}\neq 0$ while $\boldsymbol{M}=0$;

(2) $\boldsymbol{F}=0$ while $\boldsymbol{M}\neq 0$;

(3) $\boldsymbol{F}=0$ and $\boldsymbol{M}=0$.

5-5 If you give a hard boiled egg resting on a table a twist with your fingers, it will continue to spin. If you do the same with a raw egg, it will not. Can you explain why?

5-6 For a rigid body, only the work done by toque of the external force can make change of its kinetic energy while for a non-rigid system, is this true? If not, why (Recall question. 3-7(2))?

5-7 If the rear-brake of your bicycle does not work, what would happen when you rid it down a slope at a pretty high speed and suddenly put on the front brake? Why?

5-8 Suppose an astronaut floats freely in weightless conditions in a sky lab, how can he change the orientation of his body without external force exerted?

(1) If he swings one of his feet in a circle, how will his body rotate? Once he stops swinging his feet, weather or not his body continuously rotates?

(2) How about the answer if he swings his arms in a lateral circle? Explain why.

5-9 Some one thinks that when a helicopter flies off, its body would rotate in the opposite direction of its propellers rotating. However he is wrong, what mistake does he make?

5-10 When a child pumps up a swing, is there any external force (therefore an external torque) acting on the child + swing system? Explain why he will pump up the swing efficiently, if he does in such a way that he squats at the end of the swing's arc and stands up as the swing is passing through the bottom of its arc.

5-11 A tightrope walker in an acrobatic show uses a balancing pole to keep steady, how does this help?

5-12 Could an astronaut on spacecraft in outer space stop the craft by means of controlling two symmetrical jet nozzles mounted in tangential direction at the rear of the craft to inject waste gas with a certain speed in the same direction of the rotation of the craft (treat the craft + waste gas as whole system)? What is the difference between question 5-9 and question 5-12?

Problems

5-1 Fig. 5-16 shows that an elevator is supported by a cable running over a wheel of radius 0.40 m without slipping. Suppose that the elevator ascends with an upward acceleration of 0.50 m/s^2.

(1) Find the angular acceleration of the wheel.

(2) If the wheel makes 3.0 revolutions, how long does the accelerated motion last starting from rest?

(3) Find the instantaneous acceleration (tangential and centripetal) of a point on the rim of the wheel at the instant $t=2.0$ s?

5-2 In Fig. 5-17, wheel A of radius $r_A=15.0$ cm is coupled by a belt to wheel B of radius $r_B=30.0$ cm. Wheel A increases its angular speed from rest with a uniform rate of 12.0 rad/s^2, what is the rotational speed (r/min) of wheel B, after 30.0 seconds from the beginning? Assuming the belt does not slip.

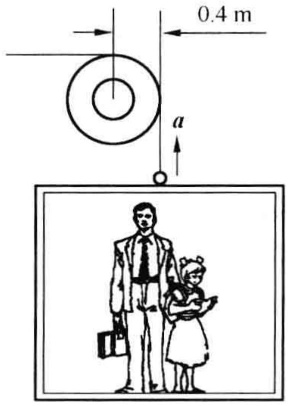


Fig. 5-16 For problem 5-1

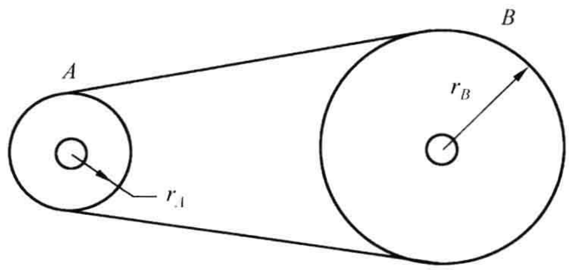


Fig. 5-17 For problem 5-2

5-3 A uniform disk rotates from rest about the vertical axis through its center at a constant angular acceleration. The rotational speed is 10 r/s at a given instant, after 100 revolutions, its speed becomes to 20 r/s. Find

(1) the angular acceleration;

(2) the time taken from the rest to reach the speed of 15 r/s;

(3) how many revolutions passed during the process in (2)?

5-4 Six particles, each with mass m, are fastened to one another by six light rods of length d, forming a hexagonal rigid body as Shown in Fig. 5-18. Calculate the rotational inertia of the combination body

(1) about the axis through any two adjacent particles;

(2) about the axis through one particle and perpendicular to the plane formed by the body.

5-5 Three uniform thin rods, each with length l, form an equilateral triangle rigid body ABC as shown in Fig. 5-19. Calculate the rotational inertia of it about its median. Suppose that the mass per unit length is λ.

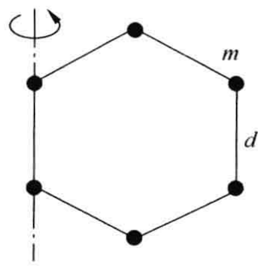


Fig. 5-18 For problem 5-4

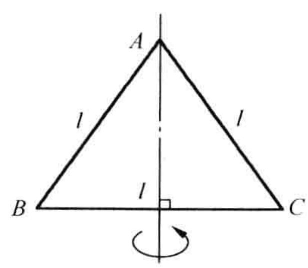


Fig. 5-19 For problem 5-5

5-6 Calculate the rotational inertia of a uniform semi-circle of radius R and mass m, about the axis shown in Fig. 5-20.

5-7 Fig. 5-21 shows an uniform disk whose mass M is 3.0 kg and whose radius R is 20.0 cm mounted on a fixed horizontal axle. A block whose mass m is 1.0 kg hangs from a light cord that is wrapped around the rim of the disk.

(1) Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.

(2) What is the time when the block falls a distance of 3.0 m?

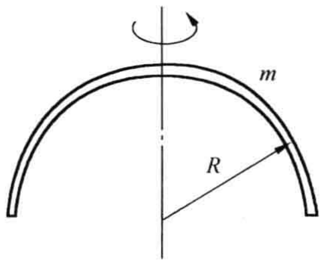


Fig. 5-20 For problem 5-6

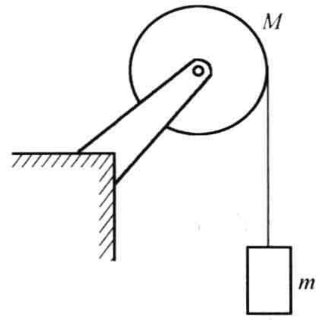


Fig. 5-21 For problem 5-7

5-8 A phonograph turntable driven by an electric motor accelerates at a constant rate from zero to 33.3 r/min in a time interval of 2.0 s. The turntable is a uniform disk of mass 1.50 kg and a radius 12.0 cm. What torque about the axis is required to drive this turntable? If the driving wheel makes contact with the turntable at its outer rim as shown in Fig. 5-22, what is the normal force that must be exerted? Suppose the frictional coefficient between the wheels $\mu=0.7$.

5-9 Two blocks each of mass m suspended from the ends of a rigid weightless rod of length (l_1+l_2) with $l_2=3\,l_1$. The rod is held at point O in the horizontal position shown in Fig. 5-23 and then released. Calculate the accelerations of the blocks as they just begin to move.

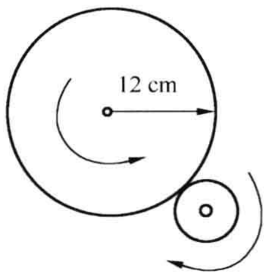


Fig. 5-22 For problem 5-8

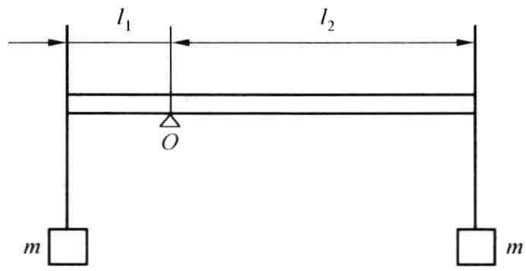


Fig. 5-23 For problem 5-9

5-10 A round uniform slab of radius $R=40$cm is placed on the horizontal plane, the frictional coefficient μ between the slab and the surface of the plane is 0.4. Suppose that the slab rotates about the vertical symmetry axis with an initial angular speed then turns through 5 revolutions before it stops. What is the magnitude of the initial angular speed ω_0 (Refer to Fig. 5-6(b) in Example 5-2, find the torque of the friction by integration) ?

5-11 In Fig. 5-24, an uniform thin stick of length l is initially standing vertically on the floor. If the stick fails, with what angular velocity will it hit the floor? Assume that the end in contact with the floor does not slip.

5-12 Fig. 5-25 shows a flywheel and its brake staff that consists of a brake bar and a shoe. The wheel has 50kg mass and 0.20 m radius, rotating with a rate of 1200 r/min. When a 100 N brake force is exerted on the end of the bar, what is the time taken to stop the flywheel? Suppose the frictional coefficient between the brake shoe and the flywheel is $\mu=0.5$. If $\mu=0.8$, what is the answer?

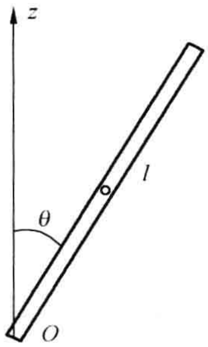


Fig. 5-24 For problem 5-11

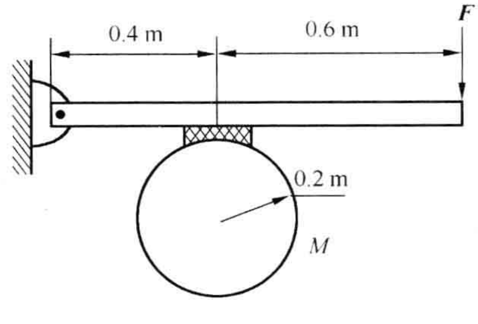


Fig. 5-25 For problem 5-12

5-13 A package of mass m_1 is dragging up along a frictionless surface inclined by an angle φ to the horizontal, a light cord attached to it is wrapped around the shaft of a wheel mounted on the top of the incline. A body of mass m_2 is attached to the rim of the wheel by another light cord as shown in Fig. 5-26.

(1) Suppose the rotational inertia of the wheel is I, the radius of the wheel is R and the radius of its shaft is r. What is the expression of angular acceleration of the wheel?

(2) If $R=0.30$ m, $r=0.10$ m, $\varphi=30°$, $m_1=5.0$ kg, $m_2=10$ kg; and m_2 passes 3.0 m during 2.0 s from the rest, calculate the rotational inertial of the wheel.

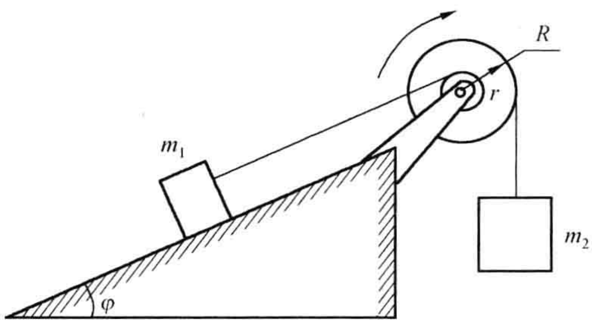


Fig. 5-26 For problem 5-13

5-14 Consider the motion in problem 5-8, calculate the work done by the torque during the acceleration, and the average power.

5-15 A flywheel of a steam engine with mass 200 kg and radius of 1m, if the steam valve is closed when the rotational rate is 150 r/m. Suppose the average frictional torque acting on the shaft of the flywheel is 5.0 N • m. Calculate

(1) the work done by the torque;

(2) the time required for stopping the flywheel.

5-16 Solve the problem 5-11 by using the rotational kinetic energy theorem.

5-17 A uniform rod of length 1.0 m pivots about a horizontal axis at the upper end, can freely rotate (refer to Fig. 5-15 in Example 5-8), which is originally at rest in the vertical position, then begins to rotate with an initial linear speed v_0 at the other end. In order to let the rod finish at least one revolution, what is the minimum initial linear speed given to the end (Suppose no friction on the axis)?

5-18 Water flowing in an open channel drives an undershot water wheel of radius 2.0 m as shown in Fig. 5-27. The water approaches the wheel with a speed 6.0 m/s and leaves with a speed 3.0 m/s. The amount of water passing by is 300 kg per second.

(1) What is the torque that the water exerts on the wheel?

(2) If the speed of the rim of wheel is 3.0 m/s, what is the power delivered to the wheel?

5-19 In the system shown in Fig. 5-28, a body of $m=5.0$ kg is connected by a rope over a pulley of rotational inertia $I=0.0100\ kg \cdot m^2$, $R=7.0$ cm to a spring of $k = 200$ N/m with one end fixed. Neglect all possible friction.

(1) When there is no deformation in the spring, and the rope is stiff, gently let the body fall down from rest till the maximum distance x_0, find x_0;

(2) then the body vibrates up and down, find the position where it has the maximum speed v, find v as well.

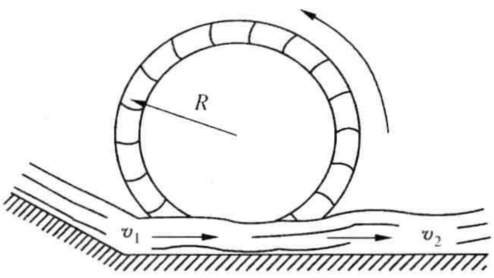


Fig. 5-27 For problem 5-18

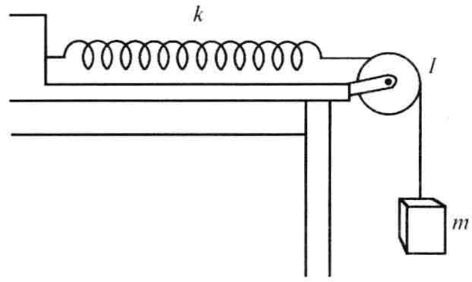


Fig. 5-28 For problem 5-19

5-20 A man sitting on a stool that can rotate freely about a vertical axis. The man, initially at rest, is holding a wheel on its central axis, while the wheel is rotating at an angular speed ω of 4.0 r/s. The axis of the wheel is vertical and the angular momentum L_i of the wheel points upward as shown in Fig. 5-29. If the man turns the wheel end for end, what will happen? Suppose the rotational inertia of the wheel about its central axis is $1.0\ kg \cdot m^2$, and the rotational inertia of the system (the man, the stool and the wheel) about the stool axis is $5.0\ kg \cdot m^2$.

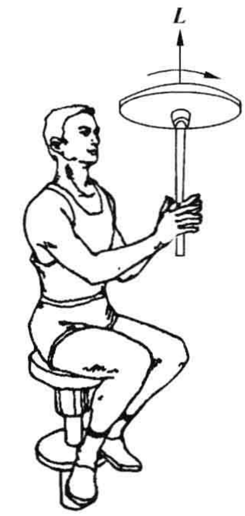


Fig. 5-29 For problem 5-20

5-21 A man of mass m_1 standing on the rim of a uniform circular platform of mass m_2 with radium R rotates together at angular speed ω in the beginning, then the man walks to the half way of radium, find the angular speed at this moment?

5-22 A uniform disk of mass M and radius R rotates about its vertical central axis with angular speed ω, a small piece of mass Δm brakes from the rim.

(1) What is the angular momentum of the disk after broken?

(2) How far is the small piece projected? If the distance from the disk to the ground is h.

5-23 A man of mass m stands at the rim of a uniform platform of mass M and radius R, mounted on a vertical frictionless shaft at its center. The whole system is initially at rest, then the man walks along the outer edge of the platform.

(1) Through what angle will the platform have rotated when the man walks one revolution and reaches his initial position on the platform.

(2) Through what angle will the platform have rotated when the man reaches his initial position relative to the ground? Compare this problem with problem 4-8.

5-24 Two flywheels A and B, are mounted on a shafts which can be connected or disconnected by a friction clutch as shown in Fig. 5-30. With the clutch disconnected, wheel A is brought up to an angular velocity of 600 r/min, wheel B is initially at rest, then the clutch is connected, accelerating B and decelerating A, until both wheels have the same angular velocity. The final angular velocity of the system is 240 r/min. It is found that 2000 J of heat are developed in the clutch when the connection is made. Find the rotational inertia of the two flywheels respectively.

5-25 A uniform vertical rod of mass M and length l pivots about an horizontal axis O at one end and can

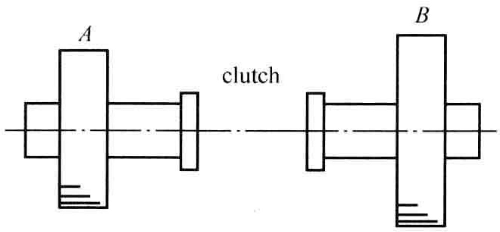


Fig. 5-30 For problem 5-24

freely rotate (Fig. 5-15), at rest in the beginning. Then a small ball of mass m flies to the end of the rod at a speed v_0 in horizontal and bounces back after collision. What is the returning speed v of the ball and the angular speed ω of the rod. Suppose the collision is elastic.

Part Two
Electromagnetism

Chapter 6

Static Electric Field in a Vacuum

There are so many electronic devices such as mobile phone, television, computers, which are widely used in our daily life, and the laws of electricity and magnetism play a central role in the operation of them. More fundamentally, the forces responsible for the formation of atoms and molecules are electric in origin. In the following chapters we will study the basic laws of electric and magnetic phenomena.

Evidence in Chinese documents suggests magnetism was observed as early as 2000 BC. The ancient Greeks observed electric and magnetic phenomena possibly as early as 700 BC. The Greeks knew about magnetic forces from observations that the naturally occurring stone (Fe_3O_4) is attracted to iron.

The systematic and quantitative study of electricity may be beginning with the establishment of Coulomb's law in 1785. In 1780 Galvani discovered electric current by accident. Not until the early part of nineteenth century did scientists establish relation between electricity and magnetism phenomena. In 1819, Hans Oersted discovered that a compass needle is deflected when placed near a circuit carrying an electric current. In 1831, Michael Faraday and, almost simultaneously, Joseph Henry showed that a current is set up in a circuit whenever the magnetic flux through the circuit is changing, and this reveals further the relation between electricity and magnetism. In 1873, James Clerk Maxwell used the theory fruits and other experimental facts as a basis for formulating the laws of electromagnetism as we know today. This theory is now called the classical theory of electromagnetism, which is the important fundament of electromagnetism.

In this chapter we will discuss mainly the property of the electric field, which are set up by the static charge relative to an observer in vacuum. For description of the property of the electric field, we introduce two important physical quantity—electric field and electric potential. We will study the Coulomb's law which is the fundamental law governing the electric force between any two point charged particles. Next we introduce how to use Coulomb's law to calculate the electric field for a given charge distribution. Then we will study the Gauss' law and its application for calculating the electric fields of highly symmetric charge distributions. Finally we discuss the relationship between electric field and electric potential.

6.1 Electric Charge and Coulomb's Law

6.1.1 Properties of electric charges

The electric phenomena were known to the ancient Greeks as long ago as 600 BC. It was known that amber, rubbed by wool, acquired the property of attracting light objects, and we say that the amber is electrified, or carries an electric charge, or is electrically charged. These terms are derived from the Greek word electron. Nowadays, the hard rubber and fur or a glass rod and silk are commonly used in demonstration.

In a series of simple experiments, it was found that there are two kinds of electric charges, which given the name positive and negative by Benjamin Franklin (1706—1790). And another important aspect of electricity that arises from experimental observation is that electric charge is always conserved in an isolated system. That is when one object is rubbed against another, charge is not created in the process. The electrified state is due to a transfer of charge from one object to the other, for example, when a glass rod is rubbed with silk, electrons are transferred from the glass to the silk, as a result, the glass has positive charge and the silk has negative charge. The further experiments lead to the fundamental results that

(1) like charges repel;

(2) unlike charges attract.

In 1909, Robert Millikan (1868—1953) discovered and confirmed that electric charge always occurs as integral multiples of a fundamental amount of charge of an electron e. In modern terms, the electric charge q is said to be quantized. That is, electric charge can write as $q=\pm ne$, n is some integer and $e = 1.60219 \times 10^{-19}\,\mathrm{C} \approx 1.6 \times 10^{-19}\,\mathrm{C}$.

6.1.2 Coulomb' law

When the dimension of a charged body is much smaller than the distances involved in a problem, such charged body is called "point charge".

In 1784, Charles Augustin de Coulomb (1736—1806) first quantitatively investigated the magnitudes of the electric forces between charged objects using the torsion balance which was invented by himself. The operating principle of the torsion balance is the same as that the type employed 13 years later by Cavendish in measuring gravitational force. Coulomb found that the magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them (Fig. 6-1).

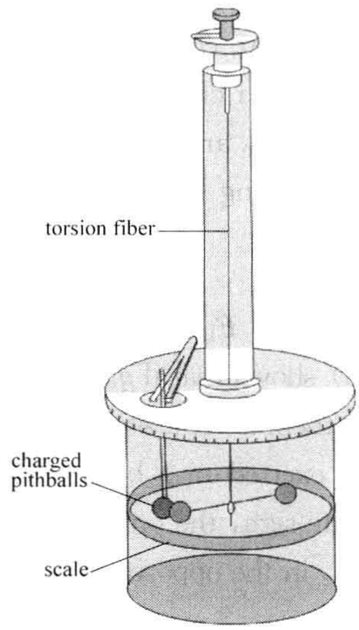


Fig. 6-1 Coulomb torsion balance

In mathematical terms, the magnitude of the force $\boldsymbol{F}$ that each of two point charges q_1 and q_2 a distance $\boldsymbol{r}$ apart exerts on the other can be expressed as

$$F = k\,\frac{|q_1 q_2|}{r^2} \tag{6-1}$$

where k is a constant. Eq. (6-1), called Coulomb's law, applies only to point charges and to spherical distributions of charges. The absolute value bars are used in Eq. (6-1) because the charges q_1 and q_2 can be either positive or negative, while the magnitude of force F is always positive. The directions of the forces the two charges exert on each other are always along the line joining them. When the charges q_1 and q_2 have the same sign, either positive or negative, the forces are repulsive (Fig. 6-2(a)); when the charges have opposite signs, the forces are attractive (Fig. 6-2(b)). The two forces obey Newton's third law; they are always equal in magnitude and opposite in direction, even when the charges are not equal.

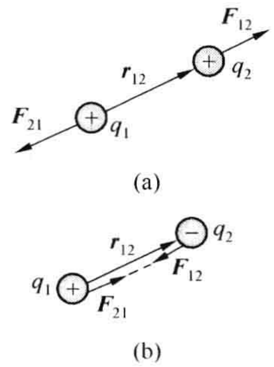


Fig. 6-2 The electric forces between two points charges

The unit of charge, coulomb (C), is derived from the fourth fundamental unit, ampere (A) of electric current in SI. One coulomb is the amount of charge that is transferred through the cross section of a wire in one second when there is a current of one ampere in the wire. From experiment, the constant k in SI has the value

$$k = 8.98755 \times 10^9 \ \mathrm{N \cdot m^2/C^2}$$

This value is taken to be $k = 9.00 \times 10^9 \ \mathrm{N \cdot m^2/C^2}$ in computation. In SI units the constant k in Eq. (6-1) is usually written as

$$k = \frac{1}{4\pi\varepsilon_0} \tag{6-2}$$

where ε_0, called the permittivity of vacuum, is

$$\varepsilon_0 = 8.85 \times 10^{-12} \mathrm{C^2/(N \cdot m^2)}$$

This appears to complicate matters, but it actually simplifies many formulas that we will encounter in later chapters. The dimension of electric current is I in SI, so that the dimensions of charge q and ε_0 are IT and $\mathrm{I^2L^{-3}M^{-1}T^4}$ respectively.

Considering both direction and magnitude of the force $\boldsymbol{F}$, Coulomb's law is expressed as

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^3}\boldsymbol{r}_{12} \tag{6-3}$$

in vector form. $\boldsymbol{F}_{12}$ is the force exerted on q_2 by q_1; $\boldsymbol{r}_{12}/r_{12}$ is a unit vector directs from q_1 toward q_2. Eq. (6-3) shows that if q_1 and q_2 have the same sign, the product q_1q_2 is positive and the electric force between them is repulsive interaction as shown in Fig. 6-2(a), and $\boldsymbol{F}_{12}$ in the same direction as $\boldsymbol{r}_{12}$, and $\boldsymbol{F}_{21}$, exerted on q_1 by q_2, points from q_2 to q_1 and in the opposite direction of $\boldsymbol{F}_{12}$. If q_1 and q_2 are of opposite sign, the product q_1q_2 is negative and the electric forces are attractive interaction between them, $\boldsymbol{F}_{12}$ in the opposite direction of $\boldsymbol{r}_{12}$ as shown in Fig. 6-2(b).

6.2 The Electric Field

6.2.1 Definition of electric field

The repulsion or attraction forces between two charge bodies A and B exist when they are some distance apart, which is similar to the gravitational force, that is action-at-a-distance. That means each charged object produces or sets up an electric field around. When another charged object B is placed at some point P around object A, the force on object B is exerted by the electric field of A. We can equally well consider that object B sets up an electric field, and that the force on object A is exerted by the electric field of B(Fig. 6-3).

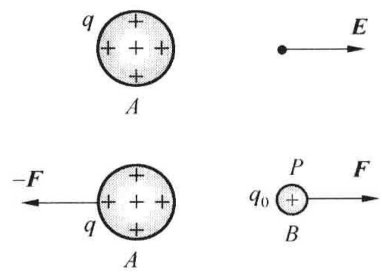


Fig. 6-3 The force on a test charge q_0 is exerted by the electric field **E** of a charged object

In order to investigate the property of the electric field produced by a given charged object, a positive change q_0, called a test charge, is placed in the field as shown in Fig. 6-3. The experiments show that the force $\boldsymbol{F}$ that acts on the test charge is different at different positions and at a given point is proportional to its charge q_0. If the charge is q_0, $2q_0$, $3q_0$, …, the force becomes $\boldsymbol{F}_0$, $2\boldsymbol{F}_0$, $3\boldsymbol{F}_0$, …, which indicates that the ratio of the force $\boldsymbol{F}$ to the charge is a definite vector, and can be used to describe the property of the electric field of the charged object. The electric field $\boldsymbol{E}$ at a point is defined as the force $\boldsymbol{F}$ on a test charge q_0 divided by q_0, that is

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q_0} \tag{6-4}$$

A test charge is a special point charge, and its amount must be less enough not disturbing the field investigated. The physical meaning of Eq. (6-4) is that the electric field $\boldsymbol{E}$ is equal to the force acting on a unit positive charge at that point.

In most instances, the magnitude and direction of an electric field vary from point to point. If the magnitude and direction are constant throughout a certain region, the field is said to be uniform in this region.

6.2.2 Electric field of a point charge

To determine the electric field of a point charge q, a test charge q_0 is placed at point P, a distance r from q, as shown in Fig. 6-3. According to Coulomb's law, the force exerted by q on the test charge is

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^3}\boldsymbol{r}$$

here $\boldsymbol{r}$ is the position vector from charge q to point P. By the definition of electric field, the electric field at point P created by q is

$$\boldsymbol{F} = \frac{\boldsymbol{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r} \tag{6-5}$$

If the charge q is positive, the electric field at point P is directed from the charge q toward point P as shown in Fig. 6-4(a). If the charge q is negative, the electric field at that point is directed from point P toward charge q, as shown in Fig. 6-4(b).

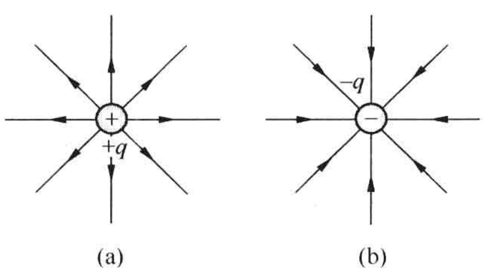


Fig. 6-4 Electric fields of a positive and a negative charge

6.2.3 Electric field due to a finite number of point charges

If there is a group of point charges, q_1, q_2, …, q_n. at distances r_1, r_2, …, r_n, from point P as in Fig. 6-5, each charge exerts a force on the test charge q_0 placed at point P and the resultant electric force is the vector sum of these forces

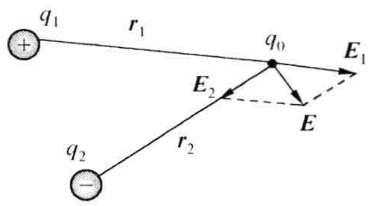


Fig. 6-5 Superposition principle of electric field.

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots + \boldsymbol{F}_n = \sum_{i=1}^{n} \boldsymbol{F}_i \tag{6-6}$$

Then from Eq. (6-5), the total electric field at point P is

$$\boldsymbol{E} = \frac{\sum_{i=1}^{n} \boldsymbol{F}_i}{q_0} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \cdots + \boldsymbol{E}_n = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 r_i^{\ 3}} \boldsymbol{r}_i \tag{6-7}$$

Here $\boldsymbol{E}_i$ is the electric field that would be produced by point charge i acting alone. Therefore the electric field at point P due to a group of point charges is the vector sum of electric fields produced by each individual point charge. Eq. (6-7) is called the **superposition principle of electric field**.

Example 6-1 The electric fields produced by an electric dipole, which consists of two charges of magnitude q but of opposite sign, separated by a distance l as shown in Fig. 6-6. The vector $\boldsymbol{l}$, whose magnitude is l and direction pointing from $-q$ to q, is called the dipole arm and the product $q\boldsymbol{l}$ is defined as the electric moment, labelled as $\boldsymbol{p}$

$$\boldsymbol{p} = q\boldsymbol{l}$$

Find the field $\boldsymbol{E}$ at point P_1, a distance x from the midpoint of the dipole on its central axis, and point P_2, a distance y along the perpendicular bisector of the line joining the charges.

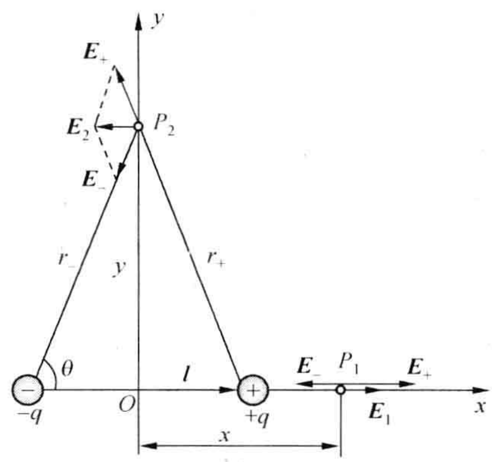


Fig. 6-6 The field of an electric dipole

Solution The dipole axis is taken to be x axis. Applying the superposition principle of electric field, we find the electric field $\boldsymbol{E}_1$ at point P_1 is

$$\boldsymbol{E}_1 = \boldsymbol{E}_+ + \boldsymbol{E}_-$$

where $\boldsymbol{E}_+$ is the electric field of positive charge q directed toward the right and $\boldsymbol{E}_-$ the electric field of negative charge $-q$ directed toward the left. Therefore, the magnitude of $\boldsymbol{E}_1$

$$E_1 = E_+ - E_- = \frac{q}{4\pi\varepsilon_0 \ (x - l/2)^2} - \frac{q}{4\pi\varepsilon_0 \ (x + l/2)^2}$$

We are usually interested in the electrical effect of a dipole only for a case of $x \gg l$ which means that point P_1 is distant. Using the binomial theorem and considering $x \gg l$, we have

$$E_1 \approx \frac{2ql}{4\pi\varepsilon_0 x^3}$$

or

$$E_1 \approx \frac{p}{2\pi\varepsilon_0 x^3}$$

in vector form.

Similarly, the electric field $\boldsymbol{E}_2$ at point P_2 is

$$\boldsymbol{E}_2 = \boldsymbol{E}_+ + \boldsymbol{E}_-$$

The magnitudes of $\boldsymbol{E}_+$ and $\boldsymbol{E}_-$ are

$$E_+ = E_- = \frac{q}{4\pi\varepsilon_0}\frac{1}{y^2 + l^2/4}$$

Their directions are shown in Fig. 6-6. Hence, the magnitude of $\boldsymbol{E}_2$, directed toward the left, is given by

$$E_2 = 2E_- \cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{ql}{(y^2 + l^2/4)^{3/2}}$$

If $y \gg l$, it can be proved that

$$\boldsymbol{E}_2 \approx -\frac{\boldsymbol{p}}{4\pi\varepsilon_0 y^3}$$

So the magnitudes of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ for distant points are proportional to p and vary as $1/x^3(1/y^3)$ and it turns out that $\boldsymbol{E}$ for a electric dipole vary as $1/r^3$ for all distant points; here r is the distance between the point and the dipole center.

6.2.4 Electric field of a continuous charge distribution

Very often the electric fields are set up by charges distributed along a line, over a surface or within a volume. When we want to find the electric field near the charged object, it can't be treated as point charge. How should we do?

To calculate the electric field created by continuous charge distributions, let's follow the procedure below. Firstly, divide the charge object into many infinitesimal elements, each of which contains a small charge Δq_1, Δq_2, $\cdots$ (called charge element), as shown in Fig. 6-7. Each small charge element can be treated as point charge.

Secondly, use Eq. (6-5) to calculate the electric field due to Δq_i at point P

$$\Delta\boldsymbol{E}_i = \frac{1}{4\pi\varepsilon_0}\frac{\Delta q_i}{r_i^2}\left(\frac{\boldsymbol{r}_i}{r_i}\right)$$

where r_i is the distance from the ith charge element to point P and $\boldsymbol{r}_i/r_i$ is a unit vector directed from the element Δq_i toward point P.

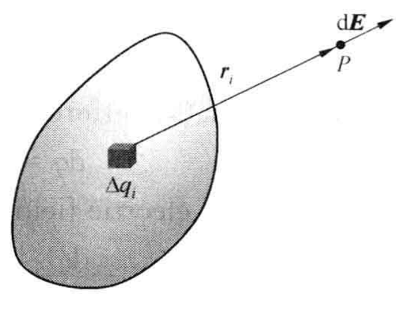


Fig. 6-7 Electric field due to continuous charge distributions

Thirdly, the total electric field at point P due to all elements is the sum

$$\boldsymbol{E} = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{\Delta q_i}{r_i{}^2}\left(\frac{\boldsymbol{r}_i}{r_i}\right)$$

where the index i refers to the ith charge element on the charged object.

Finally, because charge distributions is modeled as continuous, at the limitation that $\Delta q_i \to 0$, the above equation becomes a integration, which is over the entire charged object as follows

$$\boldsymbol{E} = \lim_{\Delta q_i \to 0}\sum_i \frac{1}{4\pi\varepsilon_0}\frac{\Delta q_i}{r_i{}^2}\left(\frac{\boldsymbol{r}_i}{r_i}\right) = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathrm{d}q}{r^2}\frac{\boldsymbol{r}}{r} \tag{6-8}$$

Let's illustrate how to calculate this integration with several examples in which the charge is distributed on a straight line or a ring.

Example 6-2 As shown in Fig. 6-8, a charge Q is distributed uniformly along a rod of length L. Find out the electric field at the some point P along the line, a distance a to the left end of the line.

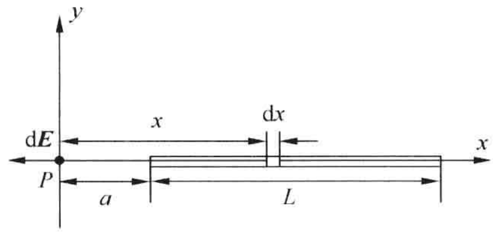


Fig. 6-8 For Example 6-2

Solution Since the charge is distributed uniformly along the rod, the charge per unit length or the linear charge density is $\lambda = Q/L$. We divide the rod into differential elements, and charge dq on the element dx is $dq = \lambda dx$. The magnitude of the electric field at P due to the element charge dq is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{x^2}$$

Because every element charge dq produces an electric field in the x direction, the sum of their contributions can be handled without the need to add vectors.

The total field at point P is obtained by integrating from $x = a$ to $x = a+L$, that is,

$$E = \int_a^{L+a} \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\left(\frac{1}{a} - \frac{1}{L+a}\right) = \frac{Q}{4\pi\varepsilon_0 a(L+a)}$$

If $Q>0$, $\boldsymbol{E}$ is in the negative direction of x axis.

Example 6-3 Positive charge Q is uniformly distributed around a semicircle of radius R, as shown in Fig. 6-9. Find the electric Field at the center of the semicircle.

Solution The coordinate axes are shown in Fig. 6-9. The linear charge density is

$$\lambda = \frac{Q}{\pi R}$$

and the charge of the differential element dl is

$$dq = \lambda dl = \lambda R d\theta$$

The magnitude of electric field dE due to dq is

$$dE = \frac{dq}{4\pi\varepsilon_0 R^2} = \frac{\lambda R d\theta}{4\pi\varepsilon_0 R^2} = \frac{Q d\theta}{4\pi^2\varepsilon_0 R^2}$$

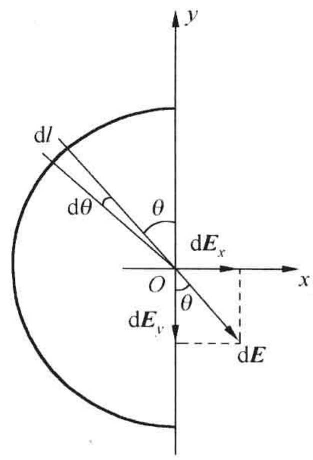


Fig. 6-9 For Example 6-3

$d\boldsymbol{E}$ can be resolved into components $d\boldsymbol{E}_x$ and $d\boldsymbol{E}_y$, as shown in Fig. 6-9. Considering the symmetry of the semicircle, the resultant electric field $\boldsymbol{E}$ points in the x axis direction, and

$$dE_x = \frac{Q d\theta}{4\pi^2\varepsilon_0 R^2}\sin\theta$$

Therefore, the electric field at point O is given by integrating from $\theta=0$ to $\theta=\pi$, that is

$$E = E_x = \int_0^{\pi} dE_x = \frac{Q}{4\pi^2\varepsilon_0 R^2}\int_0^{\pi}\sin\theta d\theta = -\frac{Q}{4\pi^2\varepsilon_0 R^2}\cos\theta\Big|_0^{\pi} = \frac{Q}{2\pi^2\varepsilon_0 R^2}$$

Example 6-4 As shown in Fig. 6-10, a thin ring of radius R has total charge Q uniformly distributed around the ring. Determine the electric field at point P, a distance x from the plane of the ring along its central axis.

Solution We take a charge element with length dl, which is at the top of the ring. Its electric field contribution $d\boldsymbol{E}$ to point P can be resolved into two components, dE_x parallel to the x axis and $dE_\perp$ perpendicular to the x axis. Because of symmetry distribution of the charge

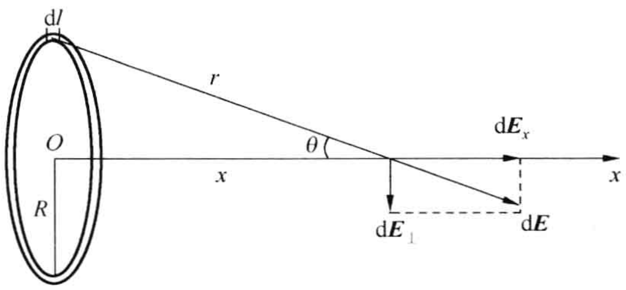


Fig. 6-10 For Example 6-4

on the ring, the perpendicular components of the fields of all charge elements cancel each other. That is true for all pairs of element charge around the ring, for example, the two element charges located at the top and at lower part of the ring.

The magnitude of electric field due to charge element dq is

$$dE = \frac{dq}{4\pi\varepsilon_0 r^2}$$

The x component $dE_x = dE\cos\theta$. Noting variables r and θ are constant for all charge elements, the electric field at point P is given by integrating dE_x, that is

$$E = E_x = \int dE_x = \int \frac{\cos\theta dq}{4\pi\varepsilon_0 r^2} = \frac{\cos\theta}{4\pi\varepsilon_0 r^2}\int dq = \frac{\cos\theta}{4\pi\varepsilon_0 r^2}Q$$

Considering the geometrical relations $r^2 = R^2 + x^2$ and $\cos\theta = x/r$, we have

$$E = \frac{x}{4\pi\varepsilon_0 (R^2 + x^2)^{\frac{3}{2}}}Q \qquad (6\text{-}9)$$

Discussion (1) If $x=0$, then $E=0$, because the contributions from two charge elements on opposite sides of the ring cancel out each other.

(2) If $x \gg R$, then $E \to Q/(4\pi\varepsilon_0 x^2)$, that is the ring can be regarded as a point charge.

6.3 Electric Field Lines, Electric Flux, and Gauss' Law

6.3.1 Electric field lines

The electric field exists in the space around the electric charge, but it is difficult for a person to imagine the field distribution at that region. It was Michael Faraday who first introduced electric field lines to picture the electric field patterns. The electric field lines with arrowheads are drawn such that the tangent of the lines at each point indicates the direction of the electric field $\boldsymbol{E}$ and the number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of the electric field in that region.

Fig. 6-11 shows some typical electric field lines. At any point near a positive charge, the electric field points radially away from the charge. Therefore the electric field lines diverge from a point occupied by a positive charge as shown in Fig. 6-11(a). Similarly, the electric field near a negative point charge points inward toward the charge, so the electric field lines point toward a negative charge. As we move away from the point charge, the electric field becomes weaker and the lines become farther apart so that the density of lines decreases.

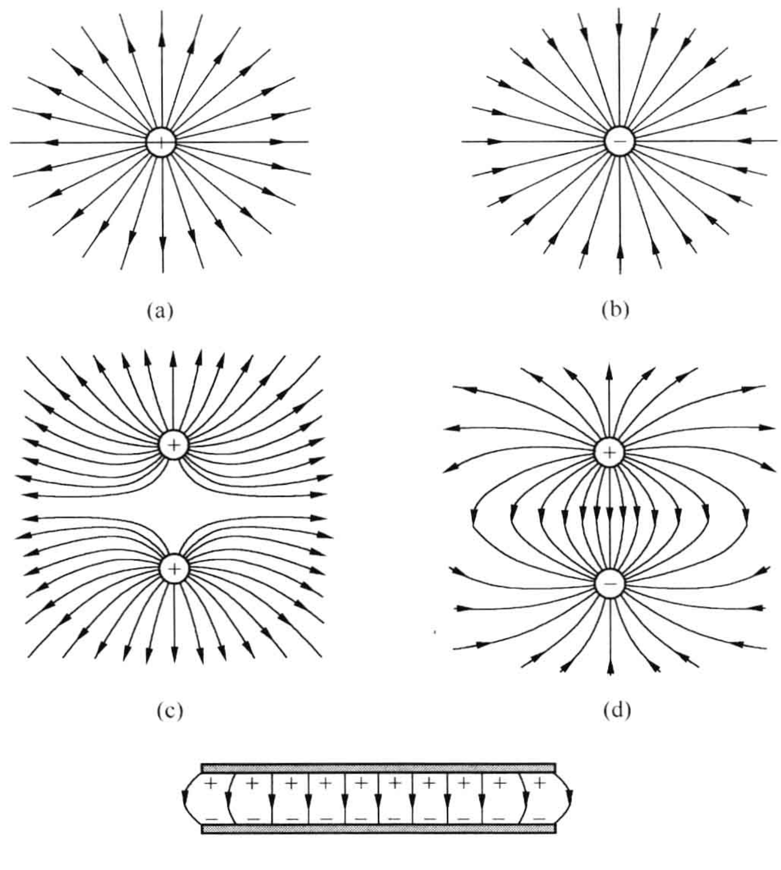


Fig. 6-11 Electric field lines for some charge distribution: (a) a positive point charge; (b) a negative point charge; (c) two identical positive point charges; (d) two equal and oppositive point charges; (e) two equal and oppositive parallel-plate charges

Form Fig. 6-11, we can conclude that the electric field lines have follow properties:

(1) The lines begin on a positive charge and terminate on a negative. In the case for a single charge, the lines will begin or end infinitely far away, as show in Fig. 6-11(a) and (b).

(2) The lines never begin or terminate at space where there is no any charge, that means the lines always has a origin(positive charge)and a end(negative charge).

(3) No two electric field lines can cross each other.

Consider two concentric spherical surfaces of radius r_1 and r_2, and there is only one point charge at the center of the spherical surfaces; it is not difficult to see that the electric field lines through the two surfaces are equal according to the properties of electric field lines that they are continuous at the space where there is no charge. The electric field lines through a surface or a closed surface is called electric flux. Next we will introduce the relationship between the electric flux through a closed surface and the charge enclosed by the surface. This relationship, known as Gauss' law, is of fundamental importance in the theory of electric fields.

6.3.2 Electric flux

The electric flux, labeled Φ_e, is the number of electric field lines through a surface. Consider an electric field that is uniform in both magnitude and direction as shown in Fig. 6-12(a).

A rectangular surface of area S, is oriented perpendicular to the field. Recalling that the electric field lines density is equal to the magnitude of electric field $\boldsymbol{E}$ and the number of the electric field lines penetrating the surface divided by the area S, so the electric flux Φ_e is

$$\Phi_e = ES \tag{6-10}$$

From the SI units of $\boldsymbol{E}$ and S, Φ_e has units of $N \cdot m^2/C$.

If the surface of area S is at angle θ to the uniform electric, the flux through it must be less than that given by Eq. (6-10). As shown in Fig. 6-12(b), notice that the number of lines that cross this area S is equal to the number of lines that cross the area $S_\perp$, which is a projection of S onto a plane oriented perpendicular to the field. Hence, the electric flux through S equals to the flux through $S_\perp$, that is the flux through S is

$$\Phi_e = ES_\perp = ES\cos\theta \tag{6-11}$$

From this result, we see that the flux through a surface of fixed S has a maximum value ES when the surface is perpendicular to the field. If $\theta=90°$, $\Phi_e=0$; if $0°<\theta<90°$, $\Phi_e>0$; if $90°<\theta<180°$, $\Phi_e<0$. Therefore, we must be in mind that the electric flux Φ_e may be zero, positive and negative.

In more general situations, the electric field is not uniform, and the surface may be an arbitary surface, such as the surface shown in Fig. 6-12(c). How can one calculate the flux through the surface? Let us divide the surface into a large number of small elements, each of area ΔS_i, and the field over the ith area element ΔS_i is approximately constant, and every area element can be regarded as a plate plane, so that the electric flux through the ith area element ΔS_i is

$$\Delta\Phi_{e,i} \approx E\Delta S_i\cos\theta_i = \boldsymbol{E} \cdot \Delta\boldsymbol{S}_i$$

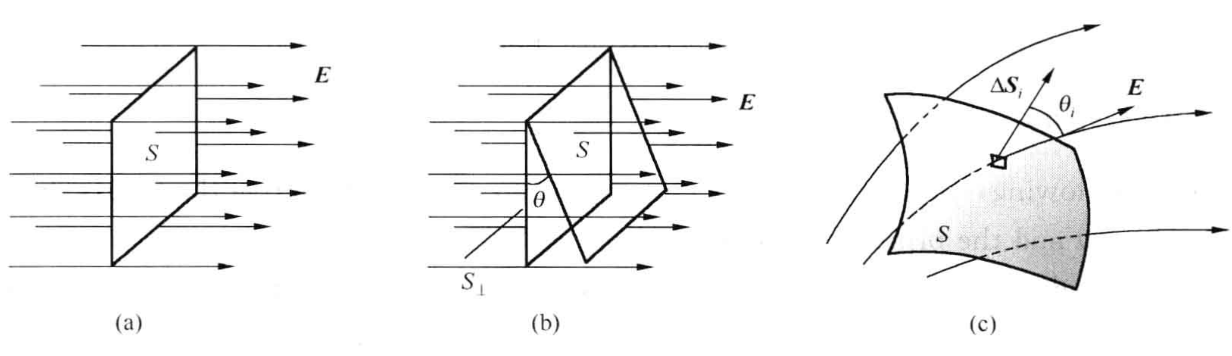


Fig. 6-12 Electric flux: (a) S perpendicular to the uniform field and (b) S not perpendicular to the uniform field; (c) arbitrary case

Here $\Delta\boldsymbol{S}_i$ is an area vector with magnitude of ΔS_i and perpendicular to the surface (that is along the normal of area element ΔS_i). Adding all $\Delta\Phi_{e,i}$, the approximative electric flux for the surface is

$$\Phi_e = \sum_i E\Delta S_i\cos\theta_i = \sum_i \boldsymbol{E} \cdot \Delta\boldsymbol{S}_i$$

If the area of each element approaches zero, the number of elements becomes infinity and the sum is replaced by an integral. Therefore, the definition of electric flux is

$$\Phi_e = \int_S E\cos\theta \mathrm{d}S = \int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} \tag{6-12}$$

If surface S is a closed one, for example the surface of a sphere, the direction of d$\boldsymbol{S}$ (the normal $\boldsymbol{n}$) is directed away from the interior of the surface, as shown in Fig. 6-13. Using the symbol $\oint_S$, the flux through a closed surface is an integral over a closed surface, that is

$$\Phi_e = \oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \oint_S E\cos\theta dS \tag{6-13}$$

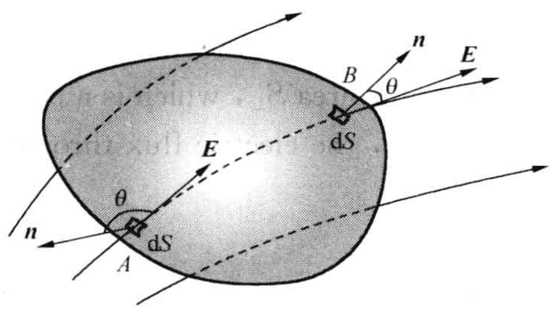


Fig. 6-13 Electric flux through a closed surface

Form Fig. 6-13, it is not difficult to see that if the field lines are crossing the surface from outside to inside, as the surface element A in Fig. 6-13, $180° > \theta > 90°$ and the flux is negative because $\cos\theta < 0$; if the field lines are crossing the surface from inside to outside, then $\theta < 90°$, the flux is positive because $\cos\theta > 0$, as the surface element B in Fig. 6-13; if the field lines are tangent to the surface element, $\theta = 90°$ and the flux is zero. Since the flux may be positive or negative, the total flux is net flux which is the number of lines leaving the surface minus the number of lines entering the surface. If there is no charge enclosed by a closed surface, from the continuity property of electric field lines, the field lines leaving and the lines entering the closed surface are equal so that the net flux is zero. If a positive charge is enclosed by a closed surface, the more lines leaving than entering, the flux is positive. If a negative charge is enclosed by a closed surface, the more lines entering than leaving, the flux is negative. This indicates that there is a relationship between the net flux through a closed surface and the charge enclosed by the surface; this relationship known as Gauss' law, is a statement of an important property of electrostatic fields.

6.3.3 Gauss' law

In the following, we take a positive charge q as an example to derive the Gauss' law from Coulomb's law and the principle of superposition of fields.

First consider a positive point charge q located at the center of a sphere of arbitrary radius r as shown in Fig. 6-14(a). We know that the magnitude of electric field everywhere on the surface of the sphere is $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$. Since the field lines are perpendicular to the sphere surface and $\theta = 0°$, we get the total flux through this spherical surface

$$\Phi_e = \oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \oint_S E\cos\theta dS = \oint_S \frac{q}{4\pi\varepsilon_0 r^2} dS = \frac{q}{4\pi\varepsilon_0 r^2}\oint_S dS = \frac{q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}$$

This means that the flux through any spherical surface centered on the point charge is same and equal to q/ε_0. Now two closed surfaces surrounding a charge q as shown in Fig. 6-14(b), S_1 is spherical surface, but S_2 is not. From the continuity of field lines, the number of lines through S_1 is equal to that through the nonspherical surface S_2; therefore, the net flux through any closed surface surrounding a point charge q is equal to q/ε_0 and is independent of the shape of that surface.

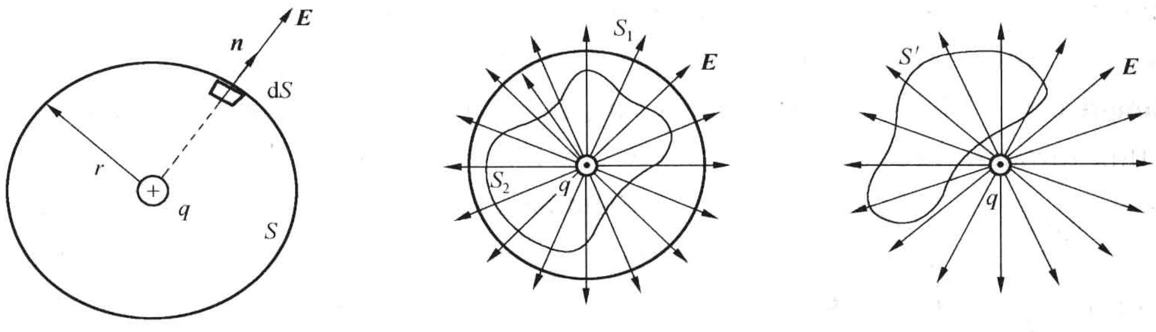


Fig. 6-14 Gauss' law

(a) q is located at the center of spherical Gaussian surface S; (b) q is inside an arbitrary Gaussian surface S_2; (c) q is outside an arbitrary Gaussian surface S'

Now consider a point charge q located outside a closed surface S' of arbitrary shape as shown in Fig. 6-14(c). As can be seen from this construction, the number of lines entering the closed surface at one side is just equal to the number of lines leaving the surface at the other side, the net flux through the closed surface S' is zero. Expressing mathematically, we have

$$\Phi_e = \oint_{S'} \boldsymbol{E} \cdot d\boldsymbol{S} = 0$$

Let's extend the arguments to a point charge system composed of charges q_1, q_2, q_3, $\cdots$, q_n, at any point. The principle of superposition gives that the total electric field is

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \cdots + \boldsymbol{E}_n$$

where $\boldsymbol{E}_i$ is the field produced by charge q_i individually. The total flux through any closed surface will be

$$\Phi_e = \oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \oint_S \boldsymbol{E}_1 \cdot d\boldsymbol{S} + \oint_S \boldsymbol{E}_2 \cdot d\boldsymbol{S} + \cdots + \oint_S \boldsymbol{E}_n \cdot d\boldsymbol{S}$$
$$= \Phi_{e1} + \Phi_{e2} + \cdots + \Phi_{en}$$

where Φ_{e1}, Φ_{e2}, $\cdots$, Φ_{en} are the fluxes due to the charges q_1, q_2, $\cdots$, q_n, individually. Form the conclusion about a single point charge, if q_i is inside the closed surface, then $\Phi_{ei} = q_i/\varepsilon_0$; if q_i is outside the closed surface, then $\Phi_{ei} = 0$. Thus the above equation becomes

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0} \sum q_{in} \qquad (6\text{-}14)$$

where $\sum q_{in}$ represents the algebraic sum of the charges inside the closed surface. Eq. (6-14) is Gauss' law which states that the electric flux through any closed surface is equal to the net charge inside the surface divided by ε_0, and the closed surface S is called Gaussian surface.

It is the fundamental law describing how charges create electric fields. It can be proved that Gauss' law is equivalent to Coulomb's law. However, it is more convenient to use in some situations. In practice, Gauss' law can be used to calculate the electric field of a system of charges or a continuous distribution of charge, especially with a high degree of symmetry such as cylindrical, planar and spherical symmetries in the following examples.

When using the Gauss' law, you should note that although the charge $\sum q_{in}$ is the net charge inside the Gaussian surface, $\boldsymbol{E}$ represents the total electric field at the surface, which includes contributions from charges both inside and outside the surface.

Example 6-5 There are two point charges $+q$ and $-q$, as Fig. 6-15 shows. Find the electric flux through three closed surfaces S_1, S_2 and S_3, respectively.

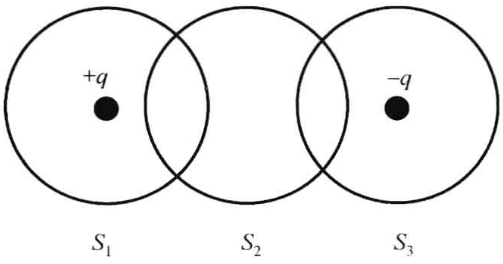


Fig. 6-15 For Example 6-5

Solution according to Gauss' law the Electric flux through closed surfaces S_1, S_2 and S_3 are q/ε_0, 0 and $-q/\varepsilon_0$, respectively. The electric flux through surface S_2 is zero, but it doesn't mean that the electric field $\boldsymbol{E}$ on the surface is zero. It only means that the number of electric field lines entering S_2 (at left side) is equal to the number of electric field lines leaving the surface (at right side) as shown in Fig. 6-11(d).

6.3.4 Application of Gauss' law

In general, the left side of Gauss' law, the integral, is difficult to finish and to solve for $\boldsymbol{E}$, but in a limited number of highly symmetric situations, if we choose the Gaussian surface surrounding the charge distribution carefully, the integral in Eq. (6-14) can be simplified and further, the electric field $\boldsymbol{E}$ is determined easily. Some examples are given below.

Example 6-6 The charge Q is uniformly distributed over the surface of a conducting sphere with radius R, as shown in Fig. 6-16. Find the Electric field at any point inside and outside the surface.

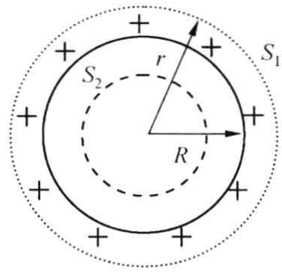


Fig. 6-16 For Example 6-6

Solution By spherically symmetric charge distribution, electric field at any point must be radial, and its magnitude must be the same at all points on a spherical surface of certain radius r. let's choose two spherical Gaussian surfaces S_1 and S_2, of radius $r_1 > R$ and $r_2 < R$, concentric with the sphere, respectively, as shown in Fig. 6-16. For Gaussian surface S_1, by symmetry, the magnitude of $\boldsymbol{E}$ is constant everywhere on the surface, and $\theta = 0°$, so we can remove E out from the integral. Notice that the total charge enclosed inside the Gaussian surface is Q, we have

$$\oint_{S_1} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = E \oint_{S_1} \mathrm{d}S = 4\pi r^2 E = \frac{Q}{\varepsilon_0}$$

yielding

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (r > R) \tag{6-15}$$

For Gaussian surface S_2, the charge enclosed inside is zero and

$$\oint_{S_2} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = E, \oint_{S_2} \mathrm{d}S = 4\pi r^2 E = \frac{0}{\varepsilon_0}$$

So we have

$$E = 0 \quad (r < R) \tag{6-16}$$

The electric field distribution is shown in Fig. 6-17.

Example 6-7 Find the electric field produced by a thin, flat, infinite sheet on which there is a uniform charge per unit area σ, as shown in Fig. 6-18.

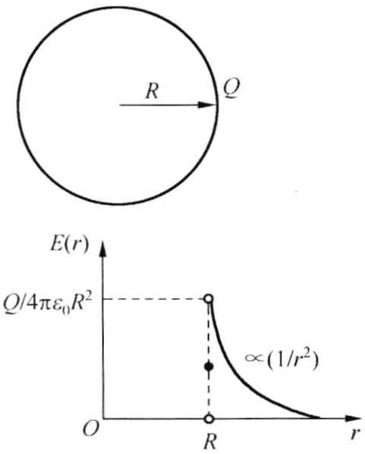


Fig. 6-17 Electric field distribution of the conducting sphere

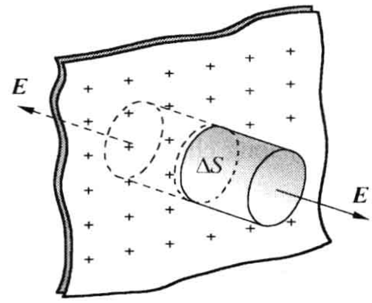


Fig. 6-18 For Example 6-7

Solution Notice that the plane of charge is infinitely large. Therefore, the electric field should be the same at all points equidistant from the plane both in magnitude and direction. And $\boldsymbol{E}$ must be perpendicular to the plane at all points and directed in opposite direction on two sides of the sheet. This implies that the electric field $\boldsymbol{E}$ has a planar symmetry. So we may take a small cylinder whose axis is perpendicular to the plane and whose ends each have an area ΔS and are equidistant from the plane. Because $\boldsymbol{E}$ is parallel to the curved surface, there is no contribution to the flux. The flux through each end of cylinder is $E\Delta S$; hence the total flux through the entire Gaussian surface is $2E\Delta S$. Notice that the enclosed charge is $\sigma\Delta S$, according to Gauss' law, we have

$$\oint_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} = 2E\Delta S = \frac{\sigma\Delta S}{\varepsilon_0}$$

This gives

$$E = \frac{\sigma}{2\varepsilon_0} \tag{6-17}$$

Because the distance from each flat end of cylinder to the sheet does not appear in the equation, that is, the field is uniform everywhere, depending only on the charge per unit area σ.

Example 6-8 There is an infinitely long line with a uniform positive linear charge density λ. Determine the electric field at any point around the line.

Solution Since the line of the charge is infinitely long, the electric field is perpendicular to the line and directed outward as shown in Fig. 6-19, that is the electric field lines are radial. Hence, the field has the same magnitude at all points equidistant from the line, that is the electric field has cylindrical symmetry. This suggests that we can apply Gauss' law to find the $\boldsymbol{E}$. Let's choose a cylindrical Gaussian surface of radius r and length l that is coaxial with the line, as shown in Fig. 6-19. For the curved part of this Gaussian surface, $\boldsymbol{E}$ is constant in magnitude and perpendicular to the surface at every points, satisfying $\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} = E\mathrm{d}S$; the flux through the two flat ends of the Gaussian surface is zero because $\boldsymbol{E}$ is parallel to them. Apply-

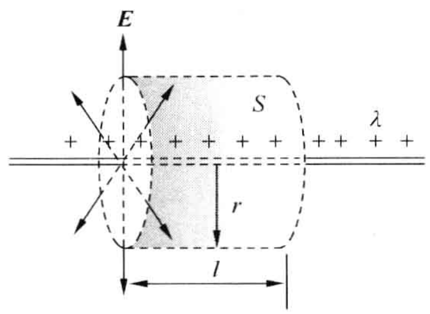


Fig. 6-19 Electric field of an infinitely long charged line

ing Gauss' law and noticing the total charge enclosed inside the Gaussian surface is λl, we have

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int E\mathrm{d}S = E\int \mathrm{d}S = 2\pi r l E = \frac{\lambda l}{\varepsilon_0}$$

From which, we solve for the magnitude of electric field

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{6-18}$$

The direction of $\boldsymbol{E}$ is radially outward from the charged line if the charge is positive and radially inward if it is negative.

From the above examples, we could conclude that the Gauss' law can be used to evaluate the electric field $\boldsymbol{E}$ for charge distribution that have spherical, cylindrical, or planar symmetry, and the Gaussian surface are spherical or cylinder closed surface. And for these symmetric arrangements of charge, it is very much easier to use Gauss's law to calculate the electric field around the charged objects.

6.4 Electric Potential

In this section, it will be proved that the Coulomb force is conservative and an electrical potential energy is defined corresponding to the Coulomb force. Furthermore, a more valuable quantity is called electric potential, defined as potential energy per unit charge, which is very much useful in the study of electricity. Electric potential is also of great practical value for dealing with electric circuits. For example, when we speak of a voltage applied between two points, we are actually referring to an electric difference between those points.

6.4.1 The work done by the electric force and potential energy

If a test charge q_0 is moved from some initial point a to some final point b in an electric field $\boldsymbol{E}$ due to a point charge q, as shown in Fig. 6-20, the electric force acting on the test charge is $q_0\boldsymbol{E}$. The element displacement is $\mathrm{d}\boldsymbol{r}$, and then the work done by the electric force is

$$W_{A\to B} = \int_{(A)}^{(B)} q_0 \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = q_0 \int_{(A)}^{(B)} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} \tag{6-19}$$

since $\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{\boldsymbol{r}}{r}\right)$, and $\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = r\mathrm{d}r$, we have

$$W_{A\to B} = q_0 \int_{r_A}^{r_B} \frac{q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = -\frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{6-20}$$

which means that the work done by the electric force due to a point charge depends only on the initial and final positions and not on the path. This indicates that the electric force of a point charge is a conservation force.

For a more general charge distribution (point charges system or a continuous distribution of charge), we may always divide the charge distribution into small charge elements and treat each charge element as a point charge, and according to the superposition principle of Eq. (6-7), the electric field due to many charges is the vector sum of the electric fields produced by the individual elements. Therefore, the total work done by the total electric field is

$$W_{AB}=\int_A^B q_0\boldsymbol{E}\cdot d\boldsymbol{r}=\int_A^B q_0(\boldsymbol{E}_1+\boldsymbol{E}_2+\boldsymbol{E}_3+\cdots)\cdot d\boldsymbol{r}$$
$$=\int_A^B q_0\boldsymbol{E}_1\cdot d\boldsymbol{r}+\int_A^B q_0\boldsymbol{E}_2\cdot d\boldsymbol{r}+\int_A^B q_0\boldsymbol{E}_3\cdot d\boldsymbol{r}+\cdots$$
$$=W_1+W_2+W_3+\cdots$$

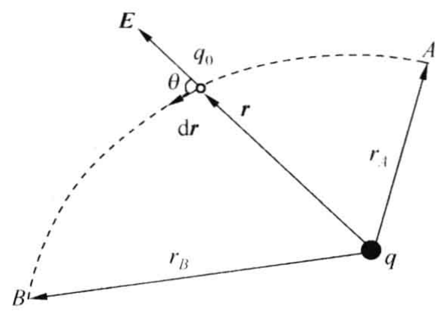


Fig. 6-20 The work done by electric force of a point charge

which is path-independent or conservative property since each term is independent of path. Therefore, it is possible to define an electric potential energy associated with electrostatic field. We can define a related potential energy by

$$W_{AB}=-\Delta E_p=-(E_{pB}-E_{pA})=q_0\int_A^B \boldsymbol{E}\cdot d\boldsymbol{r} \tag{6-21}$$

where E_{pA} and E_{pB} are called potential energy of q_0 at points A and B, respectively.

We often take the value of the potential energy to be zero at some convenient point in an electric field. If we take point B as the zero point of potential energy, that is $E_{pB}=0$, the potential energy of q_0 at point A is given by

$$E_{pA}=W_{AB}=\int_A^B q_0\boldsymbol{E}\cdot d\boldsymbol{r} \tag{6-22}$$

If we choose infinite distance as zero of potential energy, the potential energy of q_0 at point A is then

$$E_{pA}=W_{A\to\infty}=\int_A^{\infty} q_0\boldsymbol{E}\cdot d\boldsymbol{r} \tag{6-23}$$

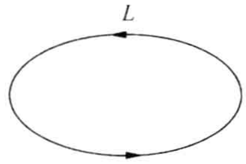


Fig. 6-21 The work done by electric field along a closed path is zero

The integral performed along a path is called either a path integral or a line integral, in which the element displacement is oriented tangent to a path through space. If the integral path is a closed one, as shown in Fig. 6-21, according to the conservative property of electric force, the integral is equal to zero, that is $W=\oint_L q_0\boldsymbol{E}\cdot d\boldsymbol{r}=0$. Since $q_0\neq 0$, we get immediately

$$\oint_L \boldsymbol{E}\cdot d\boldsymbol{r}=0 \tag{6-24}$$

This means that **the line integral of electrostatic field *E* along any closed path is zero**. This is called the **circuital theorem of electrostatic field**, which is the equivalent statement of conservative field for electrostatic field.

6.4.2 Electric potential and electric potential difference

Analyzing Eq. (6-23), the term on right hand is the potential energy proposed by the charge-field system, and depends both on the test charge q_0 and the electric field. If we divide the potential energy by the test charge q_0, we obtain a new physical quantity that depends only on the source charge distribution. This quantity is called the electric potential (or simply the potential) V_A:

$$V_A = \frac{E_{pA}}{q_0} = \int_A^{\infty} \boldsymbol{E} \cdot d\boldsymbol{r} \tag{6-25}$$

Because potential energy is a scalar quantity, electric potential also is a scalar quantity. The practical unit of potential, the joule per coulomb, is called the volt (V). The value of electric potential at some point A equals to the work done by the electrostatic force on a unit positive charge ($q_0 = +1$) when the charge is moved from point A to the zero point of potential energy (also potential). The potential at a point is one volt when it requires one joule of work to move a positive charge of one coulomb by the electrostatic force from that point to the point of zero potential. Since the electrostatic field is conservative field, we can calculate the electric potential by integral along a special and simple path as following examples illustrate these remarks.

The potential difference between two points is represented as $U_{AB} = V_A - V_B$, which is called electric potential difference between point A and point B. It is not difficulty to express U_{AB} as follows

$$U_{AB} = V_A - V_B = \int_A^{\infty} \boldsymbol{E} \cdot d\boldsymbol{r} - \int_B^{\infty} \boldsymbol{E} \cdot d\boldsymbol{r} = \int_A^{B} \boldsymbol{E} \cdot d\boldsymbol{r} \tag{6-26}$$

Comparing with Eq. (6-23), the electric potential difference two points A and B equals to the work done by electric field to remove a unit positive charge from point A to point B. In term of potential difference, Eq. (6-20) can be rewritten as

$$W_{AB} = q_0 \int_A^{B} \boldsymbol{E} \cdot d\boldsymbol{r} = q_0 U_{AB} \tag{6-27}$$

When a positive charge is placed in an electrostatic field, it moves in the direction of the field, from a point of high potential to a point of lower potential. That is its electrical potential energy decreases and its kinetic energy increases. When a negative charge is placed in an electrostatic field, it moves opposite to the direction of the field, from a point of low potential to a point of higher potential. In the process, it also undergoes a decrease in electrical potential energy and an increase in kinetic energy.

6.4.3 Electric potential of a point charge *q*

In the case of point charge, the zero point of electric potential is chosen to be at infinite distance from the charge. Applying $r_B = \infty$ and $q_0 = +1$ in Eq. (6-20), we obtain

$$V_A = \frac{q}{4\pi\varepsilon_0} \frac{1}{r_A}$$

For more general, the subscript A of r_A can be omitted and the electric potential at any distance r from the point charge q is given by

$$V = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \tag{6-28}$$

6.4.4 Electric potential of a system of point charges

When there is more than one point charge or a more general charge distribution, we take the infinity as the zero point of the electric potential. According to superposition principle of electric field, it is not difficult to confirm that the total electric potential is the algebraic sum of electric potential of individual point charge, that is

$$V=\sum \frac{q_i}{4\pi\varepsilon_0}\frac{1}{r_i} \tag{6-29}$$

where r_i is the radial distance from the source charge q_i to the field point concerned. Because potentials are scalar quantities in Eq. (6-29), it is much easier to evaluate the electric potential at some point due to several point charges than to evaluate the electric field, which is a vector quantity.

A continuous distribution of charge can be divided into small elements of charge, each of which may be treated as a point charge. The zero point of potential is also taken to be at infinity, and then the sum in Eq. (6-29) becomes an integral

$$V=\int \frac{dq}{4\pi\varepsilon_0 r} \tag{6-30}$$

where dq is a charge element and r is the distance from the charge element to the field point concerned. The integral in Eq. (6-30) must be taken over the entire charge distribution. Notice that the zero point of electric potential is at infinite distance for Eq. (6-28)~Eq. (6-30).

In the following, some examples illustrate how to evaluate the electric potential.

Example 6-9 Take the zero point of potential to be at infinity. As shown in Fig. 6-22, determine the potential at the center of a semicircle of radius R with a uniform charge per unit length λ.

Solution Divide semicircle into small charge elements. Noting the distance r from the charge element to the field point O concerned is constant of R, it is simple to apply Eq. (6-30).

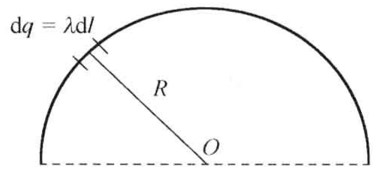


Fig. 6-22 For Example 6-9

The charge element has a charge of $dq=\lambda dl$, and we have

$$V_O=\int \frac{dq}{4\pi\varepsilon_0 r}=\int \frac{\lambda dl}{4\pi\varepsilon_0 R}=\frac{\lambda\pi R}{4\pi\varepsilon_0 R}=\frac{\lambda}{4\varepsilon_0}$$

Example 6-10 As shown in Fig. 6-23(a), a solid conducting sphere of radius R has a total charge $+Q$ distributed uniformly on its surface. Find the potential at any point inside and outside the sphere. Take the infinity as the zero point of the electric potential.

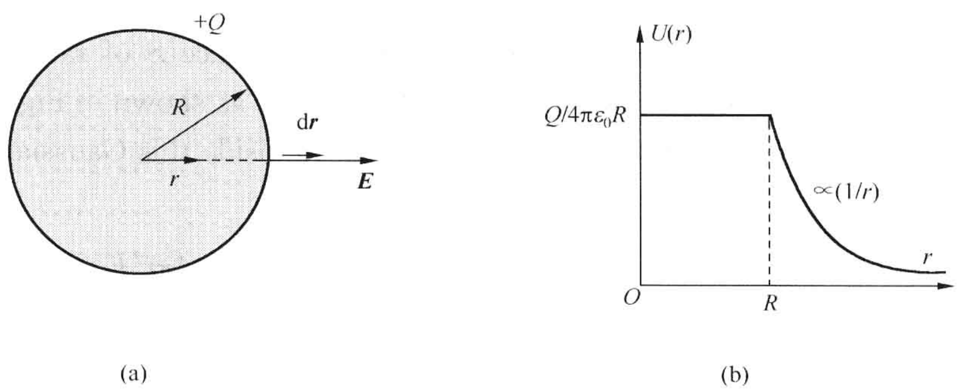


Fig. 6-23 For Example 6-10

Solution The electric field inside the conducting sphere is zero and is same as a point charge outside the sphere. It is easier to calculate the integral of Eq. (6-25) from the known electric field by integrating along one of the field lines than Eq. (6-30). From Example 6-5, the electric field distribution inside and outside the sphere is as follows

$$\begin{cases} E = 0 & (r < R) \\ E = \dfrac{Q}{4\pi\varepsilon_0 r^2} & (r > R) \end{cases}$$

The potential at any point inside the sphere is

$$V(r) = \int_r^\infty \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_r^R \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} + \int_R^\infty \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_R^\infty \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}$$

$$= \int_R^\infty \frac{Q\mathrm{d}r}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 R} \quad (r < R)$$

And the electric potential outside the sphere is

$$V(r) = \int_r^\infty \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_r^\infty \frac{Q\mathrm{d}r}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r} \quad (r > R)$$

Inside the sphere the electric field is zero and the electric potential is constant everywhere, and no work is done on a test charge by electric force moved from any point to any other in this region. Thus the electric potential distribution is the same at all points inside the conducting sphere, and equal to the value at the surface; the electric potential outside the sphere is the same as a point charge since the electric field distribution is the same as that of a point charge. The electric potential distributions are shown as in Fig. 6-23(b) .

Example 6-11 Two concentric spherical conducting shells are separated by vacuum. The inner and outer shell has charge $+Q$ and $-Q$ respectively, and the radius of the inner and outer shell are R_A and R_B, as shown in Fig. 6-24.

(1) Find the electric field everywhere inside the shells, between the shells and outside the shells;

(2) determined the potential difference between the two shells.

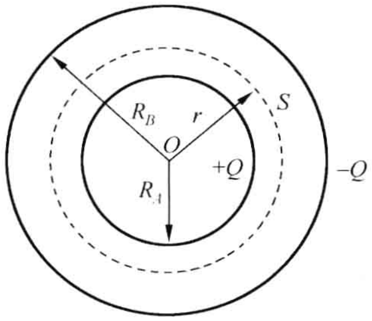


Fig. 6-24 For Example 6-11

Solution Since the charge on the conducting shell distributes itself uniformly on the surface and has the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on a coaxial sphere surface. So constructing a spherical Gaussian surface S of radius r ($R_A < r < R_B$), concentric with the sphere, as shown in Fig. 6-24, and noticing that the charge enclosed inside this Gaussian surface is $+Q$, by Gauss' law we have

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = E\oint_S \mathrm{d}S = 4\pi r^2 E = \frac{Q}{\varepsilon_0}$$

Solve for E

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (R_A < r < R_B)$$

The electric field inside the shells and outside the shells is zero.

The electric potential difference two shells can be calculated by integrating $\boldsymbol{E}$ along the radial direction

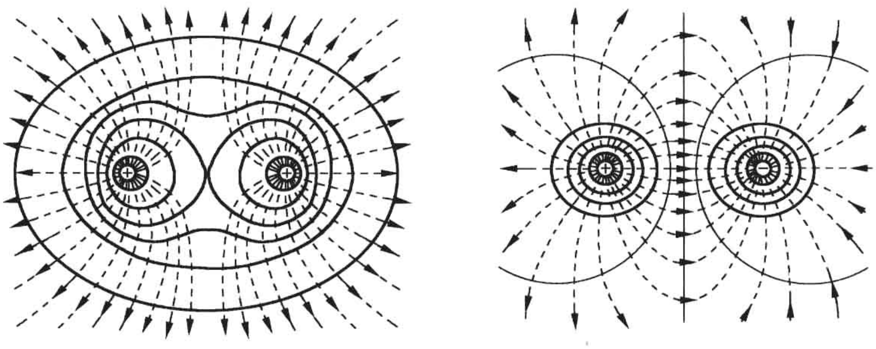
$$V_A - V_B = \int_{R_A}^{R_B} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_{R_A}^{R_B} E\mathrm{d}r = \int_{R_A}^{R_B} \frac{Q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = \frac{Q}{4\pi\varepsilon_0}\left(\frac{R_B - R_A}{R_A R_B}\right)$$

Can you apply the superposition principle of electric potential to obtain the same result?

6.5 Equipotential Surface and Potential Gradient

6.5.1 Equipotential surface

The potential distribution in an electric field may be represented graphically by equipotential surface. An equipotential surface is a surface such that the potential has the same value at all points on the surface. Two typical examples of equipotential surfaces are shown in Fig. 6-25.



(a) (b)

Fig. 6-25 Equipotential surfaces (solid lines) and electric fields (dashed lines): (a) two identical positive point charges and (b) two equal and oppositive point charges

It is easy to conclude that the equipotential surface has follow properties:

(1) The equipotential surface through any point must be at right angles to the electric field lines since no work is done to move a charge over such surface, that is the lines of field and the equipotential surface form a mutually perpendicular network. This implies that the electric field at every point on an equipotential surface is perpendicular to the surface.

(2) Two equipotential surfaces never cross each other; if it were so, there would have two values of potential at one point. In general, the field lines are curves and the equipotential surfaces are curved surfaces.

6.5.2 Potential gradient

There is an integral relation between the electric potential V and electric field $\boldsymbol{E}$, by which we can calculate electric potential by integrating with field $\boldsymbol{E}$. Conversely, there exists a differential relation between these two quantities so that we can calculate electric field $\boldsymbol{E}$ when the electric potential V is known.

As shown in Fig. 6-26, if points A and B are very close together, the potential difference $V_A - V_B$ becomes simply $-\mathrm{d}V$, and the line integral of $\boldsymbol{E}$ from A to B reduces to $\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$. That is

$$\mathrm{d}V = -\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -E\mathrm{d}l\cos\theta$$

$\boldsymbol{E}$ can be expressed as

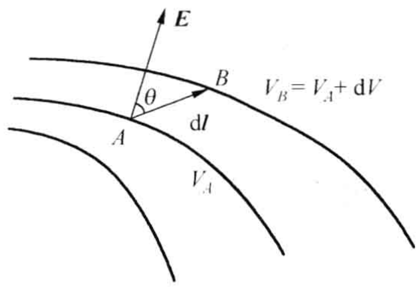


Fig. 6-26 Potential difference between two points A and B

$$\boldsymbol{E}=E_x\boldsymbol{i}+E_y\boldsymbol{j}+E_z\boldsymbol{k} \tag{6-31}$$

where E_x, E_y and E_z are components of electric field $\boldsymbol{E}$ in Cartesian coordinates. And element displacement $\mathrm{d}\boldsymbol{l}$ is expressed as

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}x\boldsymbol{i} + \mathrm{d}y\boldsymbol{j} + \mathrm{d}z\boldsymbol{k} \tag{6-32}$$

So we have

$$\mathrm{d}V = -(E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z) \tag{6-33}$$

Since the electric potential is function of (x, y, z) in general, the differential $\mathrm{d}V$ can be given mathematically by

$$\mathrm{d}V = \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z \tag{6-34}$$

From Eq. (6-33) and Eq. (6-34), we have

$$E_x = -\left(\frac{\partial V}{\partial x}\right), \quad E_x = -\left(\frac{\partial V}{\partial y}\right), \quad E_z = -\left(\frac{\partial V}{\partial z}\right)$$

Therefore, we obtain

$$\boldsymbol{E} = E_x\boldsymbol{i} + E_y\boldsymbol{j} + E_z\boldsymbol{k} = -\left[\left(\frac{\partial V}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial V}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial V}{\partial x}\right)\boldsymbol{k}\right] = -\nabla V \tag{6-35}$$

This is the differential relation between the electric field and the electric potential. Here $\nabla V = \left(\frac{\partial V}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial V}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial V}{\partial x}\right)\boldsymbol{k}$ is called the potential gradient, which is a vector and directs in the opposite direction with the electric field. Eq. (6-35) implies that $\boldsymbol{E}$ always points in the direction in which V decrease most rapidly.

Example 6-12 The electric potential V is

$$V = V(x, y, z) = x^2 + 3xy - zy \quad \text{(SI)}$$

Find the electric field at point (2,3,3).

Solution From Eq. (6-35), the electric field is given by

$$\boldsymbol{E} = -\nabla V = -(2x + 3y)\boldsymbol{i} - (3x - z)\boldsymbol{j} + y\boldsymbol{k}$$

Hence, the electric field at point (2,3,3) is

$$E(2,3,3) = -13\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k} \quad \text{(SI)}$$

6.6 The Electric Force Exerted on a Moving Charged Particle

If a charged particle enters into a region of electric field $\boldsymbol{E}$, an electric force acts on the particle and it will move in influence of electric field. Many applications are related with the movements of charged particles such as charged dirt particles in air, charged oil-droplets in the Millikan oil-drop experiment and electrons in the cathode-ray tube. In this section, we will pay our attention to the electric forces on charged particles and their motions.

6.6.1 A dipole in an electric field

As shown in Fig. 6-27, an electric dipole is in a uniform electric field $\boldsymbol{E}$ with an electric dipole moment $\boldsymbol{p}$, making an angle θ with the direction of the electric field. A pair of equal and opposite forces $q\boldsymbol{E}$ and $-q\boldsymbol{E}$ acts on the dipole so that the net electric force on it is zero. However, there is a net torque, given by

$$\boldsymbol{M} = \boldsymbol{p} \times \boldsymbol{E} \tag{6-36}$$

Its magnitude is

$$M = qEl\sin\theta = pE\sin\theta$$

This torque tends to orient the dipole in the direction of electric field.

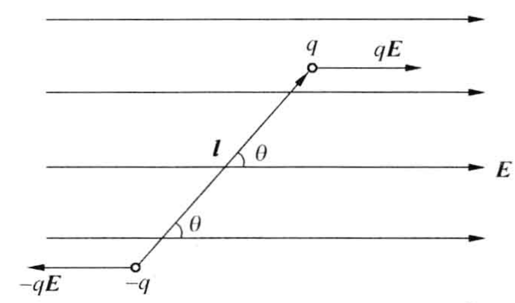


Fig. 6-27 A dipole in a uniform electric field

Because only differences in potential energy have physical meaning, we are free to choose the zero potential energy. We take the potential energy of a dipole in the uniform electric field to be zero when the angle θ is 90° so that its potential energy at any other orientation is

$$E_p = \left(-qE \cdot \frac{l}{2}\cos\theta\right) + \left(-qE \cdot \frac{l}{2}\cos\theta\right) = -pE\cos\theta$$

or

$$E_p = -pE\cos\theta = -\boldsymbol{p} \cdot \boldsymbol{E} \tag{6-37}$$

Eq. (6-37) means that an electric dipole in a uniform electric field has its smallest potential energy when $\boldsymbol{p}$ points in the direction of the field ($\theta = 0°$), which is its equilibrium position, and reaches its maximum when $\boldsymbol{p}$ point opposite to the direction of the field ($\theta = 180°$).

6.6.2 A moving charge particle in a uniform electric field

When a charged particle of q is in an electric field $\boldsymbol{E}$, there is a force $\boldsymbol{F} = q\boldsymbol{E}$ acting on this particle. By Newton's second law, we have

$$q\boldsymbol{E} = m\boldsymbol{a} \tag{6-38}$$

Here m is the mass of the particle. Given the distribution of the electric field, the orbit of the motion of the particle in the electric field can be calculated from Eq. (6-38). In the following, we deal specially with the situation on a charged particle moving in a uniform electric field.

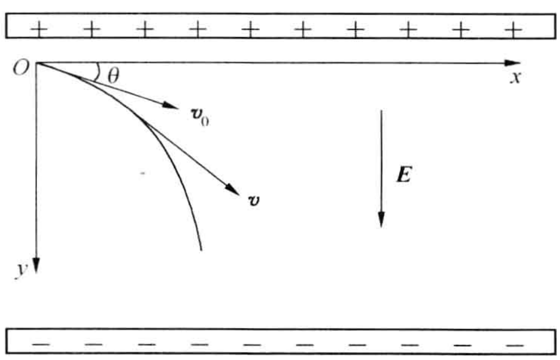


Fig. 6-28 A charged particle in a uniform electric field

Fig. 6-28 shows a positive charged particle in a uniform electric field set up by two parallel-plates carrying equal but opposite charges. Assume the initial velocity of the particle just entering the electric field has a magnitude of v_0 and an angle θ with respect to the x-axis. Because the direction of the electric field is along the y-axis, the acceleration of the particle is

$$\begin{cases} a_x = 0 \\ a_y = \dfrac{qE}{m} \end{cases}$$

Ignoring the influence of the gravity, it is easy to find the motion equation of the particle as

$$\begin{cases} x = v_0 t\cos\theta \\ y = v_0 t\sin\theta + \dfrac{qE}{2m}t^2 \end{cases} \tag{6-39}$$

Eliminating t in Eq. (6-40), we obtain the equation of its path

$$y = x\tan\theta + \frac{qE}{2mv_0^2\cos^2\theta}x^2 \tag{6-40}$$

which is the same as the projectile motion of an object in gravity field. When $\theta=90^\circ$ and $v_0=0$, the particle moves along the y-axis and its kinetic energy becomes greater with the increase of the velocity of the particle. A practical example is a cathode ray tube used for a television picture tube in which the accelerated electrons are deflected by the electric field to reach different points on the fluorescent screen.

Questions

6-1 A positively charged ball hangs from a long silk thread. We wish measure $\boldsymbol{E}$ at a point in the same horizontal plane as that of the hanging charge. To do so, we put a positive test charge q_0 at the point and measure $\boldsymbol{F}/q_0$. Will $\boldsymbol{F}/q_0$ be less than, equal to, or greater than $\boldsymbol{E}$ at the point in question?

6-2 For exploring electric field with a test charge, we have often assumed, for convenience, that the test charge was positive. Does this really make any difference in determining the field? Illustrate in a simple case of your own decision.

6-3 Electric field lines never cross, why?

6-4 A positive and a negative charge of the same magnitude lie on a long straight line. What is the direction of $\boldsymbol{E}$ for points on this line that lie (1) between the charges, (2) outside the charges in the direction of the positive charge, (3) outside the charges in the direction of the negative charge, and (4) off the line but in the median plane of the charges?

6-5 A clock face has negative point charges $-q$, $-2q$, $-3q$, …, $-12q$ fixed at the positions of the corresponding numerals. The clock hands do not disturb the field. At what time does the hour hand point in the same direction as the electric field at the center of the dial (hint: Consider diametrically opposite charges)?

6-6 A point charge q is placed at one corner of a cube of edge a. What is the flux through each of the cube faces (hint: Use Gauss' law and symmetry arguments)?

6-7 In Gauss' law $\oint_S \boldsymbol{E}\cdot d\boldsymbol{S} = \dfrac{\sum q_{in}}{\varepsilon_0}$, is $\boldsymbol{E}$ necessarily the electric field attributable to the charge q_{in}?

6-8 Suppose that a Gaussian surface encloses no net charge. Does Gauss' law require that $\boldsymbol{E}$ equal zero for all points on the surface? Is the converse of this statement true; that is, if $\boldsymbol{E}$ equals zero everywhere on the surface, does Gauss' law require that there be no net charge inside?

6-9 If you know the value of the net charge $\boldsymbol{Q}$ within some closed surface, is Gauss' law sufficient to enable you to calculate the electric field? What additional information might you need?

6-10 Do electrons tend to go to regions of high potential or of low potential? what about a positive charge?

6-11 Suppose that the earth has a net charge that is not zero. Why is it still possible to adopt the earth as a standard reference point of potential and to assign the potential $V=0$ to it?

6-12 Distinguish between potential difference and difference of potential energy. Give examples of statements in which each term is used properly.

6-13 (1) If V equals a constant throughout a given region of space, what can you say about $\boldsymbol{E}$ in that region?

(2) How can you ensure that the electric potential in a given region of space will have a constant value?

6-14 Devise an arrangement of three point charges, separated by finite distances, that has zero electric potential energy.

6-15 The calculation of the potential V of a ring of charge is asserted to be easier than the calculation of the electric field $\boldsymbol{E}$. Explain.

6-16 Imagine two spheres of equal radius and equal total charge. One sphere is uniformly charged throughout its volume. The second sphere has all of its charge distributed uniformly over its surface. How do the potentials at the surface? Compare.

Problems

6-1 Two similar tiny balls of mass m are hung from silk threads of length L and carry similar charges q as in Fig. 6-29. Assume that θ is so small that $\tan\theta$ can be replaced by its approximate equals to $\sin\theta$. (1) To this approximation show that, for equilibrium

$$x=\left(\frac{q^2 L}{2\pi\varepsilon_0 mg}\right)^{\frac{1}{3}}$$

where x is the separation between the balls.

(2) If $L=120\text{cm}$, $m=10\text{g}$, and $x=5.0\text{cm}$, what is q?

6-2 What is $\boldsymbol{E}$ in magnitude and direction at the center of the square of Fig. 6-30? Assume that $q=1.0\times 10^{-8}$ C and $a=5.0$ cm.

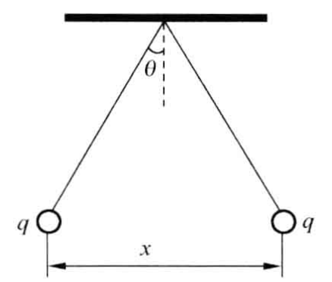


Fig. 6-29 For problem 6-1

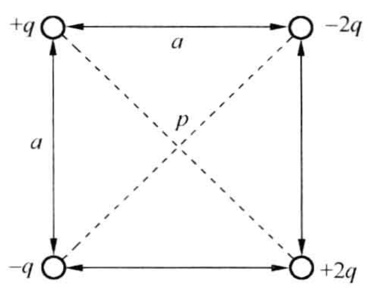


Fig. 6-30 For problem 6-2

6-3 Fig. 6-31 shows an electric quadrupole. It consists of two dipoles whose effects at external points do not quite cancel. Show that the value of $\boldsymbol{E}$ on the axis of the quadrupole for points a distance from its center (assume $z\gg d$) is given by

$$E=\frac{3Q}{4\pi\varepsilon_0 z^4}$$

where $Q\ (=2qd^2)$ is called the quadrupole moment of the charge distribution.

6-4 At what distance along the axis of a charged ring of radius R is the axial electric field a maximum?

6-5 Fig. 6-32 shows a ring of radius R, with a narrow gap of $d(d\ll R)$, with linear charge density $+\lambda$. Find the electric field at point P distance x from the center of ring along the axis.

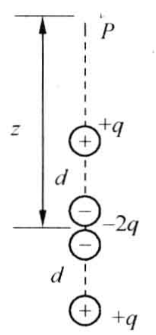


Fig. 6-31 For problem 6-3

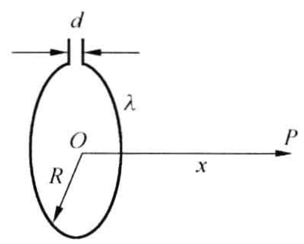


Fig. 6-32 For problem 6-5

6-6 Fig. 6-33 shows a ring of radius R, its linear charge density for one half is $+\lambda$ and another half is $-\lambda$. Find the electric field distribution along its axis.

6-7 A charged uniformly line of length a, with linear charge density λ, lies on the y-axis in Fig. 6-34. Find the electric field at point P distance x from one end of the line along the x-axis.

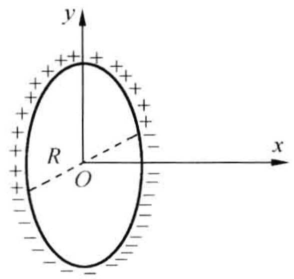


Fig. 6-33 For problem 6-6

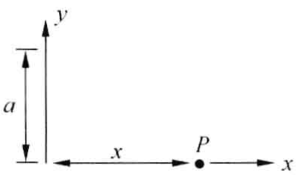


Fig. 6-34 For problem 6-7

6-8 A charged uniformly thin rod of length L, with linear charge density λ, is placed on the axis of a charged uniformly disk of radius R and its surface density being σ, as shown in Fig. 6-35. Find the electric field at a point P at x-axis in the figure.

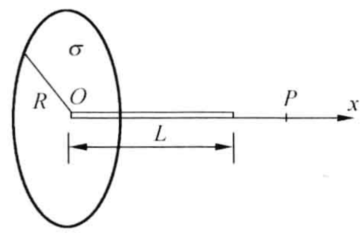


Fig. 6-35 For problem 6-8

6-9 A thin glass rod is bent into a semicircle of radius r. A charge $+Q$ is uniformly distributed along the upper half and a charge $-Q$ is uniformly distributed along the lower half, as shown in Fig. 6-36. Find the electric field $\boldsymbol{E}$ at point O, the center of the semicircle.

6-10 A cylindrical tube of length l, radius R carries a charge Q uniformly distributed over its surface. Find the field on the axis of the tube at one of its ends.

6-11 A point charge $+q$ is a distance $d/2$ from a square surface of side d and is directly above the center of the square as shown in Fig. 6-37. What is the electric flux through the square (hint: Think of the square as one face of a cube with edge d)?

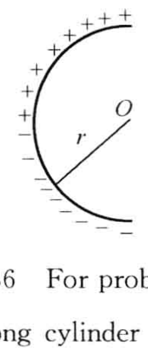


Fig. 6-36 For problem 6-9

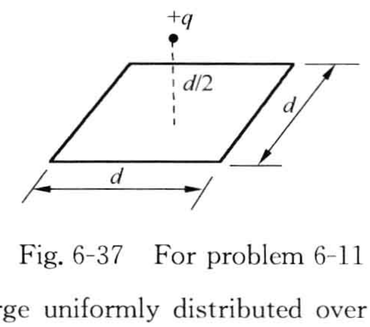


Fig. 6-37 For problem 6-11

6-12 A very long cylinder of radius R has positive charge uniformly distributed over its volume. The amount of charge is λ coulombs per meter of length of the cylinder. Find the electric field as a function of distance from the axis of the cylinder.

6-13 A long plastic pipe has inner radius a and outer radius b. Electric charge is uniformly distributed over the region $a<r<b$. The amount of charge is λ coulombs per meter of length of the pipe. Find the electric field in the regions $r<a$, $a<r<b$, and $r>b$.

6-14 A large flat nonconducting surface carries a uniform charge of density σ. A small circular hole of radius R has been cut in the middle of the sheet, as shown in Fig. 6-38. Ignore fringing of the field lines around all edges and calculate the electric field at

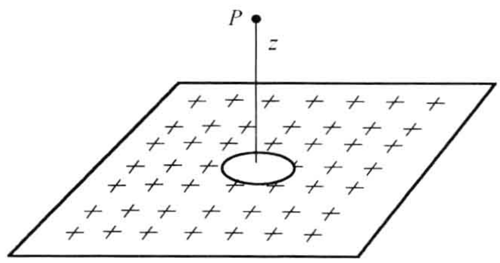


Fig. 6-38 For problem 6-14

point P, a distance z from the center of the hole along its axis (hint: use the principle of superposition).

6-15 A large plane slab of thickness d has a uniform volume charge density ρ. Find the magnitude of the electric field at all points in space both (1) inside and (2) outside the slab, in terms of x, the distance measured from the median plane of the slab.

6-16 A spherical shell has inner radius a and outer radius b. Electric charge is uniformly distributed over the region $a<r<b$, and the charge density is ρ. Use Gauss' law to find the electric field at radial point r where (1) $r<a$, (2) $a<r<b$, and (3) $r>b$.

6-17 A sold nonconducting sphere of radius R carries a nonuniform charge distribution, the charge density being $\rho=\rho_0 r/R$, where ρ_0 is a constant and r is the distance from the center of the sphere. Show that (1) the total charge on the sphere is $Q=\pi\rho_0 R^3$ and (2) the electric field inside the sphere is given by

$$E=\frac{Qr^2}{4\pi\varepsilon_0 R^4}$$

6-18 The charges and coordinates of two point charges located in the xy plane are: $q_1=+3.0\times10^{-6}$ C, $x_1=3.5$ cm, $y_1=+0.50$ cm; and $q_2=-4.0\times10^{-6}$ C, $x_2=-2.0$ cm, $y_2=+1.5$ cm.

(1) Find the electric potential at the origin.

(2) How much total work must be done to locate these charges at their given position, starting from infinite separation?

6-19 An electric charge of -9.0 nC uniformly distributed around a ring of radius 1.5m that lies the yz plane with its center at the origin. A point charge of -6.0 pC is located on the x axis $x=3.0$ m. Calculate the work done in moving the point charge to the origin.

6-20 An infinite cylindrical surface of radius R carry a uniform charge of surface density σ.

(1) Find the electric potential at one point inside and outside the cylindrical surface.

(2) Sketch the plot of $U(r)$.

6-21 The electric field inside a nonconducting sphere of radius R, containing uniform charge density, is radially directed and has magnitude

$$E=\frac{qr^2}{4\pi\varepsilon_0 R^3}$$

where q is the total charge in the sphere and r is the distance from the sphere center. Find the potential $U(r)$, taking $U=0$ (1) at the center of the sphere and (2) at $r=\infty$.

6-22 A point charge $q_1>0$ is fixed at the origin of a rectangular coordinate system, and a second point charge $q_2<0$ is fixed at $x=a$, $y=0$. The locus of all points in the xy plane $V=0$ is a circle centered on the x-axis, as shown in Fig. 6-39. Find (1) the location x_c of the center of the center of the circle and (2) the radius R of the circle.

6-23 Two semicircular rods and two short straight rods are joined in the configuration shown in Fig. 6-40. The rods carry a charge of λ coulombs per meter. Calculate the potential at O, the center of this configuration.

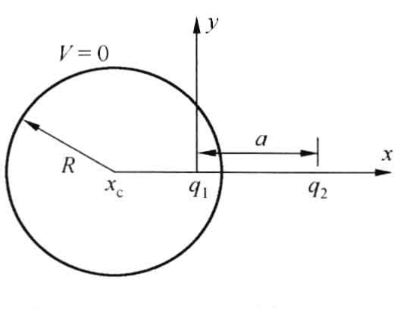


Fig. 6-39 For problem 6-22

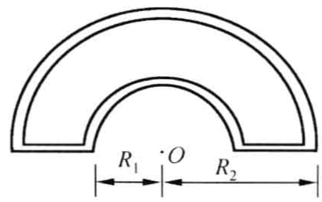


Fig. 6-40 For problem 6-23

6-24 In some region of space, the electrostatic potential is the following function of x and y

$$V = x^2 + 2xy$$

where the potential V is measured in volts and the distances in meters. Find the electric field at the point $x=2$ and $y=2$.

6-25 A charge per unit length λ is distributed uniformly along a straight-line segment of length L.

(1) Find the potential (chosen to be zero at infinity) at a point P a distance x from one end of the charged segment and in line with it (Fig. 6-41).

(2) Use the result of (1) to compute the component of the electric field at P in x direction (along the line).

6-26 On a thin rod of length L lying along the x-axis with one end at the origin ($x=0$), as in Fig. 6-42, there is distributed a charge per unit lenght by $\lambda=kx$, where k is constant.

(1) Taking the potential at infinity to be zero. Find V at P on the y-axis.

(2) Calculate the vertical component, E_y, of the electric field at P from the result of part (1) and also by direct calculation.

(3) Why cannot E_x, the horizontal component of the electric field at P, be found using the result of part (1) ?

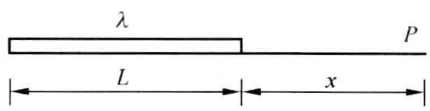


Fig. 6-41 For problem 6-25

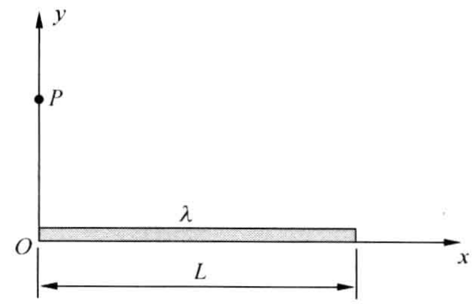


Fig. 6-42 For problem 6-26

Chapter 7

Conductors and Dielectrics in Electrostatic Field

In the last chapter, we have discussed the electric field in a vacuum, in most case, however, there are many materials around, for example, the charges are carried by conductors, and the dielectrics are used as insulator between the plates of a capacitor. When materials such as conductors and dielectrics are brought into an electric field, both the material and the electric field will have an influence on each other. We will deal with two kinds of materials, conductors and dielectrics in this chapter.

7.1 Conductors and Electrostatic Induction

7.1.1 Electrostatic induction

As we know that, a good electric conductor contains large amount of electrons called free electrons, they are not bound to any atom and are free to move around within the material. When we bring an isolated conductor B near a conductor A carrying a positive charge $+Q$, as shown in Fig. 7-1, electrons on the uncharged conductor B will be attracted by the positive charge on conductor A, leaving the near side of conductor B with a negative charge and the far side with a positive charge. This phenomenon is called **electrostatic induction; the charges redistributed on conductor *B* in Fig. 7-1 are called induced charges.**

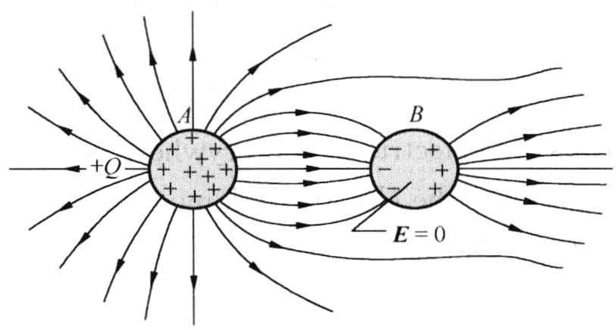


Fig. 7-1 Electrostatic induction

7.1.2 Electrostatic equilibrium condition

The induced charges separated on the conductor B set up an electric field $\boldsymbol{E}'$ which points in opposite direction with the original field $\boldsymbol{E}_0$ produced by the charge on A, the total field in conductor B is the vector sum $\boldsymbol{E}' + \boldsymbol{E}_0$. As long as $E' < E_0$, $\boldsymbol{E}' + \boldsymbol{E}_0 \neq 0$, the free electrons continue to move toward left in Fig. 7-1, $\boldsymbol{E}'$ increases consequently until $\boldsymbol{E}' + \boldsymbol{E}_0 = 0$, so that the conductor B attains at a state without net motion of electrons within it. Such state is called the **electrostatic equilibrium state.** From the analysis above, the condition that a conductor in electric field attains at electrostatic equilibrium state is that the total electric field equals zero within the conductor. We can see that some of field lines leaving the positive charged conductor A and ending on the negative charge on the near side of conductor B, and an equal number of lines leaving the far side of the conductor B, but no field lines inside conductor B.

A conductor in electrostatic equilibrium has the following properties:

(1) All points of the conductor are brought to the same potential, the surface of the conductor is an equipotential surface.

(2) The electric field at a point just outside a charged conductor is perpendicular to the surface of the conductor. To verify the first property, suppose a, b are two arbitrary points in the conductor B in which $\boldsymbol{E}=0$ everywhere under electrostatic equilibrium condition so that the potential difference $V_a - V_b = \int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$, otherwise, it would be contrary to above condition. To verify the second property, if the field vector $\boldsymbol{E}$ had a component parallel to the conductor's surface, free electrons would experience an electric force and move along the surface, in such a case, the conductor would not be in electrostatic equilibrium. Therefore, the field vector must be perpendicular to the surface.

7.1.3 The charge distribution on a conductor

(1) Any net charge on a conductor in electrostatic equilibrium state must reside on its surface. To verify this, we draw a Gaussian surface S inside the conductor as shown in Fig. 7-2, which can be very close to the conductor's surface and also can be small enough. Because the electric field everywhere inside the conductor is zero, and once it is in electrostatic equilibrium, the electric field must be zero on Gaussian surface either so that the net electric flux through this Gaussian surface is zero. From Gauss' law, we conclude that there is zero net charge inside the Gaussian surface, therefore, any net charge on the conductor must reside on the surface of the conductor.

(2) The electric field at a point just outside a charged conductor has a magnitude $E=\sigma/\varepsilon_0$ where σ is the surface charge density at that point.

To determine the magnitude of the electric field just outside a charged conductor, we apply Gauss' law again and draw a Gaussian surface in the shape of a small cylinder whose end faces are parallel to the conductor's surface as shown in Fig. 7-3. Part of the cylinder is just outside the conductor, the rest part is inside. Since the field is perpendicular to the surface of the conductor under electrostatic equilibrium, therefore, there is no flux through the cylindrical part of the Gaussian surface, and there is no flux through the flat end ΔS of the cylinder inside the conductor either, because $\boldsymbol{E}=0$ everywhere. Applying Gauss' law to the whole surface gives

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = E\Delta S = \frac{q}{\varepsilon_0} = \frac{\sigma \Delta S}{\varepsilon_0}$$

where ΔS is the small flat end just outside the conductor, σ is the surface charge density, $q=$

$\sigma\Delta S$ is the charge inside the Gaussian surface, solving for E, we have

$$E = \frac{\sigma}{\varepsilon_0} \tag{7-1}$$

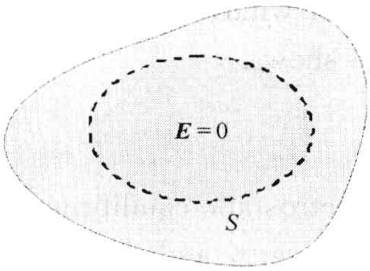


Fig. 7-2 Draw a Gaussian surface inside the conductor

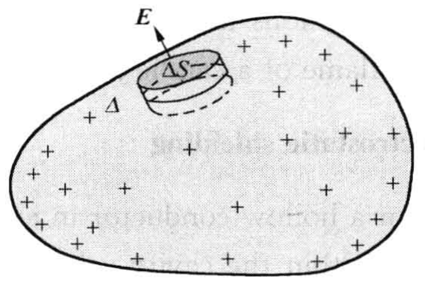


Fig. 7-3 Determining the field just outside the charged conductor

(3) If there is a cavity in the conductor, and no charge within it, then none of net charge exists on the inner surface of the conductor. In Fig. 7-4, S is the inner surface of the conductor. S_1 is a Gaussian surface over which the surface integral of $\boldsymbol{E}$ equals zero because $\boldsymbol{E}=0$ at all points on it. According to Gaussian law, the net charge within S_1 is zero. Considering the condition that there is no charge within the cavity so that the cavity inner wall S is uncharged, the net charges, if any, exist only on the outer surface of the conductor.

(4) If there is an insolated charge q inside the cavity of the conductor, draw a Gaussian surface S, applying Gaussian law and the property of conductor at electrostatic equilibrium as well as conservation of charge, it can be proved that there is induced charge $-q$ distributed on the inner surface while induced charge $+q$ on the outer surface of the conductor as shown in Fig. 7-5.

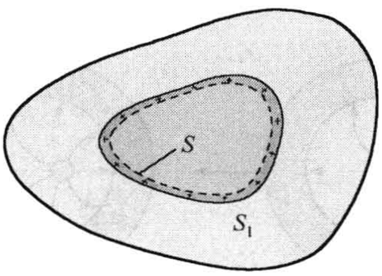


Fig. 7-4 There is a cavity in the conductor

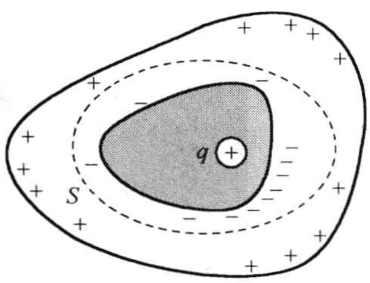


Fig. 7-5 There is an insolated q inside the cavity

(5) Tip discharge: on an irregularly shaped conductor, experiments indicate that the surface charge density is greatest at locations where the radius of curvature of the surface is the smallest, and conversely.

If an irregular shaped conductor has a sharp tip with very small radius of curvature, the discharging will occur first at the sharp tip caused by the very strong field. This phenomenon is called sharp tip discharging. An evidence is the demonstration so called "the electric wind blow out a candle" as shown in Fig. 7-6. It occurs when the electric field near the tip is strong enough (say, about 3 MV/m) to

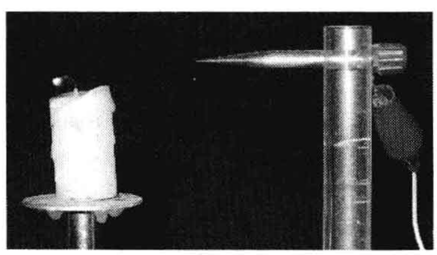
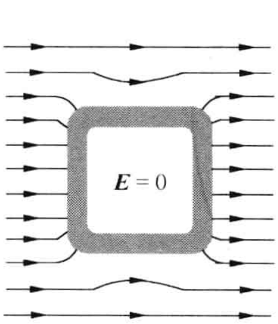


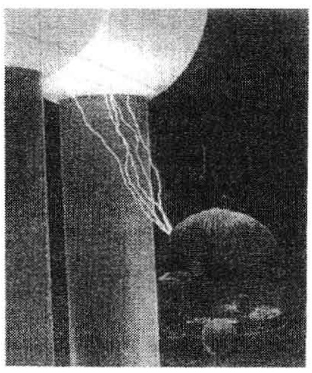
Fig. 7-6 A sharp tip discharge

ionize the air. In the beginning, a few ions are accelerated so that gain enough energy to collide, knocking more electrons out of the air molecules —that is, more and more molecules are ionized. The liberated ions will be accelerated by the field, the electrons move toward the tip while the positive ions away from the tip, forming a gust of wind, which move toward and blow out the flame of a candle just as the photo in Fig. 7-6 shows.

7.1.4 Electrostatic shielding

If we put a hollow conductor in an electric field, in electrostatic equilibrium, there is no any field line within the cavity, the field inside the cavity is zero, and the region inside the cavity is not affected by the external field, this is called the effect of **electrostatic shielding** as shown in Fig. 7-7. An electric device is placed in a conductor shell to avoid the influence of other charged object nearby, which is commonly applied in laboratories. If a charged object with q is brought into the cavity of a large conductor as shown on the left in Fig. 7-8, the electric field is zero within the shell, from Gauss' law, the electric field lines that leave the positive charge q must end on the inner surface of the cavity. And there must equal induced negative and positive charge on the inner and outside surface of the cavity. The charge on the outside surface will set up an electric field. If we connect the cavity to the ground, as shown on the right in Fig. 7-8, the potentials of the cavity and the ground are equal. The positive charge on the outside surface will flow to the ground (as the matter of fact, it is the electrons flow from the ground to neutralize the positive charges on the outside of cavity). There is no electric field in the region outside the cavity, the electric field set up by the charged object inside the cavity is then be shielded by the cavity. Usually, the metal net is used for electrostatic shielding.





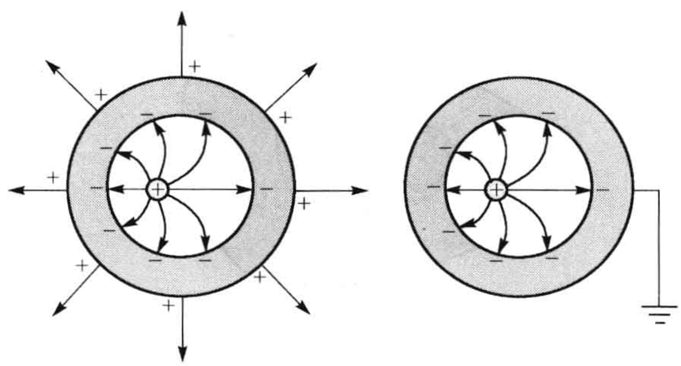


Fig. 7-7 Electrostatic shielding the outer field

Fig. 7-8 Electrostatic shielding the inner field

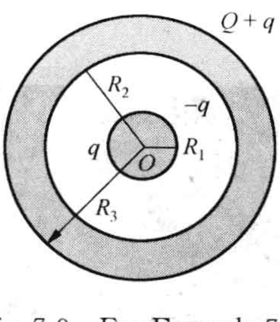


Fig. 7-9 For Example 7-1

Example 7-1 Fig. 7-9 shows a metal sphere of radius R_1 and a concentric metal spherical shell of inner and outer radii R_2 and R_3. They carry the charges of q and Q respectively. Find

(1) the charge and the field distribution;

(2) potential distribution in the space;

(3) the potential of the conductors when connected by a wire;

(4) the potential of them when the shell connected to the ground.

Solution (1) From the condition of electrostatic equilibrium and the conservation of the electric charge, the inner and the outer surfaces of the metal shell are induced $-q$ and $+q$ respectively, the total charge is $Q+q$ on the outer one. Because the spheres are concentric, the charges q, $-q$ and $Q+q$ distributed uniformly over the three spheres. From superposition principle, using Eq. (6-15) and Eq. (6-16), the field distribution is

$$E = 0 (r < R_1, R_2 < r < R_3) ; \quad E = \frac{q}{4\pi\varepsilon_0 r^2} (R_1 < r < R_2); \quad E = \frac{Q+q}{4\pi\varepsilon_0 r^2} (r > R_3)$$

(2) According to the superposition principle and result of Example 6-10, the potential distribution in four regions is given by

$$V_1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} - \frac{q}{R_2} + \frac{Q+q}{R_3}\right) \quad (r \leqslant R_1)$$

$$V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{R_2} + \frac{Q+q}{R_3}\right) \quad (R_1 < r < R_2)$$

$$V_3 = \frac{Q+q}{4\pi\varepsilon_0 R_3} \quad (R_2 \leqslant r \leqslant R_3)$$

$$V_4 = \frac{Q+q}{4\pi\varepsilon_0 r} \quad (r > R_3)$$

(3) If connected by a wire, charge q and $-q$ will neutralize, the metal sphere and shell becomes a equal potential body with charge $Q+q$ on the outer surface of radius R_3. From result of Example 6-10, its potential is

$$V_3 = \frac{Q+q}{4\pi\varepsilon_0 R_3}$$

(4) If the metal shell is connected to the ground, then only $-q$ is left on the inner surface. Using potential superposition principle, the potential of the inside sphere can be written as

$$V'_1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} - \frac{q}{R_2}\right)$$

While the metal shell becomes equal potential with the ground, that is

$$V'_3 = 0$$

In the region ($R_1 < r < R_2$) the potential becomes

$$V'_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{R_2}\right)$$

7.2 Capacitors and Capacitance

Any charged conductor can be viewed as an energy reservoir or source, if a conducting wire is connected to such a reservoir, electric charge will be transferred to perform useful work. A device has the ability to store charge is called a capacitor. Capacitors are commonly used in a variety of electric circuits, for instance, they are used to tune the frequency of radio receiver, as filters in power supplies, to eliminate sparking in automobile ignition system, to increase efficiency of alternating current power transmission, and as energy-storing devices in electronic flash units.

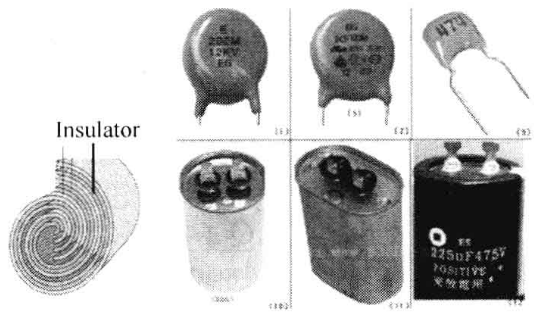


Fig. 7-10 The capacitors

7.2.1 Definition of capacitance

The most common type of capacitor consists of two conducting plates parallel to each other and separated by a distance filled with insulator layer which is very thin compared with the size of the plates, as shown in Fig. 7-10. There are three types of capacitors, parallel-plate capacitor, cylindrical capacitor and spherical capacitor.

What determines the ability to store the electric charge? Experiments show that the quantity of charge Q (the absolute valve of charge on one of the two plates) on a capacitor is linearly proportional to the potential difference between the conductors, that is

$$Q = CV$$

In which the proportionality constant C depends on the shape and separation of the given conductors, so we define capacitance of the capacitor as

$$C = \frac{Q}{V} \tag{7-2}$$

The capacitance C is always a positive quantity; the absolute valve of charge Q and the potential difference V are always positive quantities too.

Form Eq. (7-2), we see that capacitance has SI units of coulomb per volt. Named in honor of Michael Faraday, the SI unit of capacitance is farad (F): 1 F=1 C/V. The farad is a very large unit of capacitance. In practice, typical devices have capacitances ranging from microfarad (μF, $1\ \mu F = 10^{-6}$ F) to picofarad (pF, $1\ pF = 10^{-12}$ F). The dimension of capacitance is $I^2L^{-2}M^{-1}T^4$.

7.2.2 Types of capacitors and calculation of capacitance

There are three familiar geometries of the plates shape, namely, parallel plate, concentric cylinders, and concentric spheres, the corresponding capacitors are called parallel-plate capacitor, cylindrical capacitor and spherical capacitor. The capacitance of these three type capacitors will be evaluated in the following.

Example 7-2 A parallel-plate capacitor.

A parallel-plate capacitor of area S and separation d is shown in Fig. 7-11. Suppose the length of d is much smaller than the size of S. Find the capacitance of this capacitor.

Solution Assume the plates are charged with $+Q$ and $-Q$, respectively. Because d is much smaller than the size of S, we can neglect the fringe effect at the edges, taking the electric field $\boldsymbol{E}$ between the two plates to be the superposition of two infinite uniformly charged sheets with the surface charge density of $+\sigma = +Q/S$ and $-\sigma = -Q/S$, respectively. According to the results of Example 6-7, the fields produced by the two plates

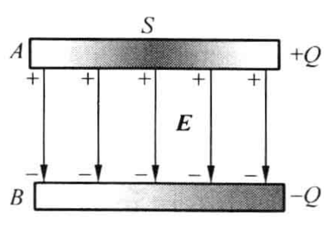


Fig. 7-11 A parallel-plate capacitor

are equal magnitude and in same direction so that the resultant field is uniform pointing from plate A to plate B. Using Eq. (6-17), we have

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

in other regions, the fields produced by the two charged sheets are equal but opposite directed, so the resultant filed is zero, and $\boldsymbol{E}$ inside the conductor plates is zero too.

The potential difference between two conductor plates can be calculated by integrating $\boldsymbol{E}$ along the field line from plate A to plate B, thus the potential difference is

$$V_{AB} = \int_A^B \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B E\mathrm{d}r = Ed = \frac{d\sigma}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 S}$$

The capacitance of the parallel-plate capacitor is then

$$C = \frac{Q}{V_{AB}} = \frac{\varepsilon_0 S}{d} \tag{7-3}$$

Example 7-3 A cylindrical capacitor. A cylindrical capacitor of length l, inner radius R_A and outer radius R_B as shown in Fig. 7-12, calculate the capacitance of it.

Solution Assuming the charge per unit length of the cylindrical plates are $+\lambda$ on the inner, $-\lambda$ on the outer one, and the length of the capacitor l is much greater than the radius and the distance between two plates. We can neglect the edge effects. From the cylindrical symmetry of the charge distribution, the field is of cylindrical symmetric. The electric field lines between two plates are directed radical outward. Thus the electric field lines are perpendicular to the curved surfaces of any coaxial cylindrical shell, and the magnitude of $\boldsymbol{E}$ at all points on the cylindrical surface of radius r is the same. So we can use Gauss' law to find the electric field between two plates. Take a coaxial cylindrical shell with radius r , $R_A < r < R_B$ as Gaussian surface S. Notice the electric flux through the plat parts (the top and the bottom ends) of the coaxial cylindrical Gaussian surface is zero because the electric field lines are parallel to them. From Gauss' law, we have

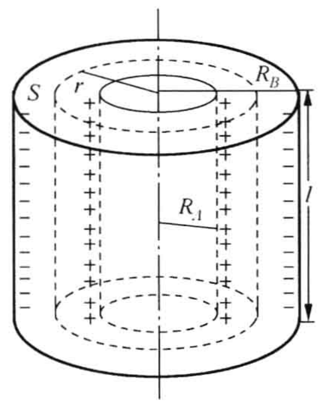


Fig. 7-12 A cylindrical capacitor

$$\oint_S E \cdot \mathrm{d}\boldsymbol{S} = \int_{S_1} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} + \int_{S_2} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} + \int_{S3} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_{S_1} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}$$

$$= E\int_{S_1} \mathrm{d}S = 2\pi r l E = \frac{q_{in}}{\varepsilon_0}$$

and

$$\frac{q_{in}}{\varepsilon_0} = \frac{l\lambda}{\varepsilon_0}$$

so that

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad (R_A < r < R_B)$$

Notice that $\boldsymbol{E}$ is parallel to $\mathrm{d}\boldsymbol{r}$ along a radial line, so we apply Eq. (6-26) to calculate the potential difference between two plates by integrating along a radial path, we have

$$V_{AB} = \int_A^B \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B E\mathrm{d}r = \int_{R_A}^{R_B} \frac{\lambda \mathrm{d}r}{2\pi\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{R_B}{R_A} = \frac{Q}{2\pi\varepsilon_0 l}\ln\frac{R_B}{R_A}$$

The capacitance of the cylindrical capacitor is

$$C=\frac{Q}{V_{AB}}=\frac{2\pi\varepsilon_0 l}{\ln\frac{R_B}{R_A}} \tag{7-4}$$

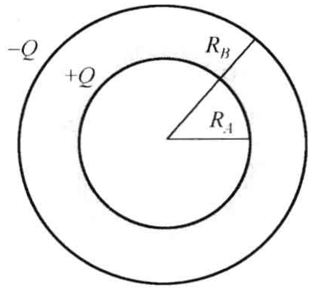


Fig. 7-13 A spherical capacitor

Example 7-4 A spherical capacitor. As shown in Fig. 7-13, a spherical capacitor consists of an inner spherical conductor of radius R_A and a concentric outer conductor sphere of radius R_B. Find the capacitance of this capacitor.

Solution Assuming the inner and the outer spheres are charged $+Q$ and $-Q$, respectively, and uniformly distributed over the sphere. Form the result of Example 6-11, the electric potential difference between the two plates is

$$V_{AB}=\frac{Q}{4\pi\varepsilon_0}\frac{R_B-R_A}{R_AR_B}$$

The capacitance of the plate capacitor is then

$$C=\frac{Q}{V_{AB}}=\frac{4\pi\varepsilon_0 R_AR_B}{R_B-R_A} \tag{7-5}$$

For an insulated conducted sphere of radius R that actually can be treated as a spherical capacitor with the inner plate at $r=R$ and outer plate at $r=\infty$. Suppose the sphere carries charge Q, from the result of Example 6-10, the potential difference is given by

$$V=\frac{Q}{4\pi\varepsilon_0 R}$$

Its capacitance is then

$$C=\frac{Q}{V}=4\pi\varepsilon_0 R \tag{7-6}$$

The results of above three examples indicate that the capacitance of a capacitor depends on the shape, geometry size as well as medium between the plates about which we will discuss later.

7.2.3 Capacitors in series and parallel

Capacitors are manufactured with certain standard capacitances and working voltages. However, in a practice application, the values of the capacitances or the maximum voltage capability over it may not satisfy what you actually require. You can obtain the values you need by combining capacitors in a series or a parallel connection or mixture of them.

When the capacitors are used in series, the working voltage will be increased. When the capacitors are used in parallel, more charge is stored for the same potential difference, and the equivalent capacitance increases. The detail will be discussed in follows.

1. Capacitors in series

Fig. 7-14 shows that two capacitors are connected in series between points A and B and connected to the terminals of a battery or two points in a circuit. There is a potential difference $V_{AB}=V_A-V_B$ across the two capacitors, and each capacitor carries the same charge Q, but the potential difference across one capacitor usually is not the same as that across the other. Suppose the capacitance are C_1 and C_2; the potential difference are V_1 and V_2, respectively, referring the definition of capacitance, we have

$$V_1 = \frac{Q}{C_1}, \quad V_2 = \frac{Q}{C_2}$$

The potential difference across the two capacitors in series is the sum of these potential differences

$$V_{AB} = V_1 + V_2 = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

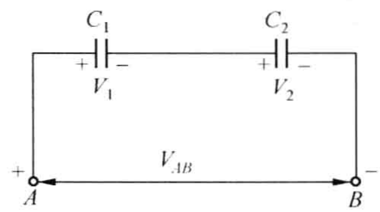


Fig. 7-14 Capacitors in series

The equivalent capacitance of two capacitors in series is that of a single capacitor that could replace the two capacitors and give the potential difference V_{AB} for the same charge Q. thus,

$$C_{eq} = \frac{Q}{V_{AB}} = \frac{Q}{V_1 + V_2} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

That is

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{7-7a}$$

For the case of more than two capacitors in series, the equivalent capacitance C is then given by

$$\frac{1}{C} = \sum_i \frac{1}{C_i} \tag{7-7b}$$

2. Capacitors in parallel

Fig. 7-15 shows two capacitors in parallel. The positive plates of the two capacitors are connected by a conducting wire to point A and the two negative plates are also connected together to point B. The potential difference is the same across each capacitor and equals to V_{AB}. Suppose the capacitances are C_1 and C_2, the charges Q_1 and Q_2 stored on the plates are given by

$$Q_1 = C_1/V_{AB}, \quad Q_2 = C_2/V_{AB}$$

The total charge stored is

$$Q = Q_1 + Q_2 = (C_1 + C_2)V_{AB}$$

The equivalent capacitance of two capacitors in parallel is the ratio of the total charge stored in the two capacitors to the potential difference which is the capacitance of a single capacitor that has the same amount of charge for a given potential difference. Then the equivalent capacitance is

$$C = \frac{Q}{V_{AB}} = \frac{Q_1 + Q_2}{V_{AB}} = C_1 + C_2 \tag{7-8}$$

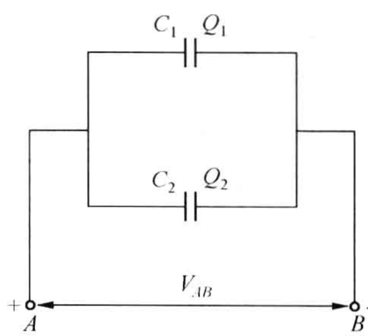


Fig. 7-15 Capacitors in parallel

Thus, the equivalent capacitance of two capacitors in parallel is the sum of the individual capacitances. In the same way, for any number of capacitors in parallel, the equivalent capacitance C is then given by

$$C = \sum_i C_i \tag{7-9}$$

Example 7-5 To repair a wash machine, a 100 μF capacitor capable of withstanding a voltage difference of 600 V is needed. The immediately available supply is a box of several

100 μF capacitors, each having a maximum voltage capability of 300 V. Give a combination with the least capacitors.

Solution If one available capacitor of $C=100\ \mu F$ having a maximum voltage capability of 300 V was connected over a voltage difference of 600 V in given circuit carelessly, it would be breakdown immediately! In order to satisfy the working voltage of 600 V, two capacitors 100 μF/300 V is connected in series, $V_2 = 2V_1 = 600$ V, but the equivalent capacitance

$$\frac{1}{C_2} = \frac{1}{C_1} + \frac{1}{C_1}$$

then

$$C_2 = \frac{C_1}{2} = 50\ \mu F$$

becomes half and can't satisfy the requirement, so two 50 μF capacitors should be combined in parallel, that is

$$C = C_2 + C_2 = 100\ \mu F$$

and the withstanding voltage $V = V_2 = 600$ V.

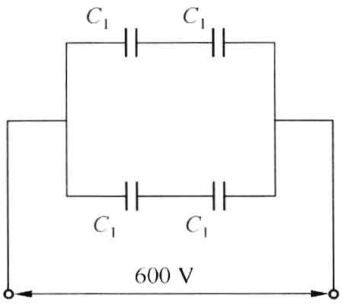


Fig. 7-16 For Example 7-4

The solution is that four 100 μF/300 V capacitors are connected in such a way as shown in Fig. 7-16. The combination has equivalent capacitance 100 μF and maximum voltage capability of 600 V.

7.3 Dielectrics

7.3.1 Capacitor with a dielectric

A category of insulating or non-conducting material, such as glass, plastics, woods, waxy paper, as well as mineral oil etc., is called dielectric. The following experiment will illustrate the effect of a dielectric on a capacitor. Consider a parallel-plane capacitor that without a dielectric between the plates has a charge Q_0 and a capacitance C_0. The potential difference across the capacitor is V_0. Fig. 7-17(a) illustrates this situation. The potential difference is measured by a device of a static voltmeter. Notice that no charge can flow through an ideal static voltmeter. If a dielectric is now inserted between the plates as in Fig. 7-17(b), experiments show that the voltmeter indicates that the voltage between the plates decreases to

$$V = \frac{V_0}{\varepsilon_r} \tag{7-10}$$

where ε_r is a number greater than 1, and is determined by the characteristic of dielectric. It is called **relative dielectric constant** or **relative permittivity**. Let the capacitance of the capacitor after filled with the dielectric or without it be C and C_0, respectively, by using

$$C_0 = \frac{Q_0}{V_0},\quad C = \frac{Q_0}{V}$$

we get

$$C = \varepsilon_r C_0 \tag{7-11}$$

Because $V_0 = E_0 d$ and $V = Ed$, substitute them into Eq. (7-10), we have

$$E = \frac{E_0}{\varepsilon_r} \tag{7-12}$$

The reason for the increase of capacitance is that the electric field between the plates of a capacitor is weakened by the dielectric. Thus for a given charge on the plates, the potential difference is reduced and the ratio $C = Q/V$ is increased.

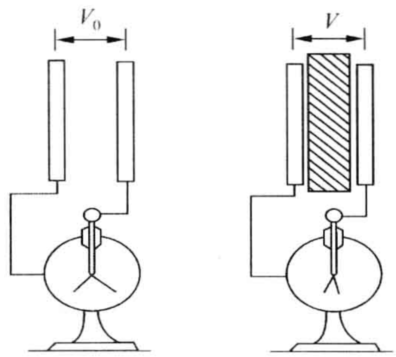


Fig. 7-17 The effect of a dielectric on a capacitor

From Eq. (7-11), the capacitances of the three capacitors Eq. (7-3), Eq. (7-4) and Eq. (7-5) become

(1) The parallel-plate capacitor

$$C = \frac{\varepsilon_r \varepsilon_0 S}{d} = \frac{\varepsilon S}{d} \tag{7-13}$$

(2) The cylindrical capacitor

$$C = \frac{2\pi\varepsilon_r\varepsilon_0 l}{\ln \frac{R_B}{R_A}} = \frac{2\pi\varepsilon l}{\ln \frac{R_B}{R_A}} \tag{7-14}$$

(3) The spherical capacitor

$$C = \frac{4\pi\varepsilon_r\varepsilon_0 R_A R_B}{R_B - R_A} = \frac{4\pi\varepsilon R_A R_B}{R_B - R_A} \tag{7-15}$$

The insulate conducted sphere

$$C = 4\pi\varepsilon_r\varepsilon_0 R = 4\pi\varepsilon R \tag{7-16}$$

In above equations, we have a new constant $\varepsilon = \varepsilon_r\varepsilon_0$ called as **permittivity** of a dielectric. From Eq. (7-11), the relative permittivity ε_r of a dielectric is a pure number with dimension $[\varepsilon_r] = 1$ so that the unit and dimension of ε are the same as that of ε_0. In the vacuum $\varepsilon_r = 1$, for all dielectrics except the vacuum, $\varepsilon_r > 1$. In Table 7-1, the relative permittivities of several materials are listed.

Table 7-1 The relative permittivities of some dielectrics

Dielectric	Relative permittivity	Dielectric	Relative permittivity
Air(1.013×10^5 Pa, 0℃)	1.000585	Transformer Oil	2.2～4.5
Wax	2.0～2.3	PVC	3.1～3.5
Paper	3.5	Porcelain	6.5
Mica	3～6	Glass	5～10
Silicon	12	Germanium	16
Glycerol	56	Pure water(20℃)	80.4
Titania ceramic	130	Strontium titanate	310

7.3.2 Molecular model of polarization charge

A dielectric weakens the electric field between the plates of a capacitor because the molecules in the dielectric produce an additional electric field in a direction opposite to that of the original field. In the view of molecular theory, there are two kinds of polarization models.

(1) Polarization of the polar dielectrics. The materials whose molecules have permanent electric dipole moments called polar dielectrics. These dipoles are randomly oriented if without

external field. In the presence of the electric field between the capacitor plates, however, these dipoles experience a torque that tends to align them in the direction of the field as discussed in section 6-6, shown in Fig. 7-18(a). The extent of alignment depends on the strength of the field and the temperature. The random thermal motion of the molecules tends to counteract their alignment, the alignment is not complete, but it becomes more complete as the magnitude of the applied field increased. The alignment of the dipoles produces an additional electric field in the opposite direction of the applied field and smaller in magnitude.

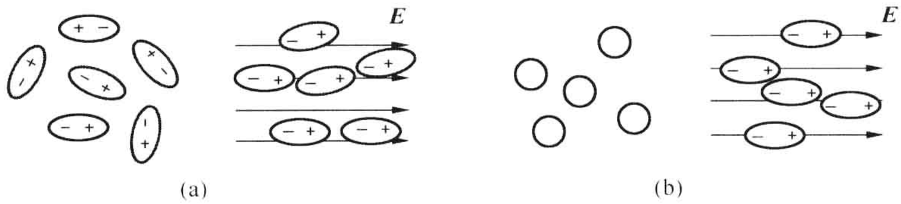


Fig. 7-18 Molecular model of polarization charge

(2) Polarization of the non-polar dielectrics. If the centers of positive charge and negative charge of a molecule are coincident in position so that it have no permanent dipole. These kinds of materials are non-polar. When brought into the applied electric filed the positive charge center and negative charge center will be stretched slightly to separate in the opposite direction so that they will acquire dipole moments being in the opposite direction of the applied field as shown in Fig. 7-18(b) .

In either case, the molecules produce an additional electric field that is in a direction opposite to that of the original field, thus weakening the original field, and the dielectric is said to be polarized by the field. The resultant field inside the dielectric has the same direction as the original but less in magnitude.

7.3.3 The electric field in dielectric

The net effect of the polarization of a homogenous dielectric is the creation of a surface charge (is called polarization charge or bounded charge) on the dielectric faces near the plates as shown in Fig. 7-19 It is this surface charges which is bound to the dielectric that produces an electric field opposite the direction of that due to the free charge on the plates. Thus, the electric field between the plates is weakened as illustrated in Fig. 7-20.

If the original electric field between the plates of a capacitor without a dielectric is $\boldsymbol{E}_0$, the surface charge on the dielectric give rise to an additional electric field $\boldsymbol{E}'$ (is called polarization electric field) in the direction opposite the original field $\boldsymbol{E}_0$, therefore the resultant electric field in the dielectric is the vector sum

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}' \tag{7-17a}$$

its magnitude is

$$E = E_0 - E' \tag{7-17b}$$

In the parallel-plate capacitor shown in Fig. 7-20 the original field E_0 is

$$E_0 = \frac{\sigma_0}{\varepsilon_0}$$

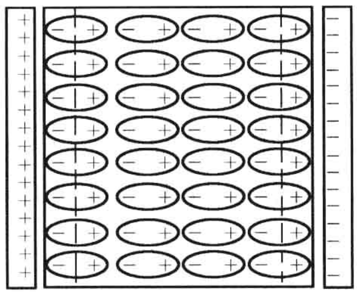


Fig. 7-19 The bounded charges presented in a dielectric when inserted in the applied field

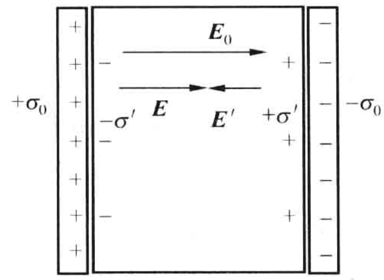


Fig. 7-20 The additional field produced by bounded charge weaken the electric field between the plates

The additional electric field in the dielectric is related to the polarization charge density σ' by equation

$$E = \frac{\sigma'}{\varepsilon_0}$$

Because of $C = \varepsilon_r C_0$, we have

$$U = Ed = \frac{U_0}{\varepsilon_r} = \frac{E_0 d}{\varepsilon_r}$$

the resultant field in dielectric is

$$E = \frac{E_0}{\varepsilon_r} \tag{7-18}$$

Substituting these relations into Eq. (7-17b) gives

$$\frac{\sigma_0}{\varepsilon_r \varepsilon_0} = \frac{\sigma_0}{\varepsilon_0} - \frac{\sigma'}{\varepsilon_0}$$

so that the surface charge (or bounded charge) density is obtained as

$$\sigma' = \sigma_0 \left(1 - \frac{1}{\varepsilon_r}\right) \tag{7-19}$$

Because $\varepsilon_r > 1$, this expression shows that the polarization charge density σ' is less than the free charge density σ_0 on the plates.

Note that Eq. (7-19) holds for the cases in which the whole space of field is full field with uniform homogenous dielectric; or the surfaces of the dielectric are equipotential surfaces. Since Eq. (7-19) is deduced from Eq. (7-18), it is holds in the same conditions.

7.4 Gauss' Law in Dielectric

7.4.1 Gauss' law in dielectric

In chapter 6 we discussed Gauss' law in a vacuum, now we will extends Gauss' law in the case that the dielectric is present. As we have known that the dielectric will be polarized and polarization charges (bounded charges) appear on the surface of the dielectric material, for general situation, the Gauss' law should be written as

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0}\sum (q_{in} + q') \qquad (7\text{-}20)$$

where q_{in} and q' are free charges on conductors and dielectric inside the Gaussian surface, and the surface charge q' is usually unknown and immeasurable, thus the above equation is not convenient to be used to calculate $\boldsymbol{E}$. We will substitute q' with q_{in} and ε_r, and the above equation will be in a new form.

Fig. 7-21 shows a parallel-plate capacitor with a dielectric filled completely between the plates, assume the charge density on the plates is σ_0 and the bounded charge density on the dielectric surface is σ'. We take a small cylinder as Gaussian surface. The up flat end embedded in the positive charged conductor plate and the bottom end embedded in the dielectric and two ends are parallel to the plate. Let the area of the end is A, The free charge q_{in} enclosed in the Gaussian surface equals to $A\sigma_0$, and the bounded charge q' is $-A\sigma'$, From Eq. (7-19), we obtain

$$q' = -A\sigma_0\left(1 - \frac{1}{\varepsilon_r}\right) = -q_{in}\left(1 - \frac{1}{\varepsilon_r}\right) \qquad (7\text{-}21)$$

Substituting it into Eq. (7-20), we can rewrite Gauss' law as

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0}\sum \frac{q_{in}}{\varepsilon_r} = \frac{\sum q_{in}}{\varepsilon_r \varepsilon_0} \qquad (7\text{-}22)$$

Now we introduce a new quantity $\boldsymbol{D}$ which is called the **electric displacement** as

$$\boldsymbol{D} = \varepsilon_0 \varepsilon_r \boldsymbol{E} \qquad (7\text{-}23)$$

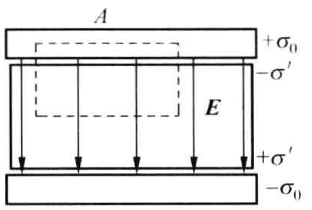


Fig. 7-21 For Gauss' law in dielectric

Then the Gauss' Law in dielectric takes the simple form as

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \sum q_{in} \qquad (7\text{-}24)$$

which states that the flux of electric displacement through any closed surface is equal to the net free charge inside the surface. This law is one of the fundamental theorems of electrostatic fields.

Although Eq. (7-24) is derived from a special case, it is generally valid even when dielectric is distributed arbitrarily in the field.

The SI unit of electric displacement is C/m^2, and its dimension is $IL^{-2}T$.

7.4.2 The electric displacement lines and electric field lines in dielectric

According to Eq. (7-24), and the state of Gauss' law in dielectric, we can say that the $\boldsymbol{D}$ lines always originate on positive free charges and terminate on negative free charges while Eq. (7-20) indicates that $\boldsymbol{E}$ lines originate on both positive free charges and polarization charges, so that terminate on both negative free charges and polarization charges. Fig. 7-22 shows the $\boldsymbol{E}$ lines and $\boldsymbol{D}$ lines in a parallel plate capacitor in which there is a dielectric slab. The number of both $\boldsymbol{E}$ lines and $\boldsymbol{D}$ lines starting from the positive free charges and ending on the negative free charges on the plates are equal, but some $\boldsymbol{E}$ lines terminate and originate on the polarization charges on the surface of the dielectric. The resultant is then all $\boldsymbol{D}$ lines pass through the dielectric in a way to ignore the polarization charges.

As a matter of fact, the $\boldsymbol{E}$ lines are rare within the dielectric because the additional electric field produced by the bounded surface charges in the opposite direction partially canceled

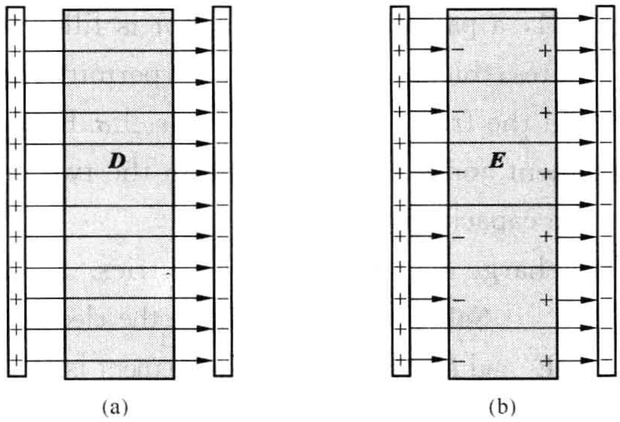


Fig. 7-22 (a) All **D** lines pass through the dielectric
(b) Some **E** lines can not pass through the dielectric

the field of free charges. The rare **E** lines in dielectric slab are due to the weakened resultant field $E = E_0 - E'$.

Example 7-6 A charged metal sphere of radius R carries a net charge q_0 and is placed in the infinite dielectric as shown in the Fig. 7-23. The permittivity of dielectric is ε. Find

(1) the electric field at any point P outside the sphere;

(2) the bounded charge on the dielectric surface contacted with metal sphere.

Solution (1) Draw a concentric Gaussian surface S of radius r. Applying Gauss' law in dielectric, we have

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = q_0$$

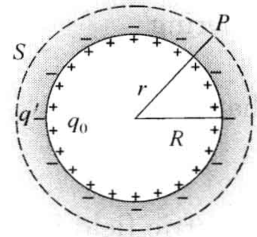


Fig. 7-23 For Example 7-5

From the spherical symmetry, we lead to $4\pi r^2 D = q_0$ the electric displacement is therefore

$$D = \frac{q_0}{4\pi r^2} \qquad ①$$

Using Eq. (7-23), the electric filed is

$$E = \frac{D}{\varepsilon} = \frac{q_0}{4\pi\varepsilon r^2} \qquad ②$$

(2) Assume the bounded surface charge appearing on the dielectric surface is q'. Applying Gauss' law Eq. (7-20) to Gaussian surface S, we have

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0}(q_0 + q')$$

Substitute Eq. ② into above integration, the left side is obtained as

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint_S E\mathrm{d}S = \frac{q_0}{4\pi\varepsilon r^2}\oint_S \mathrm{d}S = \frac{q_0}{\varepsilon} \qquad ③$$

Then we have

$$\frac{q_0}{\varepsilon} = \frac{1}{\varepsilon_0}(q_0 + q')$$

The bounded surface charge is

$$q' = q_0\left(\frac{\varepsilon_0}{\varepsilon} - 1\right) = -\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)q_0$$

Because of $\varepsilon_0 < \varepsilon$, the sign of q' is opposite to q_0.

Example 7-7 In Fig. 7-24, a parallel plate capacitor is filled with two dielectrics whose surfaces are parallel to the plates; thickness is d_1 and d_2; permittivity is ε_1 and ε_2 respectively. The area of the plates is S and the free charges density is $\pm\sigma$. Find

(1) the electric displacement and the electric field in the two dielectric;

(2) the capacitance of the capacitor;

(3) the bounded surface charge density on the dielectrics.

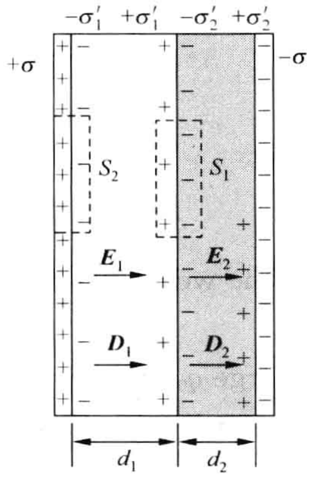


Fig. 7-24 For Example 7-6

Solution (1) Suppose the electric field in the two dielectrics is $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$; the electric displacement is $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. Draw a flat cylindrical Gaussian surface S_1 with two ends parallel to the surfaces of the dielectrics shown in Fig. 7-24. The free charge is zero inside of S_1. From Eq. (7-24), the Gauss' law in dielectric, we have

$$\oint_{S_1} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_{\text{end1}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} + \int_{\text{end2}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} + \int_{\text{the side}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = 0$$

Because $\mathrm{d}\boldsymbol{S} \perp \boldsymbol{D}$, on the side surface, so the third item equals zero, the above integral becomes

$$\int_{\text{end1}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} + \int_{\text{end2}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = -D_1 S' + D_2 S' = 0$$

Where S' is the area of the end of S_1, so that we have

$$\boldsymbol{D}_1 = \boldsymbol{D}_2 \qquad ①$$

Considering

$$\boldsymbol{D}_1 = \varepsilon_1 \boldsymbol{E}_1, \quad \boldsymbol{D}_2 = \varepsilon_2 \boldsymbol{E}_2 \qquad ②$$

So that

$$\frac{\boldsymbol{E}_1}{\boldsymbol{E}_2} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} \qquad ③$$

The electric field is inversely proportional to the permittivity or relative permittivity.

Draw another flat cylindrical Gaussian surface S_2 in which the free charge equals to $S'\sigma$ in the positive plate as shown in Fig 7-24, From the Gauss' law in dielectric, we have

$$\oint_{S_2} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_{\text{end1}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = D_1 S' = S'\sigma$$

From Eq. ①, we have

$$D_1 = D_2 = \sigma$$

From Eq. ②, we have

$$E_1 = \frac{\sigma}{\varepsilon_1}, \quad E_2 = \frac{\sigma}{\varepsilon_2}$$

(2) The potential difference between the plates is

$$V = E_1 d_1 + E_2 d_2 = \sigma\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right) = \frac{q}{S}\left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)$$

The capacitance of this capacitor is then

$$C = \frac{q}{V} = \frac{S}{\dfrac{d_1}{\varepsilon_1} + \dfrac{d_2}{\varepsilon_2}}$$

(3) Suppose the surface densities of bounded charges on the surfaces of the two dielectrics are $\pm\sigma'_1$ and $\pm\sigma'_2$, respectively as shown in Fig. 7-24, because the surfaces of the dielectrics are

equipotential, using Eq. (7-19), we have

$$\sigma'_1 = \left(1 - \frac{1}{\varepsilon_{r1}}\right)\sigma, \quad \sigma'_2 = \left(1 - \frac{1}{\varepsilon_{r2}}\right)\sigma$$

where $\varepsilon_{r1} = \varepsilon_1/\varepsilon_0$, $\varepsilon_{r2} = \varepsilon_2/\varepsilon_0$.

7.5 Energy Stored in an Electric Field

7.5.1 Energy stored in a charged capacitor

Fig. 7-25 shows a simple demonstration about the evidence that a energy is stored in a charged capacitor. At first the switch K is on the left position, the capacitor is charged by the battery which maintains a certain potential difference V and provides an energy source for charging the capacitor. After the capacitor is fully charged, as soon as we turn the switch K to the right position, the flash from the little lamp L can be observed. The energy of the flashing comes from the charged capacitor when the discharge current through it. This demonstration indicates that a charged capacitor can store a certain amount of energy.

To calculate the energy stored in the capacitor, we shall assume a charging process that an amount of the charges on the two opposite plates are gradually increase from zero to a final amount $+Q$ and $-Q$, respectively. In the beginning, no work is needed to do to transfer a small amount of charge dq from one plate to the other, but once this charge has been transferred, a small potential difference exists between the plates. Therefore, work must be done to move additional positive charge dq from low potential plate to high potential plate as shown in Fig. 7-26. As more and more charge is transferred from one plate to the other, the potential difference u increases and more work is required.

In Fig. 7-26 at some instant during the charging process, the charge on the capacitor is q, at the same instant, the potential difference across the capacitor is $u=q/C$, the work necessary to transfer an amount charge dq from the plate carrying $-q$ to the plate carrying charge $+q$ is

$$dW = u\,dq = \frac{q}{C}dq$$

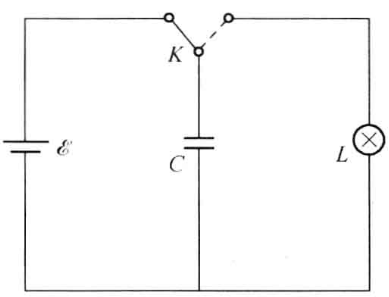


Fig. 7-25 A demonstration of the energy stored in capacitor

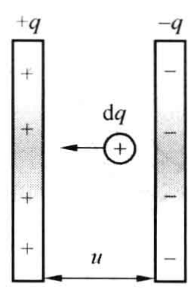


Fig. 7-26 Work done to move additional charge through potential difference u

The total work required to charge the capacitor from $q=0$ to some final charge $q=Q$ is

$$W = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} \tag{7-25}$$

Using the relation $C = Q/V$, we can express the potential energy stored in a charged capacitor as

$$W = \frac{Q^2}{2C} = \frac{1}{2}VQ = \frac{1}{2}CV^2 \tag{7-26}$$

7.5.2 Energy stored in an electric field

If we applied the capacitance of a parallel-plate capacitor to Eq. (7-26), we have then

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_0 S}{d}(Ed)^2 = \frac{1}{2}(Sd)\varepsilon_0 E^2 \tag{7-27}$$

Because Sd is the volume occupied by the electric field, the energy per unit volume, known as the energy density, is

$$w_e = \frac{W}{Sd} = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}ED \tag{7-28}$$

where $D = \varepsilon_0 E$. Although Eq. (7-28) was derived form a parallel-plate capacitor, the expression is generally valid regardless of the source of the electric field. That means there is electric energy stored in any point in the electric field. Therefore, the electric energy stored in electric field can be calculated by the following integral

$$W_e = \int_V \frac{1}{2}ED\,\mathrm{d}V \tag{7-29}$$

In which $\mathrm{d}V$ is the element volume and V is the space region over it the integration carry out, corresponding to the whole space where the field distributed.

Example 7-8 A spherical capacitor consists of a spherical conductor of radius R_A with charge $+Q$, concentric with a larger conductor sphere of radius R_B with charge $-Q$. Calculate the electric field energy stored in the capacitor.

Solution the capacitance of this spherical capacitor is $C = \frac{Q}{V_{AB}} = \frac{4\pi\varepsilon_0 R_A R_B}{R_B - R_A}$, the electric field energy stored in the capacitor can be calculated by Eq. (7-26), that is

$$W_e = \frac{Q^2}{2C} = \frac{(R_B - R_A)Q^2}{8\pi\varepsilon_0 R_A R_B}$$

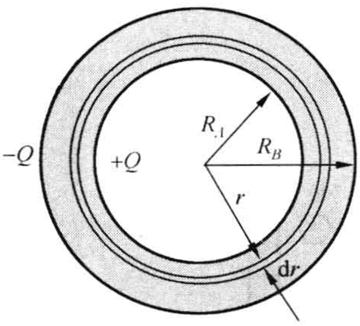


Fig. 7-27 For Example 7-6

The result can also be obtained by applying Eq. (7-29). The electric field between the two conductor plate is $E = \frac{Q}{4\pi\varepsilon_0 r^2}$ $(R_A < r < R_B)$, for other space $\boldsymbol{E}=0$. The electric energy density is then

$$w_e = \frac{1}{2}ED = \frac{1}{2}E^2\varepsilon_0 = \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

The energy density is not uniform; it varies with distance r. Take a very thin spherical shell within the inner and outer conducting spheres as volume element with radius r, surface area $4\pi r^2$ and thickness $\mathrm{d}r$, as shown in Fig. 7-27, its volume is $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ in which the energy stored is give by

$$\mathrm{d}W = w_e \mathrm{d}V = \frac{Q^2}{32\pi^2\varepsilon_0 r^4} \times 4\pi r^2 \mathrm{d}r = \frac{Q^2 \mathrm{d}r}{8\pi\varepsilon_0 r^2}$$

the total electric energy stored in the field is calculated by integrating from $r=R_A$ to $r=R_B$, that is

$$W_e = \int w_e \mathrm{d}V = \int_{R_A}^{R_B} \frac{Q^2 \mathrm{d}r}{8\pi\varepsilon_0 r^2} = \frac{Q^2}{8\pi\varepsilon_0}\int_{R_A}^{R_B}\frac{\mathrm{d}r}{r^2} = \frac{Q^2(R_B - R_A)}{8\pi\varepsilon_0 R_A R_B}.$$

The same result is obtained, but the previous method is simpler.

Questions

7-1 A positive charge is brought very near to an uncharged insulated conductor. The conductor is connected to the ground while the charge is kept near. Is the conductor charge positively or negatively or not at all if (1) the charge is taken away and then the ground connected is removed and (2) the ground connection is removed and then the charge is taken away?

7-2 Does the potential of a positively charged insulated conductor have to be positive? Give an example to prove your point.

7-3 Sketch the electric field lines between two concentric conducting spherical shells, charge $+q_1$ distributed on the inner sphere and $-q_2$ on the outer. Consider the cases $q_1 > q_2$, $q_1 = q_2$ and $q_1 < q_2$.

7-4 A positive point charge q is located at the center of a hollow metal sphere. What charge appear on (1) the inner surface and (2) the outer surface of the sphere? (3) If you bring an uncharged metal object near the sphere, will it change?

7-5 If the surface of a charged conductor is an equipotential, does it mean that charged is distributed uniformly over that surface? If the electric field is constant in magnitude over the surface of a charged conductor, does it mean that the charge is distributed uniformly?

7-6 An isolated conducting spherical shell carries a negative charge. What will happen if a positively charged metal object is placed in contact with the shell interior? Discuss the three cases in which the positive charge is (1) less than, (2) equal to, and (3) greater than the negative charge in magnitude.

7-7 Can there is a potential difference between two adjacent conductors that carry the same amount of positive charge?

7-8 A isolated metal cube is given an electrostatic charge Q. Qualitatively, how will this charge distribute itself over the cube? Consider the plane faces, the edges, and the corners.

7-9 If you put more charge on one plate of a parallel plate capacitor than on the other, how will the charges distribute?

7-10 (1) A positive charged glass rod attracts a suspended object. Can we conclude that the object is negatively charged?

(2) A positively charged glass rod repels a suspended object. Can we conclude that the object is positively charged?

7-11 Discuss the similarities and differences, when (1) a dielectric slab or (2) a conducting slab are inserted between the plates of a parallel plate capacitor.

7-12 What are the similarities and differences between the free charge and the polarization charge?

7-13 When some mineral oil is full filled in the separation of a charged parallel plate capacitor.

(1) The charging battery disconnected first;

(2) Keep the charging battery connected. Analyze the changes, if any, about the charge, the potential difference, the electric field, the capacitance and the energy stored in the capacitor.

7-14 Speculate on the disposition or change of the energy of a charged capacitor when

(1) A wire is connected across the plates of the capacitor;

(2) pull the separation of a parallel plate capacitor doubled.

7-15 Why does the energy of an isolated charged capacitor decrease when a material of dielectric slab is inserted between the plates?

Problems

7-1 A large plane conductor of thickness d has been placed in the uniform electric field, and $\mathbf{E}$ is perpendicular to the surface of the conductor. Find the surface charge density on the conductor.

7-2 A sphere of $R_1 = 10$ cm radius is connected to a sphere of $R_2 = 2.5$ cm radius as in Fig. 7-28. The electric field at the surface of the large sphere is 103(V/m). Find the net charge on the two sphere. Neglect any possible charge present on the connecting wire . Assume the distance between the two spheres is much larger then the radii of the spheres and too far to affect the fields produced by each other.

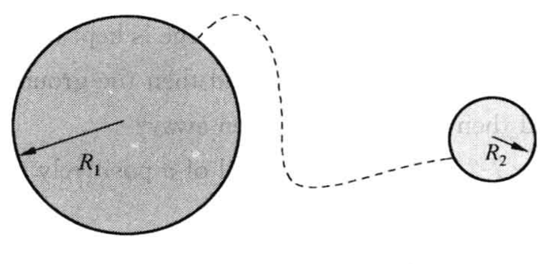


Fig. 7-28 For problem 7-2

7-3 A very large conducting sphere carries a charge of 1C. The potential at the surface of the sphere is 1 V.

(1) Calculate the radius of the sphere. How does this compare with the radius of the earth?

(2) Calculate the surface charge density.

7-4 Fig. 7-29 shows a charge $+q$ arranged as a uniform conducting sphere of radius a, and placed within the spherical conducting shell of inner radius b and outer radius concentrically. The outer shell carries a charge of $-q$. Find $E(r)$

(1) Within the sphere ($r<a$).

(2) Between the sphere and the shell ($a<r<b$);

(3) Inside the shell ($b<r<c$);

(4) Outside the shell ($r > c$);

(5) What charges appear on the inner and outer surfaces of the shell?

7-5 A concentric spherical conducting shell A with radii R_1 and R_2 carry charges of Q_1 and Q_2 as shown in Fig. 7-30. A long fine wire passes through a little hole to connect with the inner shell and a far away conducting sphere B with radius of r. This conducting sphere is uncharged before it is connected by the wire. Find the charge on the conducting sphere B after it is connected by the wire. Suppose the distance between A and B is too far to influent the fields of each other.

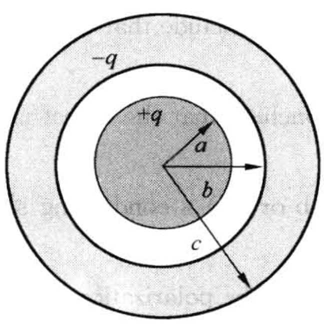


Fig. 7-29 For problem 7-4

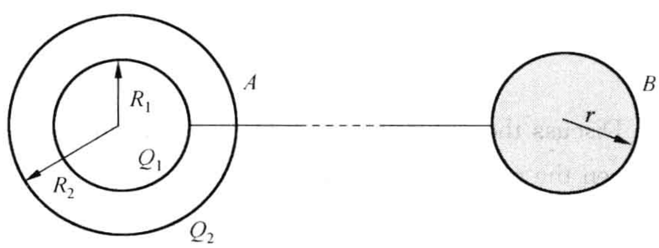


Fig. 7-30 For problem 7-5

7-6 Two large metal plates of area S face to each other. They are d apart and carry charges of q_1 and q_2, respectively as shown in Fig. 7-31. Find

(1) the surface charge density on each surface;

(2) the electric field and potential difference between two metal plates.

7-7 A metal sphere of radius R is very far from the ground, but it is connected with the ground by a thin wire. A point charge $+q$ is fixed at the point which is distance $3R$ form the center of the metal sphere. Find the induced charges on the metal sphere.

7-8 A multiplate parallel capacitor, such as that used in radios, consists of four parallel plates arranged in such away as shown in Fig. 7-32. The area of each plate is S, and the distance between adjacent plates is d. What is the capacitance of this arrangement?

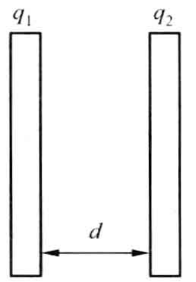


Fig. 7-31 For problem 7-6

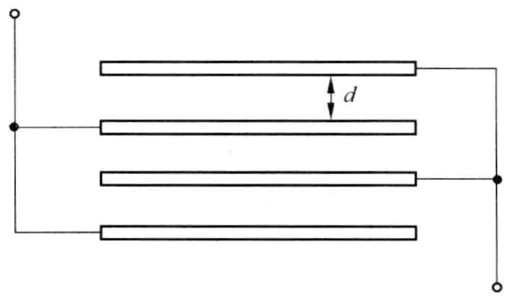


Fig. 7-32 For Problem 7-8

7-9 Find the capacitance of two parallel wires of length L and radius a whose axes are separated by a distance d and $d \gg a$.

7-10 In Fig. 7-33, $C_1 = 10\ \mu F$, $C_2 = 5.0\ \mu F$, $C_3 = 5.0\ \mu F$.

(1) Find the equivalent capacitance of the combination;

(2) If $V = 100$ V, find the charge and potential difference across each capacitor;

(3) Suppose that capacitor C_1 breaks down electrically, becoming equivalent to a conducting path. What changes in the charge and the potential difference occur for capacitor C_3?

7-11 A capacitor of 100 pF is charged to a potential difference of 50 V, the charging battery then being disconnected. The capacitor is then connected in parallel with a second (initially uncharged) capacitor, and the potential difference drops to 35 V. What is the capacitance of this second capacitor?

7-12 A air-filled parallel plate capacitor has a capacitance of 1.3 pF. The separation of the plates is doubled and wax inserted between them. The new capacitance is 2.6 pF. Find the relative permittivity of the wax.

7-13 Two parallel plate capacitors have the same plate area S and separation d, but the permittivity of the material between them are $\varepsilon + \Delta\varepsilon$ and $\varepsilon - \Delta\varepsilon$, respectively.

(1) Find the equivalent capacitance when they are connected on parallel;

(2) If the total charge on the parallel combination is Q, what is the charge on the capacitor with the larger capacitance?

7-14 A parallel plate capacitor is filled with two dielectrics as in Fig. 7-34. Show that the capacitance is given by $C = \dfrac{S(\varepsilon_1 + \varepsilon_2)}{2d}$.

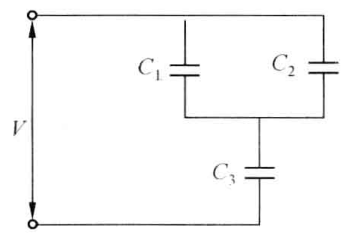


Fig. 7-33 For problems 7-10 and 7-20

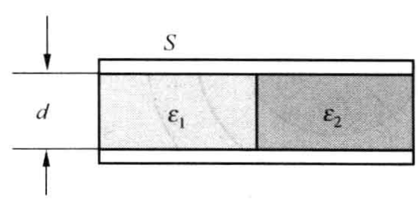


Fig. 7-34 For problem 7-14

7-15 A parallel plate capacitor is filled with two dielectrics as in Fig. 7-35. Show that the capacitance is given by $C = \dfrac{2S\varepsilon_1\varepsilon_2}{d(\varepsilon_1 + \varepsilon_2)}$.

7-16 A parallel plate capacitor has a capacitance of 100 pF, a plate area of $100cm^2$, and a mica dielectric ($\varepsilon_r = 5.4$), at 50 V potential difference. Calculate

(1) $\boldsymbol{D}$ and $\boldsymbol{E}$ in the mica;

(2) The magnitude of the free charge on the plates;

(3) The magnitude of the polarization surface charge.

7-17 A charged metal sphere of radius R_1 has a net charge of Q, which is surrounded by a concentric dielectric shell of inner radius R_1, outer radius R_2 as in Fig. 7-36. The relative permittivity of the shell is ε_r. Find the electric field and potential at radial point r where (1) $r<R_1$; (2) $R_1<r<R_2$; (3) $r>R_2$.

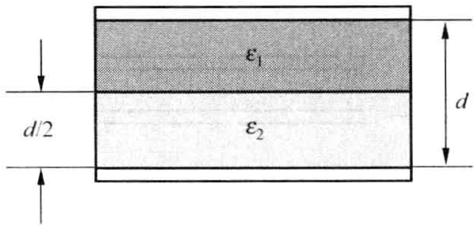


Fig. 7-35 For problem 7-15

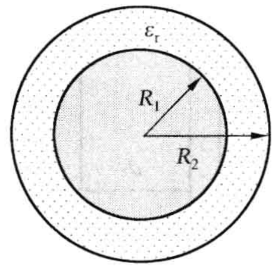


Fig. 7-36 For problem 7-17

7-18 A spherical capacitor is filled with two dielectric as in Fig. 7-37.

(1) Find the capacitance.

(2) If the capacitor is charged to a potential difference of V, find the bound surface charge density on the dielectric spherical surface at the radii of R_1, R_2 and r, respectively.

7-19 A long cylindrical copper wire of radius 0.20cm is surrounded by a cylindrical sheath of rubber of inner radius 0.20cm and outer radius 0.30cm. The rubber has $\varepsilon_r = 2.8$. Suppose that the surface of the copper has a free charge density of 4.0×10^{-6} (C/m^2).

(1) What will be the bounded charge density?

(a) On the inside surface of the rubber sheath.

(b) On the outside surface.

(2) What will be the electric field in the rubber?

(a) Near its inner surface.

(b) Near its outer surface.

(3) What will be the electric field and electric displacement just outside the rubber sheath?

7-20 Three capacitors are connected as shown in Fig. 7-33. Their capacitances are $C_1 = 6.0\ \mu F$, $C_2 = 8.0\ \mu F$ and $C_3 = 2.0\ \mu F$. If $V = 200V$. What will be the charge on each capacitor? What will be the energy stored in each?

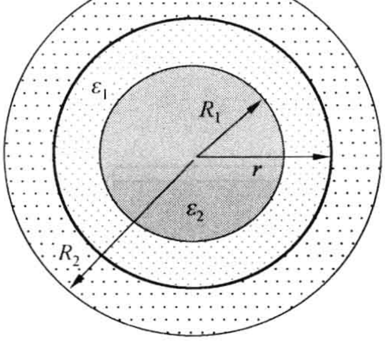


Fig. 7-37 For problem 7-18

7-21 Two capacitors of 6.0 μF and 9.0 μF are connected in two cases of parallel and series across a 1500 V potential difference, respectively. Find the charges, potential difference and energy distributing to each capacitor.

7-22 A capacitor is charged until its stored energy is 4.0 J. A second uncharged capacitor is then connected to it in parallel.

(1) If the charge distributes equally, what is now the total energy stored in the electric field?

(2) Where did the excess energy go?

7-23 A spherical shell of inner radius a and outer radius b carries charge Q uniformly distributed over its volume. What is the electric energy of this charge distribution?

Chapter 8

Magnetic Field of a Steady Current

We have known that there are electric field forces between the still charges. However, there are also non-electric property forces caused by the relative motion between the charges. That is, charges in motion relative to an observer set up a magnetic field, and this magnetic field exerts a force on second charge in motion relative to the observer. Our plan in this chapter is to discuss the property of the magnetic field, the production of the magnetic field (for steady electric currents and moving charges) and the magnetic force acted on a moving charge or steady electric current. The main contents involve the magnetic field $\boldsymbol{B}$ vector and Gauss' law in magnetic field, Boit-Savart law, Ampere's Law, Lorentz force and Ampere force and magnetic torque. We will study the properties of matters in magnetic field as well.

8.1 The Magnetic Phenomena and Ampere's Hypothesis

8.1.1 The magnetic phenomena

The first magnetic phenomena were observed by ancient Chinese people, and it was known to the Chinese as early as 121 AD that an iron rod, after being brought near a nature magnet, would acquire and retain this property of natural magnet, and such rod, when freely suspended about a vertical axis, would set itself approximately in the north-south direction. And in Han Dynasty, Si Nan, a kind of compass was invented and compass was used as aids navigation in Song Dynasty in eleventh century.

If a bar-shaped permanent magnet is free to rotate, one end always points north. This end is called a north-pole or N-pole; the other end is south pole or S-pole. The experiments show that like poles repel each other, and unlike poles attract each other. An object that contains iron, nickel and cobalt, but is not itself magnetized, is attracted by either pole of a permanent magnet. By analogy to electric field, we say that a bar magnet sets up a magnetic field in the space around it and a second body responds to that field.

The fact that a compass needle tends to align with the magnetic field at the demonstration indicates that the earth itself is a magnet. Fig. 8-1 is a sketch of Earth's magnetic field, in which the lines called magnetic field lines, shows the direction that a compass would point at each location. From Fig. 8-1, its north geographical pole is close to a magnetic south pole, which is why the north pole of a compass points north. Earth's magnetic axis is not quite parallel to its geographic axis (the axis of self-rotation), and the angle between them is about 11.2°. Scientists consider it more likely that Earth's magnetic field is due to the convection currents in the liquid part of its core. An interesting sidelight concerning Earth's magnetic field is that the field direction reverses every few million years.

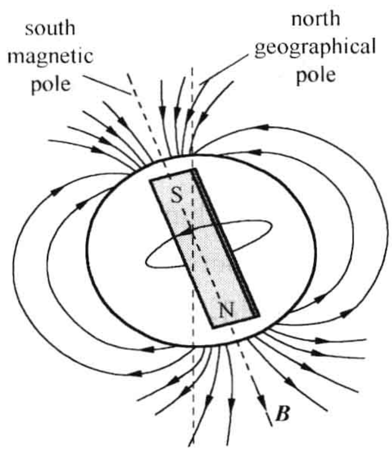


Fig. 8-1 Magnetic field of the earth

Although the force between two magnetic poles is similar to that between two electric charges, there is an important difference between electric and magnetic phenomena. Namely, Electric charges can be isolated, but magnetic poles always occur in pairs. If a magnet is broken into two parts, there will be equal and opposite poles at either part of the break point; that is, there will be two magnets, each with a north and south poles. In fact, no matter how many times a permanent magnet is cut, each piece always has a north pole and a south pole. So far, there seems to be no conclusive evidence that an isolated magnetic pole or magnetic monopole exists.

It was not until 1819 that any connection between electrical and magnetic phenomena were discovered. In that year, Denmark scientist Hans Christina Oersted observed that an electric current in a wire reflected a nearby compass needle. Shortly afterward following this discovery, French physicist A. M. Ampere also found a force acting on the wire carrying a current as it came near to a magnet, and further, showed that parallel wires having currents in the same direction attract one another. In the early 1830s, Michael Faraday and Joseph Henry independently demonstrated that an electric current can be produced in a circuit either by moving a magnetic near the circuit or by changing the current in a nearby circuit. These observations demonstrate that a changing magnetic field creates an electric field. Between 1861 and 1865, James Clerk Maxwell developed a complete theory of electricity and magnetism, which explained and united all of classical electricity and magnetism, and showed that the reverse is also true: magnetic field can be created by a changing electric field.

8.1.2 Ampere's hypothesis

The explanation of the magnetic properties of matter was first suggested by Ampere in 1820. If current is passed through a solenoid, the solenoid behaves like a magnet of the same shape, and it has a north "pole" and a south "pole". This fact led Ampere to speculate that the force between iron magnets arise from electric currents (molecular currents) in the iron, which is called the Ampere's hypothesis. He further proposed that the magnetism of a permanent magnet is due to the alignment of molecular current loops within the material. According to modern physics theory (refer to section 8-8), we know that these molecular current result partly from the motion of electrons within the atom and partly from the electron spin.

8.2 The Magnetic Field, Magnetic Field Lines, and Magnetic Flux

8.2.1 Magnetic field

In our discussion of electric field, we describe the interaction between charged objects in terms of electric field. In a similar way, magnetic field, being a vector field and denoted by the symbol $\boldsymbol{B}$, is produced by a current, a moving charge or a permanent magnet in the space surrounding them. The magnetic field can be defined in many equivalent ways based on the effects it has on its environment, and is most commonly defined in terms of the magnetic force (called the Lorentz force) that it exerts on moving electric charges.

As shown in Fig. 8-2, when moving through a magnetic field, a charged particle moving with a velocity $\boldsymbol{v}$ experiences a magnetic force, Lorentz force, which we call it the test object. Experiments on various charged particles moving in a magnetic field give the following results:

(1) The magnitude F of magnetic force exerted on the charged particle is proportional to the charge q and the speed v of the particle.

(2) When a charged particle moves parallel to the magnetic field vector, the magnetic force acting on the particle is zero.

(3) When the particle's velocity vector make any angle with the magnetic field, the magnitude of $\boldsymbol{F}$ is proportional to $\sin\theta$, $\boldsymbol{F}$ is perpendicular to the plane formed by $\boldsymbol{v}$ and $\boldsymbol{B}$.

(4) The magnetic force exerted on a negative charge is in the direction opposite the direction of the magnetic force acted on a positive charge moving in the same direction.

We can summarize these observations by writing the magnetic force in a form of vector cross product

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} \tag{8-1}$$

The direction of $\boldsymbol{F}$ can be determined by right-hand rule. Point your four fingers of your right hand along the direction of $\boldsymbol{v}$ with the palm facing $\boldsymbol{B}$ and curl them toward $\boldsymbol{B}$. Your extended thumb, which is at right angle to your fingers, points in the direction of $\boldsymbol{v} \times \boldsymbol{B}$, as shown in Fig. 8-2. Because $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, $\boldsymbol{F}$ is in the direction of your thumb if q is positive and is opposite the direction of your thumb if q is negative.

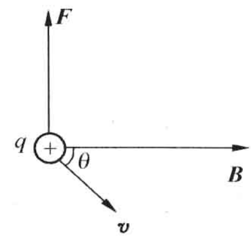


Fig. 8-2 Magnetic force exerts on a moving charge

According to Eq. (8-1), the maximum value of $\boldsymbol{F}$ is

$$F_{\max} = qvB$$

This expression is used to define the magnitude of the magnetic field as

$$B = \frac{F_{\max}}{qv} \tag{8-2}$$

The direction of the magnetic field $\boldsymbol{B}$ at any location is taken to the direction in which the north pole of a small compass needle points at that point. The SI unit of magnetic field is the tesla (T), also called the weber (Wb) per square meter. In practice, the cgs unit for magnetic field, the gauss (G), is often used, which is related to the tesla through the conversion

$$1\text{T} = 10^4\text{G}$$

8.2.2 Magnetic field lines

We can represent any magnetic field by magnetic field lines, being the same as for the electric field lines we introduced in section 6-4. We draw the lines so that the direction of the magnetic field at any point is tangent to the lines. The magnetic field lines have an arrowhead, and the tangent of the lines at each point indicates the direction of the magnetic field $\boldsymbol{B}$, as shown in Fig. 8-3. And the number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of the magnetic field in that region, that is $B=\mathrm{d}N/\mathrm{d}S$.

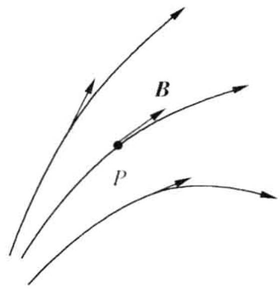


Fig. 8-3 Magnetic field lines

Magnetic field lines make it much easier to visualize and understand the complex mathematical relationships underlying magnetic field. Magnetic field lines produced by several common sources of magnetic field are shown in Fig. 8-4, in which a higher density of nearby field lines indicates a stronger magnetic field. Various phenomena have the effect of "displaying" magnetic field lines as though the field lines are physical phenomena. For example, iron filings placed in a magnetic field line up to form lines that correspond to "field lines". However, field lines are a visual and conceptual aid only and they do not exist in the actual field. Unlike electric field lines, magnetic field lines are continuous and form closed loops. In other words, magnetic filed lines do not begin or end at any point.

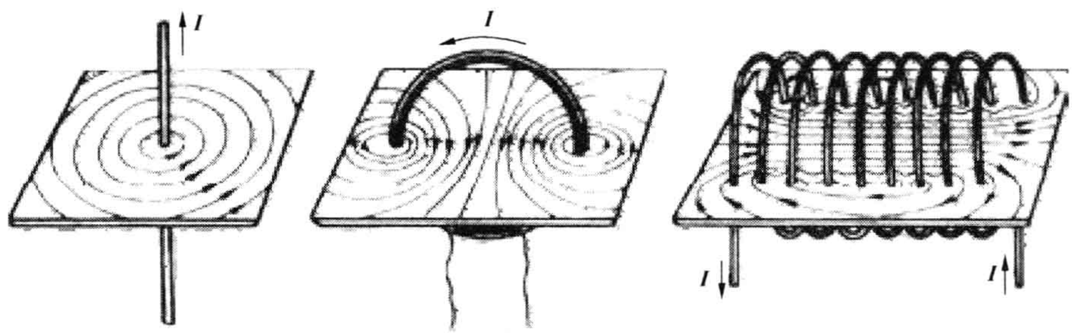


Fig. 8-4 The magnetic fields of some currents

8.2.3 Magnetic flux

The magnetic flux is the number of magnetic field lines through a surface. If the magnetic field is not uniform and the surface is in any shape as show in Fig. 8-5, it is not difficulty to see that the magnetic flux through a surface of any shape can be calculated by the integral

$$\Phi_\mathrm{m} = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \int_S B\cos\theta \mathrm{d}S \tag{8-3}$$

And the flux through a closed surface is an integral over a closed surface, that is

$$\Phi_\mathrm{m} = \oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_S B\cos\theta \mathrm{d}S \tag{8-4}$$

If we draw a closed surface anywhere in a magnetic field, as we have mentioned that the magnetic field lines do not start and end at any point, the every field line that enters the surface also exits from it. Therefore, the net flux through a closed surface is always zero, mathematically it is equivalent to

$$\Phi_m = \oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0 \quad (8\text{-}5)$$

which is called Gauss' Law for magnetism. For the case of a closed surface, d**S** points outward so that dot product in the integral is positive for **B** pointing out and negative for **B** pointing in. Sometimes, if a surface is open with a boundary, another simple surface is added to form a closed surface so Gauss' law Eq. (8-5) can be used to solve the question.

Example 8-1 Determine the magnetic flux through the semi spherical surface S in a uniform magnetic field with magnitude B as shown in Fig. 8-6.

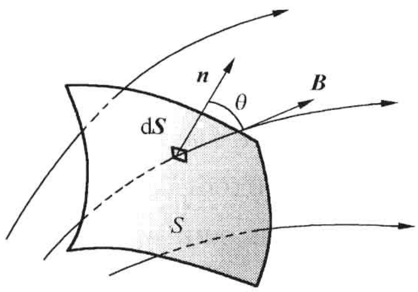


Fig. 8-5 Magnetic flux through an arbitrary surface

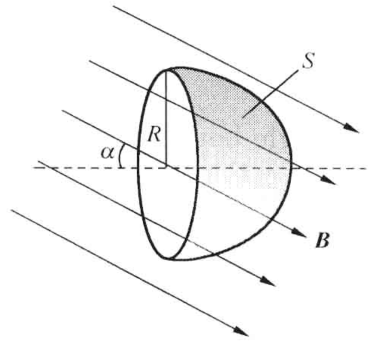


Fig. 8-6 For Example 8-1

Solution The flux through the semi spherical surface is difficulty to be calculated with integral of Eq. (8-3) because the angle θ between the element area normal vector d**S** and magnetic field **B** is not a constant. However, if we add a circle plane and form a closed surface, by Gauss' law, the net flux through any closed surface is zero, so that the flux through the semi sphere surface equals to the negative value of the circle plane. Hence, the flux of semi spherical surface is

$$\Phi_m = \pi R^2 B\cos\alpha$$

8.3 Magnetic Fields Set Up by a Current and a Moving Charge

8.3.1 Biot-Savart's law and its applications

Shortly after Oersted's discovery in 1819, two French scientists, Jean-Baptiste Biot (1774—1862) and Felix Savart(1791—1841) performed quantitative experiments on the force exerted by a straight current carrying wire on a nearby magnet. Their experimental results are as follows:

(1) The magnitude of **B** is proportional to the current and is inversely proportional to the distance from the straight current.

(2) The direction of the **B** is perpendicular to the current and located in the plane that is perpendicular to the current.

These observations are summarized in the mathematical expression known today as the Biot-Savart's law

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{I\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3} \tag{8-6}$$

where μ_0 is a constant called the permeability of vacuum, and

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T \cdot m/A} \tag{8-7}$$

Notice that the field d$\boldsymbol{B}$ in Eq. (8-6) is the field created at a point by the current of length element dl of the conductor. $I\mathrm{d}\boldsymbol{l}$ is called current element which has magnitude of $I\mathrm{d}l$, and in the direction of the current. The direction of d$\boldsymbol{B}$ is determined by the cross product of vectors $I\mathrm{d}\boldsymbol{l}$ and $\boldsymbol{r}$ ($I\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}$), and can be determined by right-hand rule. Fig. 8-7 shows the direction of d$\boldsymbol{B}$ at point P created by the element current $I\mathrm{d}\boldsymbol{l}$, and the magnitude of d$\boldsymbol{B}$ is

$$\mathrm{d}B = \frac{\mu_0}{4\pi} \frac{I\mathrm{d}l\sin\theta}{r^2} \tag{8-8}$$

where θ is the angle between vector $\boldsymbol{r}$ and the current element $I\mathrm{d}\boldsymbol{l}$.

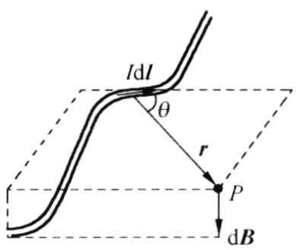


Fig. 8-7 The magnetic field d$\boldsymbol{B}$ produced by an element current $I\mathrm{d}\boldsymbol{l}$

To find the total magnetic field $\boldsymbol{B}$ created at some point P by a current-carrying conductor of finite length, we must sum up the contributions of all current elements $I\mathrm{d}\boldsymbol{l}$ of the current-carrying conductor, that is by integrating d$\boldsymbol{B}$ in Eq. (8-6)

$$\boldsymbol{B} = \int \mathrm{d}\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \frac{I\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3} \tag{8-9}$$

where the integral is taken over the entire current distribution. This expression must be dealt with special care because the integral is a cross product and therefore a vector quantity. The above vector integration is often treated by resolving d$\boldsymbol{B}$ into perpendicular components and then the components are integrated. For the case of two dimensions, that is

$$\boldsymbol{B}_x = \int \mathrm{d}\boldsymbol{B}_x,\quad \boldsymbol{B}_y = \int \mathrm{d}\boldsymbol{B}_y,\qquad \boldsymbol{B}_{/\!/} = \int \mathrm{d}\boldsymbol{B}_{/\!/},\quad \boldsymbol{B}_{\perp} = \int \mathrm{d}\boldsymbol{B}_{\perp} \tag{8-10}$$

Some examples will illustrate the method to find magnetic field $\boldsymbol{B}$ in the following.

Example 8-2 Find the magnetic field $\boldsymbol{B}$ surrounding a thin, straight wire of length L, carrying a current I, as shown in Fig. 8-8.

Solution We must find the field contribution from a current element $I\mathrm{d}\boldsymbol{l}$ and then integrate over the current distribution. The distance from some point P to the current is a, and for an arbitrary current element $I\mathrm{d}\boldsymbol{l}$ on the wire, as shown in Fig. 8-8, the direction of d$\boldsymbol{B}$ at point P due to the current element is perpendicularly into the page, represented by a cross. In fact, all the current elements produce a magnetic field directed into the page at point P. Therefore, we need only integrate the magnitude of d$\boldsymbol{B}$. From Biot-Savart's law, the magnitude of d$\boldsymbol{B}$ is

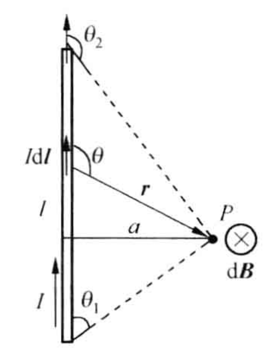


Fig. 8-8 For Example 8-2

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2}$$

where θ, l and r are three variables. From the geometry relations in Fig. 8-8, expressing r and l in terms of θ, we have

$$r = \frac{a}{\cos\theta}, \quad l = -a\cot\theta$$

From which we find the differential dl

$$dl = a\frac{d\theta}{\sin^2\theta}$$

Substituting the relations into the magnitude of $d\boldsymbol{B}$, we obtain

$$dB = \frac{\mu_0 I}{4\pi a}\sin\theta d\theta$$

Integrating dB over the wire, where the angle ranges from θ_1 to θ_2 as defined in Fig. 8-8, we have

$$B = \int dB = \frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2}\sin\theta d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_1 - \cos\theta_2) \tag{8-11}$$

As the length of the wire becomes infinite, we see that $\theta_1 = 0$ and $\theta_2 = \pi$, and Eq. (8-10) becomes

$$B = \frac{\mu_0 I}{2\pi a} \tag{8-12}$$

which is the magnitude of the magnetic field around a long (infinite) straight wire carrying a current I and a is the perpendicular distance from the wire. Its direction can be determined by the right-hand rule: grasp the wire in your right hand with your out-stretched thumb pointing in the direction of the current and your fingers will then naturally curl around in the direction of the magnetic field. Therefore, the magnetic field lines are circles of different radii about the wire.

Example 8-3 A circular current loop of radius R and carrying a steady current I is shown in Fig. 8-9. Find the magnetic field at point P on the axis of the loop a distance x from its center.

Solution Fig. 8-9 shows the magnetic field contribution $d\boldsymbol{B}$ at point P due to a current element at the top of the loop. This field vector can be resolved into components $d\boldsymbol{B}_x$ and $d\boldsymbol{B}_\perp$, and $d\boldsymbol{B}_x$ is parallel to the axis of the loop and $d\boldsymbol{B}_\perp$ is perpendicular to the axis. Consider the element magnetic field contributions from another current element at the bottom of the loop. Because of the symmetry of the situation, the perpendicular components of the field $d\boldsymbol{B}_\perp$ due to current elements at the top and bottom of the loop cancel out. This cancellation occurs for all pairs of elements around the loop on the opposite positions, so we can calculate magnitude of $\boldsymbol{B}$ by algebraic sum of the magnitude of the parallel components $d\boldsymbol{B}_x$. From geometry, every element current $Id\boldsymbol{l}$ is perpendicular to the vector $\boldsymbol{r}$, and r is a constant. Therefore, for any element, $|Id\boldsymbol{l} \times \boldsymbol{r}| = Idlr\sin 90° = Irdl$, and the magnitude of $d\boldsymbol{B}$ due to the current element is given by

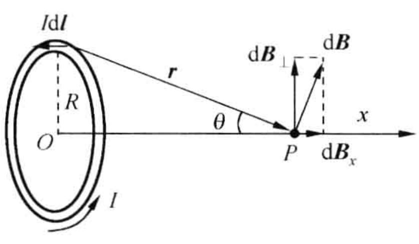


Fig. 8-9 For Example 8-3

$$dB = \frac{\mu_0 Idl}{4\pi r^2}$$

The x component dB_x is

$$dB_x = \frac{\mu_0 I dl}{4\pi r^2}\sin\theta$$

Form Fig. 8-9, $r^2 = R^2 + x^2$ and $\sin\theta = \dfrac{R}{\sqrt{R^2+x^2}}$. Integrating around the loop, we have

$$B = B_x = \int dB_x = \frac{\mu_0 I}{4\pi r^2}\sin\theta \oint dl = \frac{\mu_0 IR^2}{2\,(R^2+x^2)^{\frac{3}{2}}} \qquad (8\text{-}13)$$

To find the magnetic field at the center of the loop, we set $x=0$ in Eq. (8-13) and have

$$B = \frac{\mu_0 I}{2R} \quad \text{(at center of the loop)} \qquad (8\text{-}14)$$

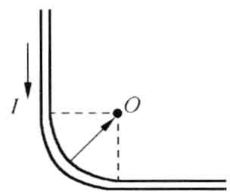


Fig. 8-10 For Example 8-4

Example 8-4 As shown in Fig. 8-10, the wire carries a current I, consisting of two very long, straight sections and a quarter-circle with radius R. What are the magnitude and direction of the magnetic field at the center O of the quarter-circle section?

Solution The magnetic field is the vector sum of three sections of current, two half infinite currents and a quarter-circle current. The directions of their magnetic fields are the same and directed out of the page. The magnetic field at point O is given by

$$B_O = \frac{\mu_0 I}{4\pi R} + \frac{1}{4}\frac{\mu_0 I}{2R} + \frac{\mu_0 I}{4\pi R} = \frac{\mu_0 I}{8R}\left(\frac{4+\pi}{\pi}\right)$$

8. 3. 2 Magnetic field of a moving charge

A point charge q, moving with velocity $\boldsymbol{v}$, produces a magnetic field $\boldsymbol{B}$ in space given by

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\boldsymbol{r}}{r^3} \qquad (8\text{-}15)$$

where $\boldsymbol{r}$ points from the charge q to the point as shown in Fig. 8-11 and its magnitude is

$$B = \frac{\mu_0}{4\pi}\frac{qv\sin\theta}{r^2} \qquad (8\text{-}16)$$

Eq. (8-15) and Eq. (8-16) show that

(1) The magnitude of $\boldsymbol{B}$ is proportional to the product of the charge q and the speed of $\boldsymbol{v}$ and varies inversely with the square of the distance from the charge.

(2) The magnitude of $\boldsymbol{B}$ is proportional to $\sin\theta$, where θ is the angle between the velocity $\boldsymbol{v}$ and the position vector $\boldsymbol{r}$ from the charge to the point.

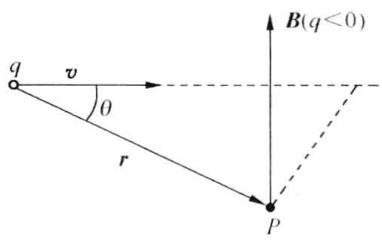


Fig. 8-11 The magnetic field of a moving charge q

(3) The direction of $\boldsymbol{B}$ is perpendicular to both the velocity $\boldsymbol{v}$ and the position vector $\boldsymbol{r}$, determined by the right-hand rule and the sign of the charge q. If $q>0$, the direction of $\boldsymbol{B}$ is the same as that of $\boldsymbol{v}\times\boldsymbol{r}$; if $q<0$, the direction of $\boldsymbol{B}$ is opposite to that of $\boldsymbol{v}\times\boldsymbol{r}$.

Eq. (8-16) implies that the magnetic field is zero along the line of motion of the charge.

Example 8-5 In the Bohr model of the hydrogen atom, the electron of charge $-e$ in the ground state is revolving in a circular orbit with a speed of 2.19×10^6 m/s. Its orbital radius r is 5.29×10^{-11} m. Find the magnitude of the magnetic field $\boldsymbol{B}$ due to the moving electron at the center of the orbit.

Solution From Eq. (8-16), since $\sin\theta = 1$, we have

$$B = \frac{\mu_0}{4\pi}\frac{ev}{r^2} = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}) \cdot (1.6 \times 10^{-19}\ \mathrm{C}) \cdot (2.19 \times 10^6\ \mathrm{m/s})}{4\pi \cdot (5.29 \times 10^{-11})^2}$$

$$= 12.5\mathrm{T}$$

8.4 Ampere's Law

Gauss' law for electric fields can be used to calculate the electric field easily in situations with a highly symmetric charge distribution. Is there a law for magnetic field that allows us to find the magnetic fields caused by highly symmetric current distributions? However, Gauss' law for magnetic field is not a relation between magnetic field and current distribution; it states that the flux of $\boldsymbol{B}$ through any closed surface is always zero, whether or not there are currents within the surface. So Gauss' Law for $\boldsymbol{B}$ can't be used to determine the magnetic field $\boldsymbol{B}$ produced by a particular current distribution.

8.4.1 Ampere's Law

Recalled that the electric field is conservative field, that is, the integral of $\boldsymbol{E}$ around any closed path is zero. By contrast, the magnetic field lines are closed so that if we perform the line integral of $\boldsymbol{B}$ along one of the magnetic field lines, the result is not zero. It can be proved that

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \sum I \qquad (8\text{-}17)$$

which is called Ampere's (circuital) law. Here, $\sum I$ is the net current penetrates the area bounded by a closed path l, called an Amperian loop. A plus sign or a minus sign to each of the currents $\sum I$ are determined by the right-hand rule: curling your right hand around the closed path l with the fingers along the direction of integration, you assign a plus sign to a current through the loop in the general direction of your out-stretched thumb and a minus sign to a current generally in the opposite direction.

To testify Ampere's Law, let's consider the magnetic field produced by a long, straight wire carrying a current I. We know that the field at a distance r from the conductor has magnitude $B = \dfrac{\mu_0 I}{2\pi r}$, and that the magnetic field lines are circles centered on the wire. Let's take the line integral of $\boldsymbol{B}$ around such circle with radius r, which is overlapping completely with the magnetic field lines. As Fig. 8-12(a) shows, at every point on the circle, $\boldsymbol{B}$ and $\mathrm{d}\boldsymbol{l}$ are parallel, and so $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B\mathrm{d}l$. Since r is constant around the circle and B is constant, we can take B out of the integral. The remaining integral $\oint_l \mathrm{d}l$ is just the circumference of the circle, so

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \oint_l B\mathrm{d}l = B\oint_l \mathrm{d}l = \frac{\mu_0 I}{2\pi r}2\pi r = \mu_0 I$$

The integral is thus independent of the radius of the circle and is equal to μ_0 multiplied by the current passing through the area bounded by the circle.

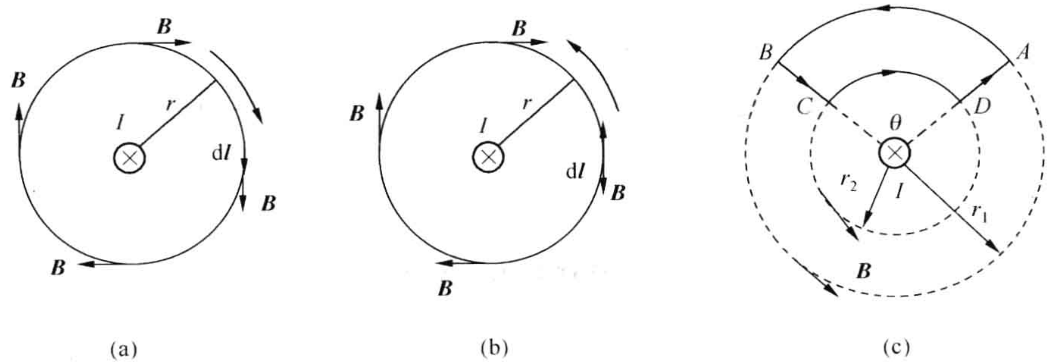


Fig. 8-12 The line integral of **B** along a closed path in the magnetic field of a long straight current

In Fig. 8-12(b), the situation is the same, but the direction of the line integral is in the opposite direction. Now **B** and d**l** are anti-parallel so that $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = -B\mathrm{d}l$, and the integral equals $-\mu_0 I$. Hence, the integral equals μ_0 multiplied by the current passing through the area bounded by the integration path, with a positive or negative sign on the direction of the current relative to the direction of integration.

If the current is not passing the area bounded by the integration path $ABCDA$ as shown in Fig. 8-12(c), the path can be divided into AB, BC, CD and DA. Along the circular arc AB of radius r_1, $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \frac{\mu_0 I}{2\pi r_1}\mathrm{d}l$; along the circular arc CD of radius r_2, $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mu_0 I}{2\pi r_2}\mathrm{d}l$; and **B** is perpendicular to d**l** at each point on the straight sections BC and DA, so $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B\mathrm{d}l\cos\left(\frac{\pi}{2}\right) = 0$. The total line integral is then

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \int_A^B \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} + \int_B^C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} + \int_C^D \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} + \int_D^A \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \int_A^B \frac{\mu_0 I}{2\pi r_1}\mathrm{d}l + 0 + \int_A^B \left(-\frac{\mu_0 I}{2\pi r_2}\right)\mathrm{d}l + 0$$

$$= \frac{\mu_0 I}{2\pi r_1}(\theta r_1) - \frac{\mu_0 I}{2\pi r_2}(\theta r_2) = 0$$

Even through there is a magnetic field everywhere along the integration path, the line integral $\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}$ is zero if there is no current passing through the area bounded by the path.

For more strict mathematical derivation, it can be generalized to any non-plane closed path and any shaped steady current, the above relation between the current and the line integral of its magnetic field **B** always holds.

Ampere's law is the fundamental law describing how electric currents create magnetic fields in the surrounding empty space. Like Gauss' law of electric field, the Ampere's law can be used to obtain the magnetic field in situations that the magnetic field has a high degree of symmetry. If the symmetry is great enough, the B can be taken out of the line integral and the integral $\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}$ becomes very simple so that the B can be solved. Two examples are given as following.

8.4.2 Application of Ampere's Law

Example 8-6 A long straight tightly wound solenoid, which is circular in cross section as shown in Fig. 8-13, has n turns of wire per unit length and carries a current I. Find the magnetic field within the solenoid.

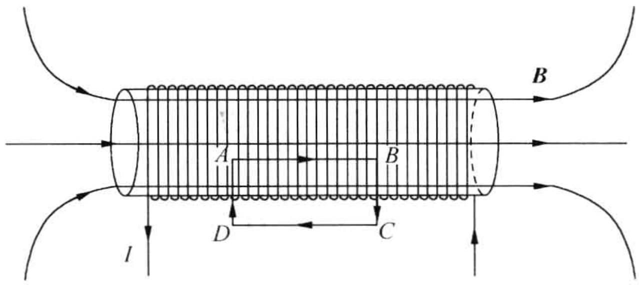


Fig. 8-13 For Example 8-5

Solution The field lines within the solenoid are approximately parallel, indicating a nearly uniform field, and the magnetic field is weak in the region near the outside of the solenoid. We then choose a rectangle $ABCDA$ as Amperian loop, as shown in Fig. 8-13. Section AB, with length L_{AB}, is parallel to the axis of the solenoid. Section CD is outside but very near the solenoid, and the magnetic field is negligible small. Along sections BC and DA, because $\boldsymbol{B}$ is perpendicular to these section, then $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = 0$. The integral around Amperian loop $ABCDA$ therefore reduces to

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \int_A^B \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = BL_{AB}$$

Since the number of turns in length L_{AB} is nL_{AB} and each of these turns passes once through the rectangle Amperian loop $ABCDA$, the total current enclosed by the rectangle is then $nL_{AB}I$. And the direction of the current is right-hand rule with the circular direction of Amperian loop $ABCDA$, then the current $nL_{AB}I$ is positive, and Ampere's law gives

$$BL_{AB} = \mu_0 nL_{AB}I$$

Solving for B, we have

$$B = \mu_0 nI \tag{8-18}$$

This result also proves that the field is uniform over entire cross section at the center of the solenoid's length.

Example 8-7 Fig. 8-14 shows a doughnut-shaped toroidal solenoid, also called a toroid, tightly wound with N turns of wire carrying a current I. Find the magnetic field at all point.

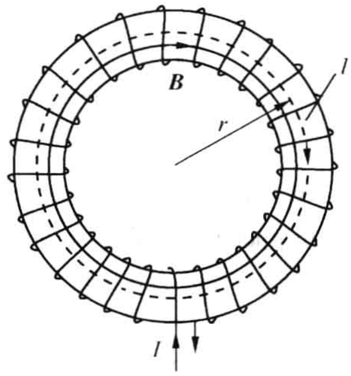


Fig. 8-14 For Example 8-6

Solution By circle symmetry, the field lines (the solid line in the figure) must be circles concentric with the axis of the toroid. The field is zero everywhere outside the toroid. To find the field, we can use Ampere's Law. We take the circle path l that is overlapped completely with the magnetic field lines as Amperian loop l(the dashed line in the figure). The direction of $\mathrm{d}\boldsymbol{l}$ is parallel to $\boldsymbol{B}$ at all points on the path, so $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B\mathrm{d}l$. Since the magnitude of $\boldsymbol{B}$ is constant on the loop, B can be moved out the integral, and the integral $\oint_l \mathrm{d}l = 2\pi r$. Therefore, the integral around Amperian loop l gives

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \oint_l B\mathrm{d}l = B\oint_l \mathrm{d}l = 2\pi rB$$

Each turn of the wire passes through the area bounded by Amperian loop l, and the total current through the area is NI, where N is the total number of the turns. Then, from Ampere's Law, we obtain

$$2\pi rB = \mu_0 NI$$

Finally, we have

$$B = \frac{\mu_0 NI}{2\pi r} \tag{8-19}$$

It is different from that of solenoid. Here the magnetic field $\boldsymbol{B}$ is not constant over the cross section of the toroid, and the magnitude of $\boldsymbol{B}$ is larger at the inner side than at outside within the toroid.

8.5 Motion of a Charged Particle in a Magnetic Field

When a charged particle moves in a magnetic field, it is acted on by Lorentz force of Eq. (8-1), and its motion is determined by Newton's law. An important characteristic of the Lorentz force is that it is always perpendicular to the velocity of the charged particle. Therefore, the magnetic force on a charged particle changes the direction of velocity but not its magnitude, and it does no work on the particle and does not affect the kinetic energy of the particle.

Let's discuss a special case that the magnetic field is uniform. We first consider the simple situation: the charged particle's velocity $\boldsymbol{v}$ perpendicular to the uniform magnetic field with a magnitude of B. So the Lorentz force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ has magnitude $F = qvB$ and the force $\boldsymbol{F}$ is perpendicular to vector $\boldsymbol{v}$ and $\boldsymbol{B}$, as shown in Fig. 8-15. The force is always perpendicular to $\boldsymbol{v}$, so it is centripetal force, and then the particle moves along a circle at the constant speed of v. Assume the radius of the orbit is r, then the centripetal acceleration is v^2/r, and Newton's second law gives

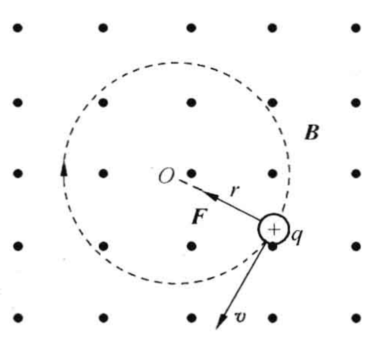


Fig. 8-15 A charged particle moves in a magnetic field directed out of the plane of the page and indicated by the dots

$$F = qvB = m\frac{v^2}{r} \tag{8-20}$$

where m is the mass of the charged particle. Solving for radius r, we find

$$r = \frac{mv}{qB} \tag{8-21}$$

That is, the radius of the path is proportional to the linear momentum of the particle and inversely proportional to the magnitude of the charge of the particle and the magnitude of the magnetic field. The angular speed of the particle is

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{8-22}$$

The period of the motion is equal to the circumference of the circle divided by the speed of the particle

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \tag{8-23}$$

Eq. (8-22) and Eq. (8-23) show that the angular speed of the particle and the period of the circular motion do not depend on the speed of the particle or on the radius of the orbit. For a given magnetic field, all particles with the same mass-charge ratio have the same period T.

If the direction of the initial velocity is not perpendicular to the field $\boldsymbol{B}$, as shown in Fig. 8-16, the velocity can be resolved into two components, the parallel component $v_{//}$ to the magnetic field and the component $v_{\perp}$ perpendicular to the field. Along the direction of the magnetic field, no force acts on the particle, the component $v_{//}$ remains constant. However, the particle is acted by Lorentz force due to the component $v_{\perp}$ of the velocity, which causes the particle moving in a circle. So, the resultant motion path of the particle is a helix, and the radius of the helix is given by Eq. (8-21).

The radius r and the pitch of the helix are

$$r = \frac{mv\sin\theta}{qB} \tag{8-24}$$

$$h = v_{//}\,T = \frac{2\pi mv\cos\theta}{qB} \tag{8-25}$$

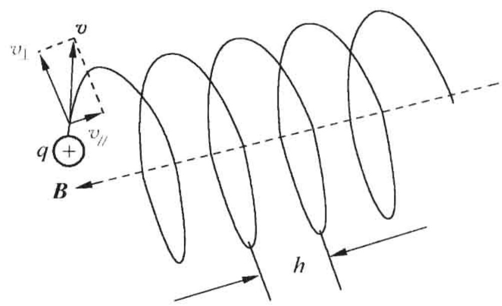


Fig. 8-16 The helix path of a moving particle

Motion of a charged particle in a non-uniform magnetic field is quite complicated. The radius of the helix may vary with the magnitude of the field and the axis of the helix may be not a straight line, so it may be deflected along the direction of magnetic field. Fig. 8-17 shows the earth's non-uniform magnetic field traps charged particles coming from the space in dot-shaped region around the earth. These regions are called the Van Allen radiation belts. The fact that the charged particle can be trapped in a non-uniform magnetic field is used to design the magnetic bottle, which is used to confine very hot plasmas with temperatures of the order of 10^6 K, as shown in Fig. 8-18.

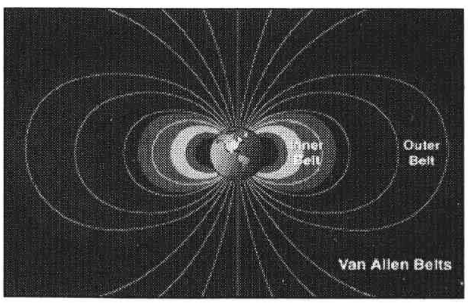


Fig. 8-17 The Van Allen radiation belts

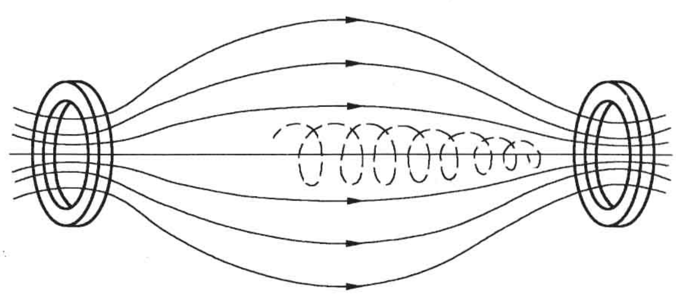


Fig. 8-18 Magnetic bottle used to confine the hot plasmas

8.6 Magnetic Force on Current-carrying Conductors

When a current-carrying conductor lies in a magnetic field, magnetic forces are exerted on the moving charges within the conductor. These forces are transmitted to the material of the conductor through the collision with the lattice array of the conductor. That is there is a force on the wire acted by the magnetic field, which is called Ampere force. The moving coil in the galvanometer uses magnetic forces on the conductor is a good example.

We know that the magnitude of magnetic force $\boldsymbol{F}$ (usually is called Ampere force) on a straight current wire in a uniform magnetic field $\boldsymbol{B}$ is

$$F = BlI\sin\theta$$

where l is the length of the conductor, I is the current in the wire, and θ is the angle between the direction of $\boldsymbol{B}$ and current. The magnetic force can be expressed in a cross product of vectors $\boldsymbol{B}$ and $I\boldsymbol{l}$, that is

$$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B}$$

where $\boldsymbol{l}$ is a vector in the direction of the current with a magnitude equal to the length of the wire. The direction of $\boldsymbol{F}$ is determined by the right-hand rule.

For a conductor in an arbitrary shape in the non-uniform magnetic field, the conductor can be divided into infinite current elements $I\mathrm{d}\boldsymbol{l}$, as shown in Fig. 8-19, and in the location of element current, the magnetic field can be regarded as uniform. Then the element force exerted on the current $I\mathrm{d}\boldsymbol{l}$ is given by

$$\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{l} \times \boldsymbol{B} \tag{8-26}$$

So we can integrate $\mathrm{d}\boldsymbol{F}$ along the wire to find the total force on the conductor to be

$$\boldsymbol{F} = \int \mathrm{d}\boldsymbol{F} = \int I\mathrm{d}\boldsymbol{l} \times \boldsymbol{B} \tag{8-27}$$

The integral is a line integral.

Example 8-8 Fig. 8-20 shows that a wire segment in the shape of semicircle of radius R and carrying a current I is placed in a uniform magnetic field $\boldsymbol{B}$ that perpendicularly points into the plane of the page. What is the resultant magnetic force acts on the wire?

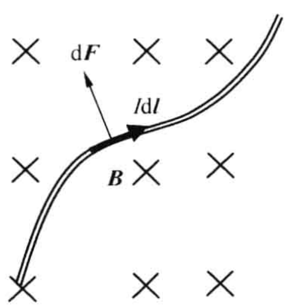


Fig. 8-19 Ampere force

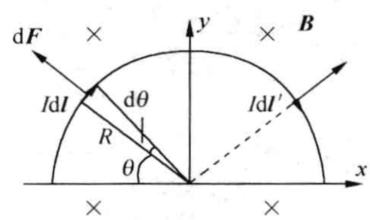


Fig. 8-20 For Example 8-7

Solution A current element of the arc of length $\mathrm{d}l = R\mathrm{d}\theta$ is exerted by a force $\mathrm{d}\boldsymbol{F}$ directing radically outward from the center. Note that the component $\mathrm{d}F_x$ in x direction is canceled by another current element located on the symmetrical position (the current element $I\mathrm{d}\boldsymbol{l}'$ on right side). Considering that the current element $Id\boldsymbol{l}$ and magnetic field $\boldsymbol{B}$ are perpendicular, thus the magnitude of $d\boldsymbol{F}$ is given by $dF = IBR d\theta$, and the component in y direction is $dF_y = dF\sin\theta$. The total force on the semicircle wire is then

$$F = \int dF_y = \int_0^{\pi} BIR\sin\theta d\theta = -BIR\cos\theta\Big|_0^{\pi} = 2BIR$$

And the direction of the $\boldsymbol{F}$ points in the positive y direction.

8.7 Magnetic Torque on a Current-carrying Loop

An electric motor is a kind of machine that is widely used in industry and daily life, which transforms the electric energy to the mechanical energy. The motor's current carrying loops are driven by a torque due to the Ampere force acting on the loops located in a magnetic field.

Fig. 8-21 shows a rectangular loop of wire with sides of lengths l_1 and l_2, carrying the current I in a uniform magnetic field $\boldsymbol{B}$. The orientation of the loop plane can be described conveniently by a unit vector $\boldsymbol{n}$ that is perpendicular to the plane of the loop. The direction of $\boldsymbol{n}$ is determined by the right-hand rule as follows: when the fingers of right hand curl around the loop, with the figures pointing in the direction of the current, the thumb points in the direction of $\boldsymbol{n}$, as illustrated in Fig. 8-21. From the figure, the normal of the plane of the loop makes an angle θ with the direction of the field $\boldsymbol{B}$ at the time t. For simplicity, the other part of the loop is omitted. Forces on the side of BC and DA, represented by $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$, are of the same magnitude but opposite in direction, so the net force and net torque cancelled each other. The forces on side AB and CD, represented by $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, respectively and having a value of BIl_1, are also of the same magnitude, but lie along different lines, thus the net force of them is zero. However, the net torque about an axis (since they form a couple, it is convenient to choose an axis to be parallel to AB or CD and passes through the centers of BC and DA), is not zero so that the loop will rotate. The magnitude of the torque is

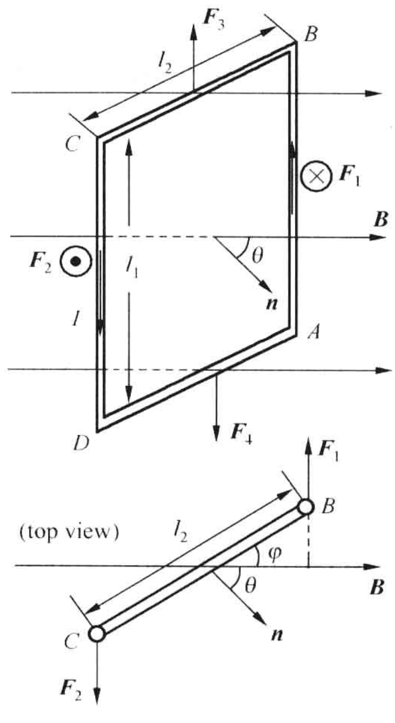


Fig. 8-21 Magnetic torque of a current carrying loop

$$M = BIl_1 l_2 \sin\theta = BIS\sin\theta$$

For a coil with N turns, the torque is

$$M = NBIS\sin\theta$$

This torque tends to twist the loop so that its plane is perpendicular to $\boldsymbol{B}$.

Let's define magnetic moment of the loop as

$$\boldsymbol{p}_m = NIS\boldsymbol{n} \tag{8-28}$$

where $\boldsymbol{n}$ is the unit normal vector of the loop plane. So, we can express the net torque vector $\boldsymbol{M}$ in a cross product form

$$\boldsymbol{M} = \boldsymbol{p}_{\mathrm{m}} \times \boldsymbol{B} \tag{8-29}$$

The magnitude of $\boldsymbol{M}$ is

$$M = p_{\mathrm{m}} B \sin\theta$$

Eq. (8-28) and Eq. (8-29), derived for a rectangular loop, holds in general for a coil of any plane shape with an area of S.

It is not difficulty to see that the magnetic torque $\boldsymbol{M}$ has maximum value when $\boldsymbol{p}_{\mathrm{m}}$ and $\boldsymbol{B}$ are perpendicular, and the $\boldsymbol{M}$ is zero when $\boldsymbol{p}_{\mathrm{m}}$ and $\boldsymbol{B}$ are parallel or anti parallel, thus the torque $\boldsymbol{M}$ tends to rotate the current-carrying loop plane to the position perpendicular to $\boldsymbol{B}$. In practice, it is the magnetic torque to make a motor rotate and work; it is the magnetic torque to make ammeter and voltmeter work.

When a small permanent magnet such as a compass needle is placed in a magnetic field, there are two forces: $\boldsymbol{F}_1$ on the north pole in the direction of $\boldsymbol{B}$ and an equal but opposite force $\boldsymbol{F}_2$ on the south pole. These two forces produce no translational motion but they do produce a torque that tends to rotate the magnet so that it lines up with the field like a current-carrying loop. That is because the magnetic moment of a bar magnet is due to microscopic current loops that result from the motion of electrons in the atoms of the magnet.

*8.8 The Hall Effect

The moving charges in a magnetic field will deflect in transverse direction by action of Lorentz force. So, when the a current-carrying wire is in an external magnetic field, the moving charges in the wire are driven by the Lorentz forces toward the transverse edge of the conductor to result a transverse electrostatic field, and further cause a transverse potential voltage between the opposite edges of the conductor. This effect is called Hall effect , and the transverse voltage is called Hall voltage .

To describe this effect, let's consider a conductor in the form of a flat strip, as shown in Fig. 8-22(a). The current is in the right direction, and there is a uniform magnetic field $\boldsymbol{B}$ perpendicular to the plane of the strip, into the page. The drift velocity of the moving positive charges has magnitude v_{d}, and the magnetic force is upward. The positive charges are driven to the upper edge of the strip by Lorentz force $F = qv_{\mathrm{d}}B$ and produce a transverse electric field $\boldsymbol{E}$. This accumulation continues until the transverse electrostatic field $\boldsymbol{E}$ becomes large enough to cause an electric force $q\boldsymbol{E}$ that is equal and opposite to the magnetic force. After that, a steady state is reached, and we have

$$qE = qv_{\mathrm{d}}B \quad \text{or} \quad E = v_{\mathrm{d}}B \tag{8-30}$$

This electric field causes a transverse potential difference between opposite edges A and B of the strip, that is Hall voltage or the Hall emf. The polarity of the Hall emf depends on whether the moving charges are positive or negative, compare Fig. 8-22(a) with Fig. 8-22(b).

Suppose there are n charged particles per unit volume. The width of the conductor is d and the thickness is b. Assume that all the particles move with the same drift velocity with magnitude v_{d}. In a time interval $\mathrm{d}t$, each particle moves a distance $v_{\mathrm{d}}\mathrm{d}t$. The amount of the

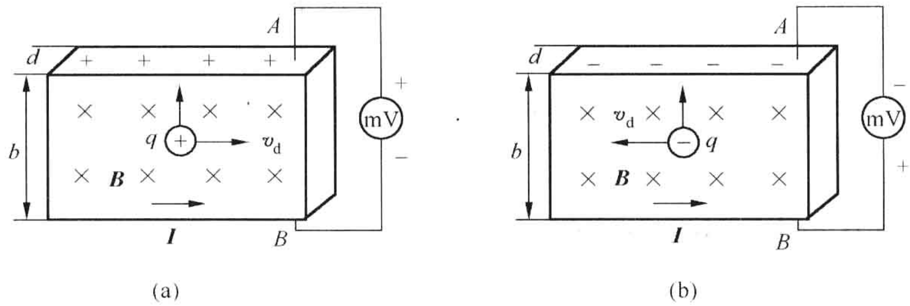


Fig. 8-22 Hall Effect: (a) positive carriers; (b) negative carriers

charge dQ passes the cross-section of the conductor during time dt is nAv_ddt and the current is

$$I = \frac{dQ}{dt} = nqv_dA \tag{8-31}$$

Notice that $A = db$ is the cross-sectional area and $V_{AB} = Eb$. Eliminating v_d between these equations, we find

$$V_{AB} = \frac{1}{nq}\frac{IB}{d} \tag{8-32}$$

Let $K = 1/(nq)$, which is called Hall coefficient.

If we know Hall coefficient K, by measuring I, V_{AB} and d, we can compute the magnetic field $\boldsymbol{B}$; this is a major application of Hall effect. And the device is called Hall sensor. On the other hand, the Hall effect permits us to measure the charge concentration n as well as the drift speed v_d of the charged particles. The charge density n of a semiconductor is smaller than that of a metal, so the Hall sensor is almost made from semiconductors.

8.9 Magnetic Material

8.9.1 The classifications of magnetic media

In chapter 7 we have known that when a dielectric is placed in an external electric field, it will be polarized, and the electric field will then be influenced and varied. Similarly, when a magnetic medium is brought into a magnetic field, it will be magnetized, and the magnetic field will also be influenced and changed. For example, when an iron core is inserted into a solenoid, the magnetic field will increase greatly with the same current in the windings.

Suppose $\boldsymbol{B}'$ is the additional magnetic field set up by the magnetized medium and $\boldsymbol{B}_0$ is the external magnetic field which would be present if the magnetic medium were not in place. According to superposition principle of magnetic field, the actual magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}' \tag{8-33}$$

The direction of $\boldsymbol{B}'$ varies with the mediums; if the direction of $\boldsymbol{B}'$ is the same as that of $\boldsymbol{B}_0$, the substance is called paramagnetic medium, such as, Mn, Cr, Pt, N. etc.; if the direction of $\boldsymbol{B}'$ is opposite to that of $\boldsymbol{B}_0$, the material is called diamagnetic medium, and Hg, Cu, Bi, H. etc., belong to this family.

The magnitude of additional magnetic field $\boldsymbol{B}'$ of all of diamagnetic mediums and most paramagnetic mediums is much smaller than the value of $\boldsymbol{B}_0$. There are some special paramagnetic substances such as iron, carbon steel, cobalt steel, Alnieo2, Alnieo5 and ferrite, for which the magnitude of the additional magnetic field $\boldsymbol{B}'$ is much greater than that of $\boldsymbol{B}_0$. They are called ferromagnetic materials.

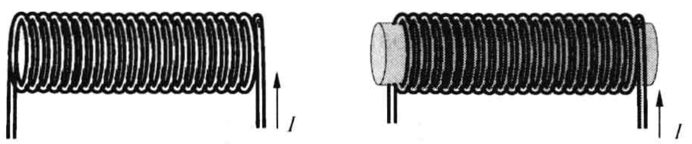
8.9.2 Permeability and relative permeability

Similar with the definitions of permittivity and relative permittivity, the ratio of the magnitude of $\boldsymbol{B}$ to the magnitude of $\boldsymbol{B}_0$ is termed relative permeability of material and presented by μ_r, so that

$$\mu_r = \frac{B}{B_0} \tag{8-34}$$

The quantity μ_r is a pure number without unit as ε_r of dielectric.

The value of μ_r can be measured, for example, by the experiment illustrated in Fig. 8-23(a) and Fig. 8-23(b). When the solenoid is empty (without core) in Fig. 8-23(a), the magnitude of $\boldsymbol{B}_0$ is $B_0 = \mu_0 nI$, where n is the number of the turns of windings wound on the solenoid per unit length, I is the current carried by the winding. Then the magnetic medium sample to be tested is inserted into the solenoid, as Fig 8-23(b) shows, with the current I unchanged, and the magnitude of $\boldsymbol{B}$ in the specimen can be measured by a special method (for example, by magnetic induction method discussed in next chapter). From Eq. (8-34), the relative permeability μ_r can be determined using the experimental values of $\boldsymbol{B}_0$ and $\boldsymbol{B}$.



(a) (b)

Fig. 8-23 Solenoid: (a) without core; (b) with magnetic medium

From Eq. (8-34), we have

$$B = \mu_r B_0 = \mu_r \mu_0 nI \tag{8-35}$$

If we define

$$\mu = \mu_r \mu_0 \tag{8-36}$$

Then Eq. (8-35) becomes

$$B = \mu nI \tag{8-37}$$

Here μ is called the permeability of material. The unit of μ is the same as that of μ_0, namely, H/m.

The relative permeability of some materials under room temperature is given in Table 8-1. From this table, you can see that, for paramagnetic materials, μ_r is slightly larger than 1; for diamagnetic materials, μ_r is slightly smaller than 1; for ferromagnetic materials, μ_r is not a constant, which means that the property of the ferromagnetism is more complicated than those of paramagnetic and diamagnetism.

Table 8-1 **Relative permeability of some materials under room temperature**

Paramagnetic	μ_r	Diamagnetic	μ_r	Ferromagnetic	μ_r
Mg	1.000012	Bi	0.999834	Iron	$\leqslant 1.99\times 10^4$
Al	1.000023	Hg	0.999968	Cool cut silicon steel	$\leqslant 7.96\times 10^3$
W	1.000068	Ag	0.999974	Iron-cobalt soft	$6.94\times 10^4\sim$
Ti	1.000071	Cu	0.999990	magnetic alloy	8.95×10^4

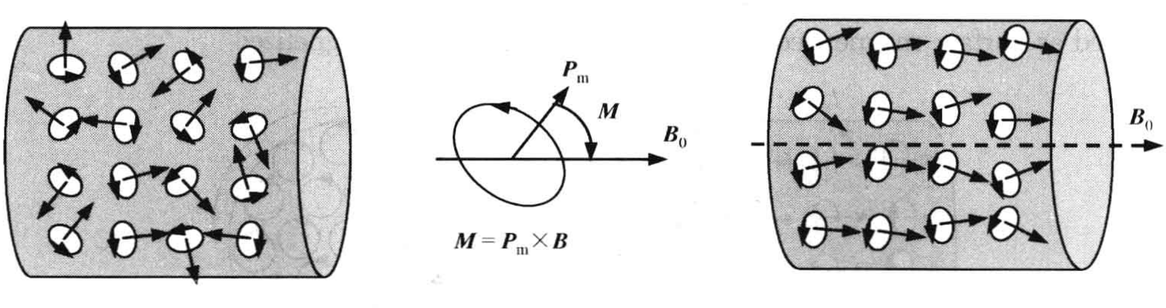
In the following, we will explain the magnetization mechanism of paramagnetic and diamagnetic materials on the micro scale, and that of ferromagnetism will be discussed in section 8-10.

8.9.3 Molecular theory of paramagnetism and diamagnetism

According to modern quantum theory, we know that the electrons revolve about the nuclei while spinning. Furthermore the nuclei itself is also in the motion of spin. Thus each motion is equivalent to a tiny current loop corresponding to a magnetic moment, so that the total magnetic effect of all these equivalent tiny current loops of an atom or a molecular can be regarded as an equivalent current called molecular current. The magnetic moments of a molecular current is called molecular magnetic dipole moment, represented by $\boldsymbol{P}_{\mathrm{m}}$.

1. Alignment of the magnetic dipole moment in a magnetic field

For paramagnetic materials, the equivalent molecular magnetic moments $\boldsymbol{P}_{\mathrm{m}}$ are not equal to zero and interact with each other only very weakly, but they are randomly oriented due to the thermal motion of molecules, so that when there is no external magnetic field, paramagnetic material exhibits no net magnetic effect, as shown in Fig. 8-24(a). As it is placed in an external field $\boldsymbol{B}_0$, the magnetic dipole moment of every atom or molecule is acted by a magnetic torque (Fig. 8-24(b)), which forces them to align with the external magnetic field, as illustrated in Fig. 8-24(c). Since the thermal motion of molecules is always a disrupting factor, the alignment is not perfect. The result is that the magnetic field at every point in such material is added up by an additional magnetic field $\boldsymbol{B}'$, as shown in Fig. 8-24(c). Since $\boldsymbol{B}'$ is smaller in magnitude comparing with that of $\boldsymbol{B}_0$, the magnitude of the resultant magnetic field ($\boldsymbol{B}'+\boldsymbol{B}_0$) is slightly larger than that of $\boldsymbol{B}_0$.



(a) (b) (c)

Fig. 8-24 (a) No external magnetic field; (b) Magnetic torque; (c) The alignment of the magnetic dipole moments in a magnetic field $\boldsymbol{B}_0$

2. The induced magnetic dipole moment of electron in a magnetic field

For the diamagnetic substances, the magnetic effects of the electrons, including both their spins and orbital motions, exactly cancel, so that the atom or molecule is not magnetic; in other words, its magnetic moment is zero. However, if an external magnetic field is applied to a diamagnetic substance, a weak magnetic moment is induced in the direction opposite to the applied field. When an external magnetic field is applied, the electrons experience an additional Lorentz force. This added magnetic force combines with the electrostatic force to increase the speed of the electron whose magnetic moment is antiparallel to the field and to decrease the speed of the electron whose magnetic moment is parallel to the field. As a result, the two magnetic moments of the electrons no longer cancel and the substance acquires a net magnetic moment that is opposite to the applied field, as illustrated in Fig. 8-25, causing diamagnetic substances to weakly be repelled by a magnet.

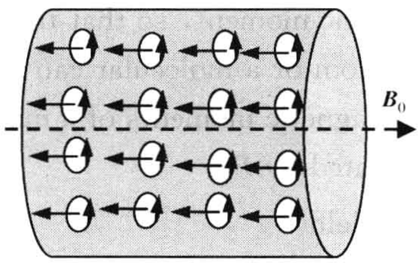


Fig. 8-25 The induced magnetic dipole moment of electron in a magnetic field

It should be pointed out that, the diamagnetic effect is present in all atoms, including paramagnetic media, in which this effect is much weaker than the effect of the alignment of their magnetic moment in external field, so that the diamagnetic effect is masked by the far stronger paramagnetic behavior, when a paramagnetic substance is concerned.

Since each magnetic effect arises from the action of a certain current, we can regard the additional magnetic moment of every molecular as an equivalent current, called molecular current. The molecular current may occur in every points of the material that is placed in an external magnetic field; at the interior points of the material, the net molecular current is zero because the currents in adjacent loops are in opposite directions and cancel with each other; on the surface of material, however, the currents are uncompensated and the entire assembly of loops is equivalent to a surface current, as shown in Fig. 8-26. For the paramagnetic materials, the direction of the equivalent surface current is opposite to that of diamagnetic materials. So macroscopically, there are equivalent surface currents moving around the surface of magnetic material that is placed in an external magnetic field. The surface current is termed as surface magnetized current, and the material is magnetized.

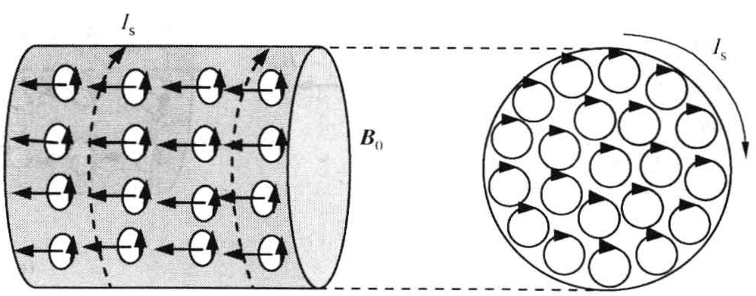


Fig. 8-26 The equivalent surface magnetization currents of diamagnetic medium

These equivalent surface magnetized currents are analogous to bound surface charges on a polarized dielectric, but with the difference that while the $\boldsymbol{E}'$ due to bound charges is always opposite to $\boldsymbol{E}_0$ produced by free charges, but the $\boldsymbol{B}'$due to surface magnetization currents may be in the same or the opposite direction of the $\boldsymbol{B}_0$ established by the current in the conductor.

8.10 Ampere's Law and Gauss' Law for Magnetism

8.10.1 Ampere's law for magnetic field in a medium

In the last chapter, we have introduced Ampere's Law

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \sum I \tag{8-38}$$

in which the currents on the right side include actually all currents produced by any reason, and $\boldsymbol{B}$ on the left side means the resultant field contributed by them. In the case that a magnetic medium is present, the surface magnetized current and the current in conductor (current formed by moving charges) should be both taken into account, so Ampere's law Eq. (8-38) can be rewritten as

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 (\sum I_0 + \sum I_s) \tag{8-39}$$

where $\boldsymbol{B}$ is the resultant magnetic field at $\mathrm{d}\boldsymbol{l}$ along the Amperian loop l, I_0 and I_s are conduction current and surface magnetized current that enclosed by Amperian loop l, respectively.

Eq. (8-39) is not convenient to use since I_s is unknown and immeasurable. Now we will introduce a new parameter to simplify Eq. (8-39), just as what we have done in deriving Gauss' law for dielectric in chapter 7.

We can use a special case to generalize Ampere's law in presence of magnetic medium. In Fig. 8-27, assuming that a uniform paramagnetic material is fully filled in a long straight solenoid, and the current carried in winding is I_0, the magnetic material is then magnetized. Suppose that the positive direction of surface magnetized current I_s is counterclockwise as seen from the cross section on the left.

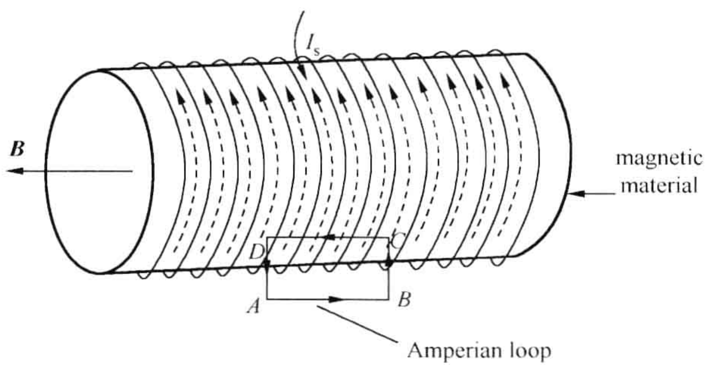


Fig. 8-27 Ampere's law in presence of magnetic medium

The surface magnetized current can be treated as an equivalent solenoid that consists of as many turns as those of the conduction currents, and each turn carries magnetized current I_s

which is corresponded to the conduction current I_0 carried by one turn in the actual solenoid as shown in the Fig. 8-27. Put in the other way, the distribution of the surface magnetized current is just like a solenoid coaxed with the winding solenoid, so that, the magnetic fields set up by them can be written as

$$B_0 = \mu_0 n I_0, \quad B' = \mu_0 n I_s$$

respectively. Applying superposition principle of magnetic field, we have the resultant field (note $\boldsymbol{B}_0$ and $\boldsymbol{B}'$ are in the same direction in this case)

$$B = B_0 + B' = \mu_0 n(I_0 + I_s) \tag{8-40}$$

On the other hand

$$B = \mu n I_0 = \mu_r \mu_0 n I_0 \tag{8-41}$$

Combining Eq. (8-40) and Eq. (8-41), yields

$$I_s = (\mu_r - 1) I_0 \tag{8-42}$$

That is, the equivalent surface magnetized current is the $(\mu_r - 1)$ times of I_0. The magnetized current per unit length is then written as

$$J_s = nI_s = (\mu_r - 1) n I_0 \tag{8-43}$$

Now, we are ready to eliminate I_s in Eq. (8-39). Applying Eq. (8-39) to the loop *abcda* in Fig. 8-27, we have

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \sum (I_0 + I_s) \tag{8-44}$$

Substituting Eq. (8-42) into Eq. (8-44), yields

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{l} = \mu \sum I_0 \tag{8-45}$$

Let

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} \tag{8-46}$$

Eq. (8-45) becomes

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \sum I_0 \tag{8-47}$$

Eq. (8-47) is the **Ampere's law for magnetism**, which means that the line integral of $\boldsymbol{H}$ around a closed path is equal to the conduction current only across any surface bounded by the path L. Although Eq. (8 -47) is derived from a special case, it is of universal significance.

The vector $\boldsymbol{H}$ in Eq. (8-46) and Eq. (8-47) is called **magnetic field intensity.** The SI unit of $\boldsymbol{H}$ is Ampere per meter, symbolized as A/m. From Eq. (8-46) one should note that the direction of $\boldsymbol{H}$ is the same as that of $\boldsymbol{B}$; hence, like the magnetic field $\boldsymbol{B}$, the magnetic field intensity $\boldsymbol{H}$ can also be presented by lines of magnetic field intensity. In any magnetic field, the distribution of lines of $\boldsymbol{H}$ and $\boldsymbol{B}$ are the same except that the density of them is not the same (the factor μ).

After introducing the new parameter $\boldsymbol{H}$, Ampere's law for magnetic field in magnetic medium involves only conduction current I_0 on the right side, and the surface magnetized current does not appear, but represented by μ. So that, it is convenient to use Eq. (8-47) to solve $\boldsymbol{H}$, and then $\boldsymbol{B}$ is calculated by Eq. (8-46) for the special symmetry magnetic field with magnetic material present. The role of $\boldsymbol{H}$ in magnetism is similar with that of $\boldsymbol{D}$ in dielectrics and both $\boldsymbol{H}$ and $\boldsymbol{D}$ are auxiliary parameters. In fact, it is $\boldsymbol{B}$ rather than $\boldsymbol{H}$ that determines the force acting on moving charges or current.

For the magnetic field in vacuum, Eq. (8-46) becomes

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \tag{8-48}$$

8.10.2 Gauss' law for magnetic medium

Since there are no sources or sinks of $\boldsymbol{B}$, even though a magnetic material is present, thus the magnetic flux Φ_m through any closed surface must be zero, that is

$$\Phi_m = \oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0 \tag{8-49}$$

Eq. (8-49) is the Gauss' law for magnetic field in media. It has the same form as Eq. (8-4), and the closed surface is called Gaussian surface.

Example 8-9 In Fig. 8-28, a specimen of ferromagnetism with permeability μ is formed in a toroid ring and a wire carrying the current I_0 is uniformly close packed over the toroid. Suppose that the number of winds per unit length is $n = 100$, $\mu = 2.0 \times 10^{-4}$ H/m, $I_0 = 5.0$A and the cross section of toroid is quite small comparing with its size. Find out:

(1) the magnetic intensity $\boldsymbol{H}$ and magnetic field $\boldsymbol{B}$ in the toroid;

(2) the surface magnetized current per unit length.

Solution (1) From symmetry, the lines of $\boldsymbol{H}$ form a set of circles inside the toroid. The magnitude of $\boldsymbol{H}$ at every point on a circle is the same, and its direction is tangent to the circle. So we choose a circle path l of radius r as an Amperian loop and traverse it in the clockwise direction. Applying Ampere's law Eq. (8-47) to this loop, yields

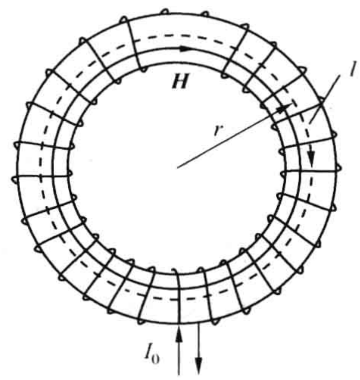


Fig. 8-28 For Example 8-9

$$\oint_l \boldsymbol{H} \cdot d\boldsymbol{l} = H 2\pi r = NI_0$$

thus

$$H = \frac{NI_0}{2\pi r} \quad (R_1 < r < R_2)$$

If the toroid is thin comparing with its average radius, the magnitude of $\boldsymbol{H}$ can be regarded as constant within the toroid, and $N/2\pi r$ approximately equals to the number of turns per unit length. So

$$H \approx nI_0$$

and

$$B \approx \mu n I_0$$

Substituting $n = 100$, $\mu = 2.0 \times 10^{-4}$ H/m and $I_0 = 5.0$ A into these results yields

$$H = 5.0 \times 10^2 \text{ A/m}$$

$$B = 0.10 \text{ T}$$

(2) From superposition principle of magnetic field

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}'$$

Since $\boldsymbol{B}_0$ and $\boldsymbol{B}'$ point in the same direction, thus

$$B = B_0 + B'$$

that is

$$\mu n I = \mu_0 n I + \mu_0 J_s$$

which leads to

$$J_s = \left(\frac{\mu}{\mu_r} - 1\right) n I = 7.95 \times 10^4 \text{ A/m}$$

The result shows that the value of J_s is much larger than the conduction current per unit length, which has value $nI = 5.0 \times 10^2$ A/m. As a consequence, the additional magnetic field $\boldsymbol{B}'$ set up by J_s, is much larger than the magnetic field $\boldsymbol{B}_0$ set up by conduction current. This may be quite beyond your exception, but it is just the special charateristic of ferromagnetism, which will be discussed in the next section.

* 8.11 Ferromagnetism

8.11.1 Properties of ferromagnetic material and experimental method

Some substances, such as Fe, Ni, Co, and some alloys of them, exhibit special characteristics.

(1) Their permeability is much greater than those of paramagnetic and diamagnetic materials; furthermore, it is not a constant, in other words, the relation between B and H is not a linear function.

(2) When the external magnetic field is removed, there is a magnetic field remained.

(3) Every ferromagnetic material has a critical temperature called the **Curie temperature**, above which the material becomes paramagnetic. The material showing these properties is called **ferromagnetism.**

To study the properties of a ferromagnetic material it is convenient to form the specimen into a toroidal ring, as shown in Fig. 8-29. N turns of wire are wrapped uniformly around it, which are usually called primary coil, and the current carried by primary coil is called **magnetizing current.** The magnetic field intensity H is determined only by magnetizing current I_0 ($H = nI_0$ see Example 8-9). The magnetic field $\boldsymbol{B}$ can be measured by using a secondary coil connected to a ballistic galvanometer BG in Fig. 8-29.

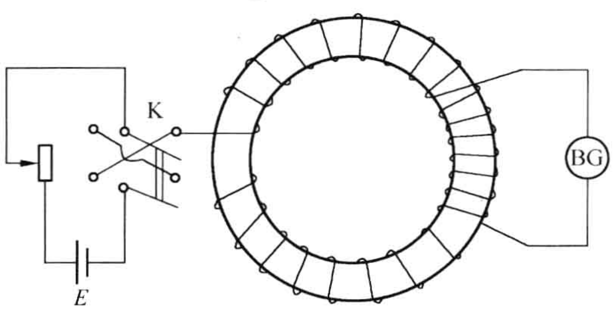


Fig. 8-29 Experiment diagram for studying the property of ferromagnetic material

The experiment shows that, the relation between the magnetic field **B** and magnetic field intensity **H** in a ferromagnetic material is complicated. And it is not possible to express B as a simple function of H, instead, the relation between these two quantities is presented by a graph of B versus H, called the magnetization curve of the material.

The magnetization curve and permeability μ versus H of annealed iron are shown in Fig. 8-30. The permeability, which is equal to the ratio of B to H, can be calculated at any point of the curve with the corresponding magnitudes of magnetic field intensity **H** and magnetic field **B**. It is clear that the permeability μ is not a constant, and the magnetic field B increases slowly when H is relatively small, and then increases rapidly as H increases a little more. There is very little further increase in magnitude of **B** while **H** has reached a certain value, and the iron is said to become magnetically saturated, as point S shows in Fig. 8-30.

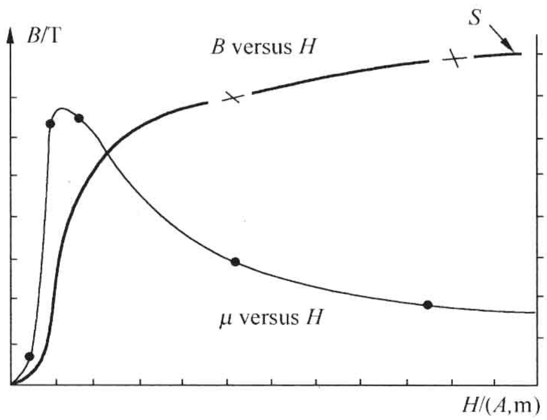


Fig. 8-30 Magnetization curve and permeability curve of annealed iron magnetization curve of the material

8.11.2 Hysteresis and hysteresis loop

The magnetization curves for ferromagnetic material do not retrace themselves, as we increase and then decrease the current in primary coil. Supposing the sample is initially unmagnetized and the magnetic field intensity is increased steadily from zero (by increasing the magnetizing current in primary coil), until the value of **B** becomes saturated, the curve follows the path OAB in Fig. 8-31. If the magnetizing current is now reduced to zero, while magnetic field **B** has not been reduced to zero, but has a certain value remained as point D shows in Fig. 8-31. The curve is path BCD rather than BAO. The behavior of the material, as the fact that B-H curve for decreasing H does not coincide with that for increasing H, that is, the property that the magnetic field B falls behind the magnetic intensity H in changing with the current is called **hysteresis**.

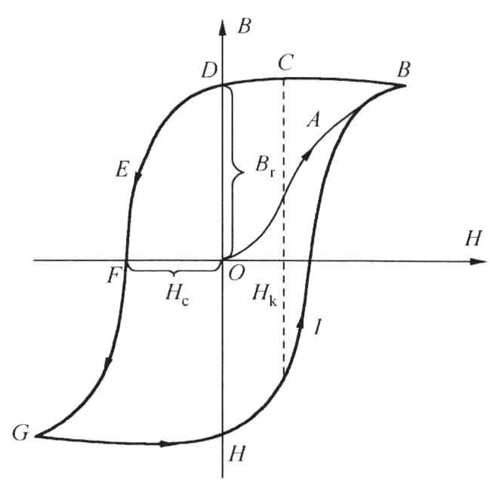


Fig. 8-31 A hysteresis loop

The ordinate OD in Fig. 8-31, representing the magnetic field B remained in the specimen when the magnetic intensity H has been reduced to zero (that is the magnetizing current has been cut off), is called **remanence or retentivity** of the specimen and designated by B_r. After the magnetization process $OABCD$, the sample has become a permanent magnet.

The magnetized sample can be demagnetized by reversing the magnetizing current. When we reverse the magnetizing current and increase it steadily from zero, the curve is represented by path DEF, and the abscissa OF represents the reversed magnetic field intensity H needed to demagnetize the magnetic induction to zero after it has been magnetized, called the coercive force or the coercivity H_c.

If we increase the reversing current further, the specimen will reach another reversing saturated state along path FG. While the reversing current is decreased to zero, the curve follows the path GH. At the point H, the ferromagnetic material becomes a permanent magnet again but in the opposite direction.

The curve goes along path HIB, if the current again increases from zero in positive direction. After several cycles of changing of $\boldsymbol{H}$, the curve tracing out a closed loop $BCDFGHIB$ in the Fig. 8-31, is called hysteresis loop.

Thus, $\boldsymbol{B}$ in the specimen seems to depend not only on $\boldsymbol{H}$, but also on the magnetization history as well. For example, while $\boldsymbol{H}$ has positive value H_k as shown in Fig. 8-31, the magnitude of $\boldsymbol{B}$ at that point may have three possible values, which are determined by the magnetization history.

8.11.3 Hard magnetic material and soft magnetic material

It is evidently required that a material for permanent magnets should have both a large retentivity B_r and a large coercive force H_c, so that the magnet will be strong and not be wiped out by external field. The materials with both a large B_r and H_c are called **hard magnetic material**, such as carbon steel and cobalt steel. The hysteresis loop of hard magnetic material is shown in Fig. 8-32(a).

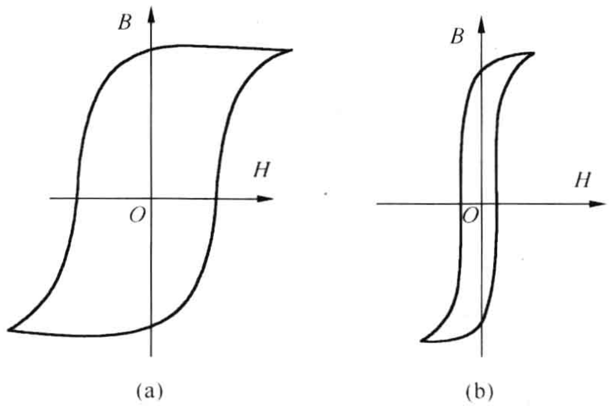


Fig. 8-32 (a) Hysteresis loop of hard magnetic material;
(b) Hysteresis loop of soft magnetic material

In some kinds of electrical apparatus, such as transformers and motors, masses of ferromagnetic material are located in magnetic fields whose directions are continually reversing.

There is dissipation of energy within the material each time the material is caused to cycle around its hysteresis loop. It has been shown that the energy dissipated per unit volume within each cycle is proportional to the area enclosed by the hysteresis loop. Hence, the hysteresis loop of the material which to be subjected to magnetic field whose direction is alternately reversing should be narrow to minimize energy losses.

The material with high permeability and small hysteresis loss is called **soft magnetic material.** The hysteresis loop of soft magnetic material is shown in Fig. 8-32(b) . This material is suitable for alternating magnetic field.

The shape of hysteresis loop of hard magnetic material is fat and that of soft magnetic material is thin. The typical magnitude of coercive force H_c for former is ($10^4 \sim 10^6$) A/m, and for the latter about 1 A/m. Some parameters of hard magnetic and soft magnetic materials are given in Table 8-2, where $(BH)_m$ is a parameter presenting the maximum energy dissipated per unit volume in each cycle, μ_0 is the initial permeability and μ_m is the maximum permeability.

Table 8-2 Some parameters of hard magnetic and soft magnetic materials

Hard magnetic material				
Material	Composition present	H_c/(A/m)	B_r/T	$(BH)_m$/(J/M³)
Carbon steel	Fe98. 1, Mn C0. 9	4×10^3	1. 00	1.6×10^3
Alnico5	A18, Nl14 Co24, Cu3, Fe51	4.4×10^4	1. 22	3.6×10^4
Samarium-Cobalt alloy	$SmCO_5$	6.93×10^5	0. 98	7.91×10^3
Barium ferrite	$BaO \cdot 6Fe_2O_3$	1.44×10^5	0. 45	3.6×10^4
Soft magnetic material				
Material	Composition present	μ_0 /(H/m)	μ_m /(H/m)	H_c/(A/m)
Com	Fe99. 95	1.2×10^{-3}	2.5×10^{-2}	7. 2
Hot cut silicon steel	Si4, Fe96	0.56×10^{-3}	1.0×10^{-2}	1. 8
Cool cut silicon steel	Si3. 3 Fe96. 7	0.75×10^{-3}	1.2×10^{-2}	16
Nickel-zinc ferrite	NiO 30, ZnO 20 $Fe_2O_3$50	0.13×10^{-3}	—	240

8. 11. 4 Magnetic domains

The behavior of ferromagnetism can be explained on the basis of the concept—**magnetic domains.** For ferromagnetic materials, strong coupling occurs between neighboring atoms to form large groups of atoms called **magnetic domains** in which the alignment of the atomic magnetic moments of spin are essentially perfect due to quantum effect. Therefore, the ferromagnetic materials, such as iron, are made up of a number of magnetic domains. In the unmagnetized state, however, the domains are randomly oriented so that they largely cancel each other as far as their external magnetic effects are concerned, so the ferromagnetic material as a whole, is magnetically neutral, as shown in Fig. 8-33(a) .

When the external magnetic field is increased steadily from zero, two effects take place:

(1) The domains that are oriented favorably with external field increase in size at the expense of others.

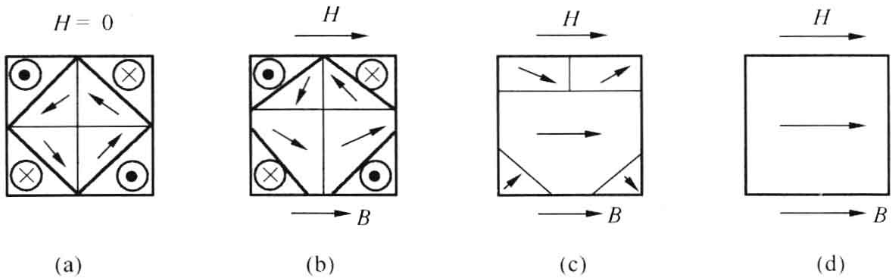


Fig. 8-33 (a) Magnetic domains are randomly oriented in an un-magnetized material; (b) and (c) All domains turn favorably with external field; (d) When external field large enough, all domains oriented favorably with external field

(2) The orientation of a domain may swing around as a unit, becoming closer to the direction of the external magnetic field, as shown in Fig. 8-33(b) and Fig. 8-33(c). They cause the material to become magnetized.

Both effects contribute to the magnetization curve of Fig. 8-30 and path *OA* of hysteresis loop in Fig. 8-31. While the external magnetic field is increased large enough, all domains will be oriented favorably with external field as shown in Fig. 8-33(d), and then the sample is magnetically saturated. That is the magnitude of ***B*** increases slowly since the magnetic field set up by ferromagnetic material stops increasing anymore.

When the applied magnetic field increases and then decreases back to its initial value, the domains may not return completely to their original state, but remain some "memory" of initial process. In other words, the changes of domains are irreversible; this is the reasonable explanation for the hysteresis behavior of ferromagnetism.

The existence of domains can be shown by spreading a finely divided magnetic powder on polished surface of the specimen. The powder particles are collected along the boundaries between domains and may be examined under a microscope. The sizes of the domains may vary widely, depending on the size of specimen and whether it is a single crystal or a polycrystalline. Typical values are from 10^{-6} to 10^{-2} cm^2, which means that a domain may, contain from 10^{17} to 10^{21} molecules. Fig. 8-34 shows domain picture of cobalt-nickel ferrite on a (100) surface.

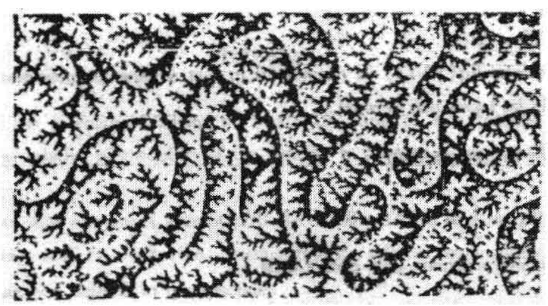


Fig. 8-34 Domain picture of cobalt-nickel ferrite on a (100) surface

The friction occurring in the action of swing around of domains is responsible for the energy dissipation in reversing of magnetic field.

Every ferromagnetic material has a critical temperature called the **Curie temperature**, above which the material becomes paramagnetic. As the **Curie temperature** is reached, the domains are disrupted due to the effects of thermal motion with energy larger enough, the character of ferromagnetic material then disappears and it becomes paramagnetic. For example, the Curie temperature of iron is 770℃, hot silicon steel 690℃ and Alnico permanent magnetic alloy 1030～1180℃.

Questions

8-1 Why do we not simply define the direction of the magnetic field $\boldsymbol{B}$ to be the direction of the magnetic force that acts on a moving charge?

8-2 If an electron is not deflected in passing through a certain region of space, can we be sure that there is no magnetic field in that region?

8-3 What are differences and analogies in representing the property of the field lines between the magnetic field and electric field?

8-4 From the Eq. (8-6), discuss the conclusion of that the magnetic field set up by a segment of straight wire carrying a current is zero at the point in the extension of the wire.

8-5 Apply Ampere's law to the four paths, shown in Fig. 8-35, to calculate separately the value of the line integral of $\boldsymbol{B}$, set up by a closed current I.

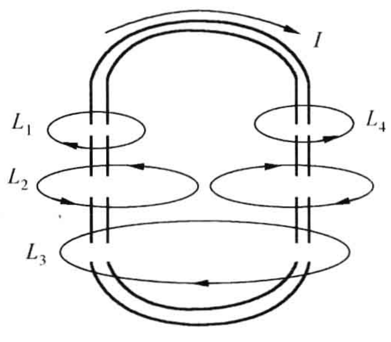


Fig. 8-35 For question 8-5

8-6 Could the magnetic field $\boldsymbol{B}$, in the Ampere's law $\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \sum I$, be set up by the current which doesn't flow through the Amperian loop.

8-7 If the windings on a solenoid are considered to be helical, rather than being equivalent to a cylindrical current sheet, then the magnetic field outside is not zero strictly but is a circling field, just the same as the field of a straight wire. Explain it.

8-8 Can we apply the Ampere's law to solve the problem on the magnetic field of a segment of straight current or circle current?

8-9 Of the three vectors in the equation $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, which are always at right angles? Which may have any angle between others?

8-10 Give an example of two moving charged particles with velocity such that the mutual magnetic force do or do not obey Newton's third law.

8-11 A conductor, even though it is carrying a current, has zero net charge, Why? Does a magnetic field exert a force on it?

8-12 A current is sent through a vertical spring from its lower end at which a weight is hanging? What will happen?

8-13 Two long straight wires pass near one another at right angles. If the wires are free to move, describe what will happen when currents are sent through both of them.

8-14 Fig. 8-36 shows that an infinitely long straight wire lies along the axis of a circular loop. Both the wire and the loop carry currents I. In this case no mutual force exert on between them. If the straight wire is shifted to coincide with the dotted line MN, discuss the forces and torque exerting on the wire and the loop.

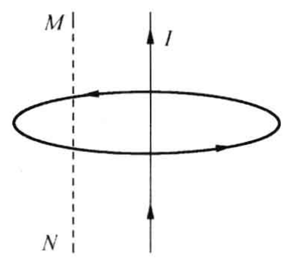


Fig. 8-36 For question 8-14

8-15 Tell the differences between conduction current I_0 and surface magnetization current I_s. In Eq. (8-33), what is the magnetic field $\boldsymbol{B}$, $\boldsymbol{B}_0$ and $\boldsymbol{B}'$ produced by?

8-16 Compare Ampere's law and Gauss' law in vacuum and for magnetism, give your conclusions.

8-17 One method to determine whether a material is para- or diamagnetic is suspending the specimen of needle shape in a uniform magnetic field, while the needle become rest.

(1) If the needle parallel to the field.

(2) If the needle at right angles with the field.

Which situation responds to paramagnetic and diamagnetic? Why?

Problems

8-1 A uniform magnetic field $\boldsymbol{B}$ in a certain region is 2T and its direction is that of the positive x-axis, as shown in Fig. 8-37.

(1) What is the magnetic flux across the surface $ABCD$ in the figure?

(2) What is the magnetic flux across the surface $BEFC$?

(3) What is the magnetic flux across $AEFA$?

8-2 A frustum of a cone, with top radius R_1 and bottom radius R_2, is placed in a uniform magnetic field, as shows in Fig. 8-38. The flux through the bottom surface is ϕ. Find the magnetic flux through the side surface of the frustum.

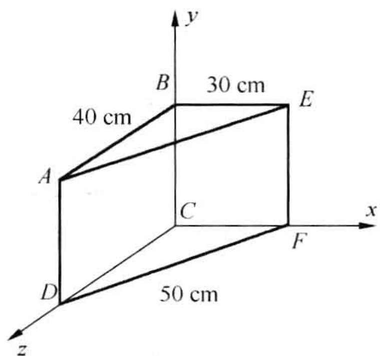


Fig. 8-37 For problem 8-1

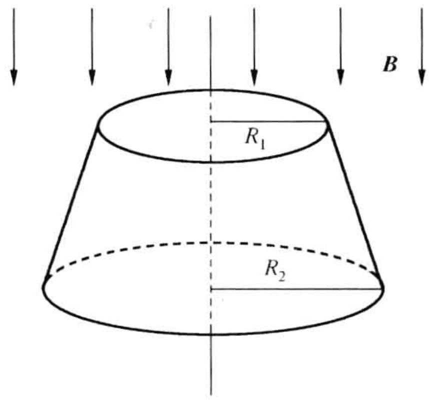


Fig. 8-38 For problem 8-2

8-3 Two semi-infinite straight wires, carrying a current I, connect with a rectangle resistive wires of width l at edge points A and B, as shown in Fig. 8-39, calculate magnetic field $\boldsymbol{B}$ at the center point O.

8-4 Find the magnetic field $\boldsymbol{B}$ at point O in Fig. 8-40(a) and Fig. (b).

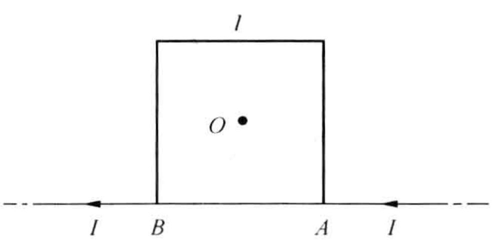


Fig. 8-39 For problem 8-3

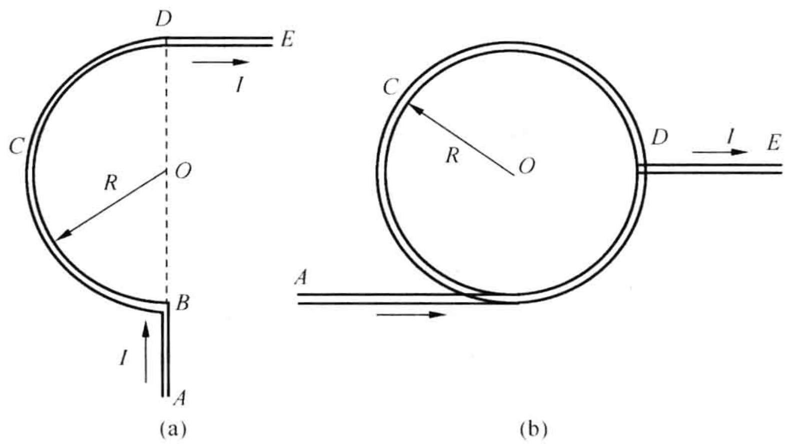


Fig. 8-40 For problem 8-4

8-5 Fig. 8-41 shows that a closed current flows along the some edges of a cube. Calculate the magnetic field $\boldsymbol{B}$ at point O of the cube.

8-6 A wire carrying current I has the configuration shown in Fig. 8-42. Two semi-infinite straight sections AB and DE are connected to a circular conductor ring of radius R and uniform resistance. The arcs BCD and $BD$are 3/4 and 1/4 circular, respectively. Find the magnetic field $\boldsymbol{B}$ at point O?

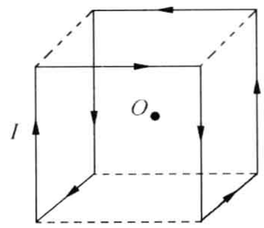


Fig. 8-41 For problem 8-5

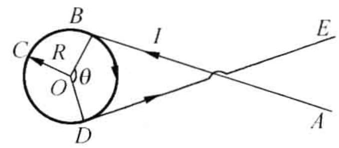


Fig. 8-42 For problem 8-6

8-7 Fig. 8-43 shows an arrangement known as a Helmholtz coil: it consists of two circular coaxial coils each of N turns and radius R, separated a distance R. They carry equal current I in the same direction.

(1) Find the magnetic field at any point of the x-axis.

(2) Show that $\mathrm{d}B/\mathrm{d}x$ and $\mathrm{d}^2B/\mathrm{d}x^2$ are both zero at $x=0$, midway between the coils.

8-8 A very long and thin strip of copper of l carries a current I uniformly distributed over the strip in Fig. 8-44. Calculate the magnetic field at a distance z above the middling of this trip (hint: imagine the strip to be constructed from many long thin parallel wires).

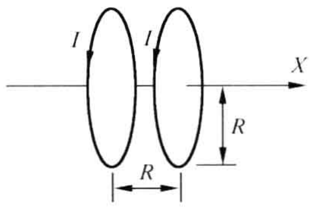


Fig. 8-43 For problem 8-7

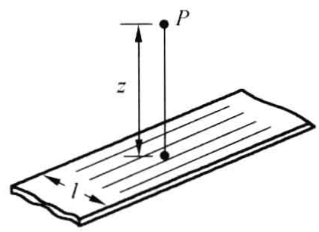


Fig. 8-44 For problem 8-8

8-9 A coil of N turns is wound along the surface of a cone of top angle 2θ in Fig. 8-45. Each turn carry a current I. Find the magnetic field at the top point of the cone.

8-10 A thin semi-circle arc of radius R carrying a charge Q distributing uniformly along the arc. If the arc rotates about the axis OO' with angular velocity ω, as shown in Fig. 8-46, calculate the magnetic field at the center O of the arc.

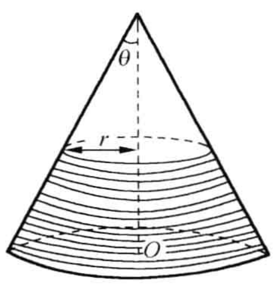


Fig. 8-45 For problem 8-9

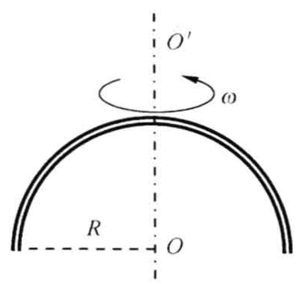


Fig. 8-46 For problem 8-10

8-11 The long straight wire AB in Fig. 8-47 carries a constant current I. What is the magnetic flux Φ_m through the rectangular area $CDEF$, in terms of I, l, a and b?

8-12 Fig. 8-48 shows a long straight conductor of radius R and carries a current I, the current is uniformly distributed on the cross section of the conductor. Find the expression for $B(r)$ in the ranges (1)$r<R$, (2)$r>R$.

8-13 The current density inside a long solid cylindrical wire of radius a is in the direction of the axis and varies linearly with radial distance r from the axis according to $J=J_0 r/a$. Find the magnetic field inside the wire.

8-14 Fig. 8-49 shows a cross section of an infinite conducting sheet with a current per unit length j emerging from the page at right angles.

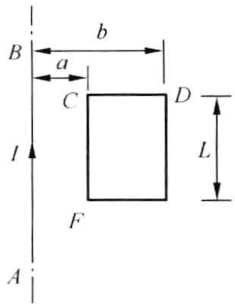


Fig. 8-47 For problem 8-11

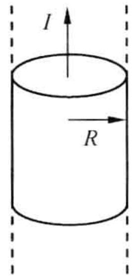


Fig. 8-48 For problem 8-12

(1) Use the right-hand rule and symmetry arguments to convince yourself that the magnetic field **B** is constant for all points P above the sheet (and for all points P' below it) and is directed as shows.

(2) Use Ampere's law to prove that $B=0.5\mu_0 j$.

8-15 A solenoid is 33 cm long and is wounded two layers of wire. The inner layer consists of 300 turns, the outer layer of 250 turns. The current is 3A in the same direction in both layers. What is the magnetic field at a point near the center of the solenoid.

8-16 A long solenoid of n turns per unit length carries a current I, and a long straight wire lying along the axis of this solenoid carries a current I'. Find the net magnetic field within the solenoid, at a distance r from the axis. Describe the distribution of the magnetic field lines.

8-17 A toroid having a square cross section, as shown in Fig. 8-50 has N turns and carries a current I. Find

(1) The magnetic field inside the toroid at the range $R_1<r<R_2$;

(2) The flux through the cross section.

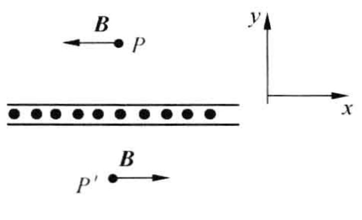


Fig. 8-49 For problem 8-14

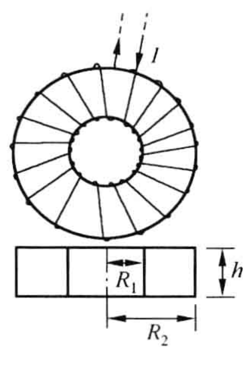


Fig. 8-50 For problem 8-17

8-18 A positive point charge of magnitude q is a distance d from a long straight wire carrying a current I and is traveling with speed v perpendicular to the wire. What are the direction and magnitude of the force acting on it if the charge is moving (1) toward, or (2) away from the wire?

8-19 (1) In a magnetic field with $B=0.50$ T, for what path radius will an electron circulate at 0.10 the speed of light? (2) What will be its classic kinetic energy in eV?

8-20 A beam of electrons with the velocity of $\boldsymbol{v}$ enters into a region that have both electric field **E** and magnetic field **B**. The field **E** is at right angles to the field **B** and $\boldsymbol{v}$, and its magnitude is $E=\dfrac{vB}{\sqrt{2}}$.

(1) What is the angle between the **B** and $\boldsymbol{v}$ when the beam has no deflection?

(2) What is the radius and pitch of the electron orbit (helix) when the electric field is removed (Assume $v=6.0\times10^7$ m/s, $B=0.050$ T)?

8-21 An electron moves in a circular path of radius 1.2 cm and perpendicular to a uniform magnetic field. The velocity of the electron is 10^6 m/s. What is the total magnetic flux encircled by the orbit?

8-22 Show that, in terms of the Hall electric field $\boldsymbol{E}$ and the current density J, the number of charge carriers per unit volume is given by $n=JB/(eE)$.

8-23 Fig. 8-51 shows a portion of a silver ribbon with $z_1=2\text{cm}$ and $y_1=1\text{mm}$, carrying a current of 200A in the positive x-direction. The ribbon lies in a uniform magnetic field, in the y direction, of magnitude 1.5T. If there are 7.4×10^{28} free electrons per m^3, find

(1) The drift velocity of the electrons in the x-direction;

(2) The magnitude and direction of the electric field in the z-direction due to the Hall effect;

(3) The Hall potential difference.

8-24 Fig. 8-52 shows a long wire carrying a current of 30 A. The rectangular loop carries a current of 20 A. Calculate the resultant force acting on the loop. Assume that $a=1.0\text{c m}$, $b=8.0$ cm, and $L=30$ cm.

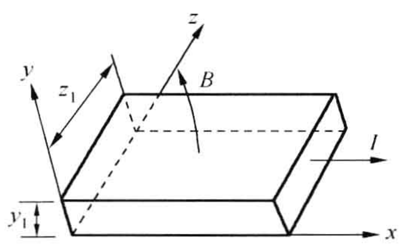


Fig. 8-51 For problem 8-23

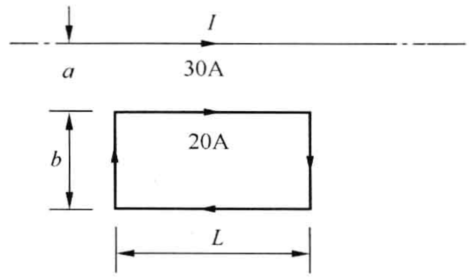


Fig. 8-52 For problem 8-24

8-25 Three long straight wires are placed equidistantly on a plane and parallel to each other. The currents and mutual forces on each wire show in Fig. 8-53. Assuming $F_2/F_3=3/4$ and $I_1/I_2=2/3$, find I_3/I_1 and F_3/F_1.

8-26 A circular loop of wire of radius R is placed in a uniform magnetic field $\boldsymbol{B}$ which directs perpendicular to the plane of loop as shown in Fig. 8-54. If the wire carries a current I, what is the tension on the wire?

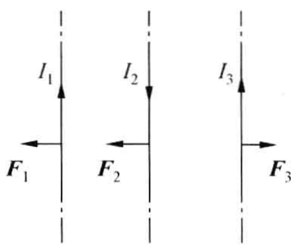


Fig. 8-53 For problem 8-25

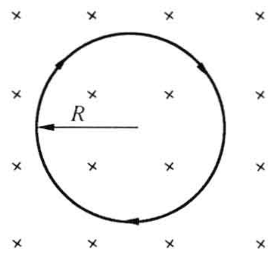


Fig. 8-54 For problem 8-26

8-27 Fig. 8-55 shows a rectangular loop of 20 turns of wire, with width 5.0 cm and length 10 cm. It carries a current of 0.10 A and is hinged at one side. It is mounted with its plane at an angle of 30° to the direction of a uniform magnetic field of 0.50 T. Calculate the torque about the hinge line on the loop.

8-28 Calculate the magnetic moment of the closed current that shown in Fig. 8-56.

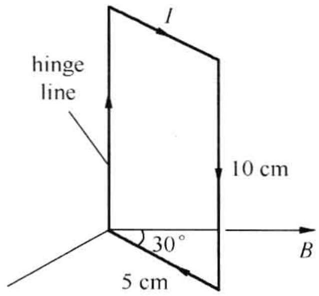


Fig. 8-55 For problem 8-27

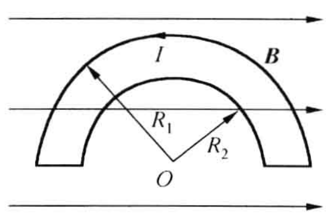


Fig. 8-56 For problem 8-28

8-29 A solenoid has a length of 20 cm, on which 220 turns of wire is uniformly wounded. The current carried by the wire is 5 A. What should be the relative permeability of the core to produce a magnetic field of 0. 2 T at the center of the solenoid?

8-30 $N=100$ turns wire that carrying a current of 0. 3 A is uniformly wounded on a toroid of radius $r=0.10$ m, and a ferromagnetic material is fully filled in the toroid. Assume that the relative permeability the core is $\mu_r=5000$. Calculate the average magnetic field $\boldsymbol{B}$ in the specimen.

8-31 A circular iron plate with a radius $r=10$cm is placed perpendicular to a uniform magnetic field of $B_0=0.4$ T. Assume the magnetic field in the center of plate is $B_c=0.1$ T.

(1) Give the distribution pattern of equivalent surface magnetization current.

(2) Find out the value of equivalent surface magnetization current.

(3) Calculate the magnetic field $\boldsymbol{B}$ at the point P, which is apart from center of plate 0. 40 m on axis, as shown in Fig. 8-57.

8-32 A straight conductor of radius a, carrying a current I, is wrapped by a layer of magnetic medium with thickness of b and relative permeability of μ_r, as shown in Fig. 8-58. Find out

(1) The magnetic intensity $\boldsymbol{H}$ and magnetic field $\boldsymbol{B}$ within the medium.

(2) The equivalent surface magnetization current moving on inter-surface and out-surface of the magnetic medium.

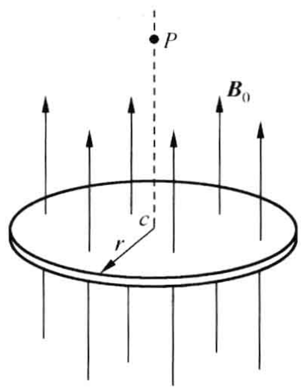


Fig. 8-57 For problem 8-31

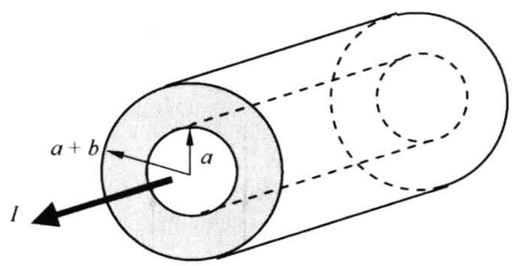


Fig. 8-58 For problem 8-32

Chapter 9

Electromagnetic Induction

In chapter 8 we have discussed Oersted's discovery of the link between magnetism and electricity. Soon after Oersted's work, other scientists attempted to find out whether an electric current could be produced by the action of a magnetic field. From the point of view of symmetry, it is quiet reasonable. In 1831 Michael Faraday, after several years hard researching, discovered that an electromotive force is set up in a closed electric circuit located in a magnetic field whenever the total magnetic flux through the circuit is changing. At about the same time, the American physicist Joseph Henry made a similar discovery. This phenomenon is called electromagnetic induction. Since Faraday published his result first, he got the priority of the discovery. Based on the discovery of Faraday and Henry, the first electric generator which converted mechanical energy directly to electric energy was soon invented in Manchester, England. The phenomenon of electromagnetic induction and Oersted's discovery reveal that there are some deep relations between phenomena of electric and magnetic.

At last, we will see, a changing electric field induces a magnetic field. The law describing this induction effect of electric field was established by James Clerk Maxwell, the great Scottish physicist, who thereby achieved a wide-ranging unification of the laws of electricity and magnetism, known as Maxwell's equations. The significant achievement of Maxwell's equations is that it predicts the existence of electromagnetic wave. We will examine such electromagnetic waves which are the electromagnetic field spreading in the form of polarized waves in the space and moving with the speed of light! This must deepen our understanding about the nature of light. Maxwell equations led to the discovery of radio wave, became the basis of the principles of radio broadcast, television transmitters and receivers, telephones, electromagnets, radars as well.

9.1 Nonelectrostatic Force, Source, and Electromotive Force

9.1.1 Nonelectrostatic force

If we connect each end of a wire (with certain resistance) to a metal plate, one plate being charged positively and the other negatively, as Fig. 9-1 shows, and the plates are parallel to each other, the charge carries (in most cases the moving charges are electrons, but it will make no difference to use either positive charges or electrons in discussion) will be forced to move by the action of electrostatic field created in the wire due to the potential difference between the plates. The current arises from the motion of positive charges, but it will stop soon since the excess charges on plates A and B will decrease and the potential difference between them will reduce and eventually become zero.

This led us to understand that if we want the current to be continuous and constant, the positive charge arrived at plant B should be moved from plate B back to plate A to keep the potential difference. But the electrostatic force $\boldsymbol{F}_e$, which is due to the charges on two plates, always drives the positive charges from high potential spot to low potential spot. If there exists only electrostatics force, the positive charges will never be moved from plate B (of low potential) to plate A (of high potential), so another force $\boldsymbol{F}_n$ is required to act on the positive charges in the direction opposite to the electrostatic force $\boldsymbol{F}_e$, as shown in Fig. 9-2. The force $\boldsymbol{F}_n$ is therefore of nonelectrostatic character, and is called nonelectrostatic force. In order to keep the potential difference, the magnitude of $\boldsymbol{F}_n$ should be at least equal to the magnitude of $\boldsymbol{F}_e$.

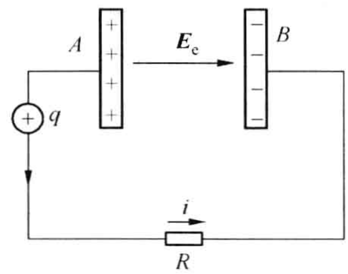


Fig. 9-1 The current will stop soon if there were only electrostatic force

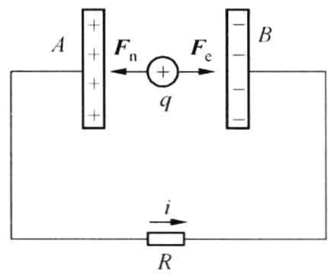


Fig. 9-2 A nonelectrostatic force $\boldsymbol{F}_n$ is needed to move charges against the effort of Electrostatic force $\boldsymbol{F}_e$ in the source

9.1.2 Source and electromotive force

A device with the ability to maintain potential difference between two points is called **a source of electromotive force , or briefly a source. In other words, there exists a nonelectrostatic force in a source.**

Whatever the origin of nonelectrostatic force, since it exerts forces on charges, analogy with the electrostatic field, we can define an equivalent nonetectrostatic field $\boldsymbol{E}_n$ as

$$\boldsymbol{F}_n = q\boldsymbol{E}_n \quad \text{or} \quad \boldsymbol{E}_n = \frac{\boldsymbol{F}_n}{q} \tag{9-1}$$

In other hand, the source of electromotive force does work on the charge passing through it and supplies electrical energy to the charge, which converts chemical, mechanical, or other forms of energy into electrical energy. The work performed by nonelectrosratic force on per unit positive charge from source terminal B to A within source is defined as the electromotive force, abbr. emf, denoted by $\mathscr{E}$, that is

$$\mathscr{E} = \int_A^B \boldsymbol{E}_n \cdot d\boldsymbol{l} \tag{9-2}$$

Eq. (9-2) applies only to a kind of source that the nonelectrostatic force of which occurs in a part of circuit, such as the battery. There exists another kind of source, in which the noneletrostatic force appears at every points of a closed loop l, so the closed loop is a source as a whole, and its emf should be calculated by the following integral

$$\mathscr{E} = \oint_l \boldsymbol{E}_{\mathrm{n}} \cdot \mathrm{d}\boldsymbol{l} \tag{9-3}$$

The SI unit of emf is the same as that of potential or potential difference, namely volt. Similar to potential difference, emf is also a scalar, but like the current, it is conventional to assign to emf the same direction as the moving direction of positive charges inside the source; thus it points from low potential spot to high potential spot.

It is important to note that the electromotive force is not a force in the mechanics sense, but a work done by nonelectrostatic force on per unit positive charge inside a source to produce an electric potential difference. A source of emf can be thought of as a sort of charge pump that pumps the charge from a region of low electrical potential energy to a region of high electrical potential energy, much like a water pump that pumps water from low to high regions of gravitational potential energy.

The origin of nonelectrostatic force depends on the nature of the source. For examples, the force in a dry cell is a chemical character; in a DC generator, the force results from the motion of charged particles or Lorentz force; and in the windings of a transformer it results from the effect of a magnetic field that is changing with time.

9.1.3 Complete circuit Ohm's law

As we shall see later, the electrostatic field within a source, and hence the potential difference between its terminals depends on the current in the source. The nonelectrostatic field, and hence the emf of the source, in many cases is a constant, independent of the current. As shown in Fig. 9-3, the terminals of a source are connected by a resistor R, and the source and wire are then said to form a complete circuit. Suppose the internal resistance of the source is r (for a real source). In a time interval $\mathrm{d}t$, a charge $\mathrm{d}q$ ($=i\mathrm{d}t$) will move through the source, and the amount of work performed by nonelectrostatic field is given by

$$\mathrm{d}W = \mathscr{E}\,\mathrm{d}q = \mathscr{E}i\mathrm{d}t \tag{9-4}$$

During this same interval $\mathrm{d}t$, the amount of energy appearing on the resistors R and r is $P\mathrm{d}t$, and from Joule's law

$$P\mathrm{d}t = (i^2R + i^2r)\mathrm{d}t$$

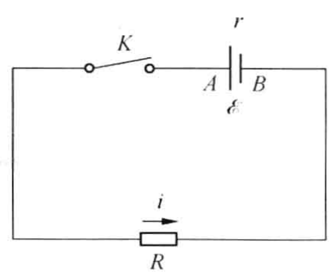


Fig. 9-3 A complete circuit

From the principle of conservation of energy, the work done by the nonelectrostatic force and electrostatic force on moving charges along the whole circuit must be equal to the thermal energy that appears on the resistors R and r. Since the net work done by electrostatic force along a closed circuit is zero, we have

$$\mathrm{d}W = \mathscr{E}\,i\mathrm{d}t = P\mathrm{d}t = (i^2R + i^2r)\mathrm{d}t$$

Solving for i, we obtain

$$i = \frac{\mathscr{E}}{R+r} \tag{9-5}$$

As the potential difference between terminals A and B (Fig. 9-3) equals the potential difference of the external resistance R, thus

$$V_{AB} = iR \tag{9-6}$$

From Eq. (9-5) and Eq. (9-6), we find

$$V_{AB} = \mathscr{E} - ir \tag{9-7}$$

Eq. (9-7) is called Complete circuit Ohm's law. When the circuit is open ($i=0$), Eq. (9-7) reduces to

$$V_{AB} = \mathscr{E} \text{ (for an open circuit)} \tag{9-8}$$

Since the emf of source in many cases is a constant, from Eq. (9-7) we could see that the potential difference V_{AB} varies with the current while the internal resistance r of the source is not zero. Therefore the electrostatic force $\boldsymbol{F}_e$ inside a source depends on the current in the source.

9.2 Faraday's Law of Induction

9.2.1 Conclusion from two demonstrational experiments

Two simple experiments will guide us to an understanding of Faraday's law of induction.

The first experiment: Fig. 9-4 shows the terminals of a wire loop of many turns are connected to a sensitive galvanometer G that can detect the presence of a current in the loop. Usually, the galvanometer will not deflect because there is no any source in the circuit; however, if we push a bar magnet toward the loop, a curious thing happens. While the magnet is moving (and only while it is moving), the meter deflects, showing that a current has been set up in the loop. Furthermore, the faster we move the magnet, the greater the deflection. When we stop moving the magnet, the deflection stops and the needle of the meter returns to zero. If we move the magnet bar away from the loop, the meter again deflects while the magnet is moving, but in the opposite direction, which tells us that the current in the loop is in the opposite direction.

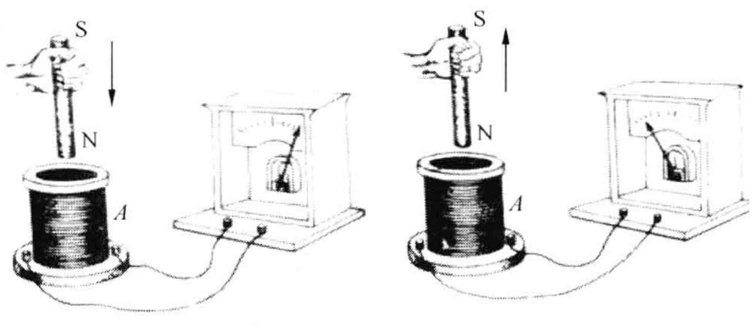


Fig. 9-4 When the magnet bar is moving, the galvanometer G deflects, showing that a current has been set up in the loop

If we invert the end of magnet with its south pole towards the loop, the experiments work just as before except that the direction of deflection of the needle is reversed. Further tests would convince you that what matter is the relative motion of the magnet and the loop. It makes no difference whether we move the loop towards the magnet or the magnet towards the loop. The current that appears in the loop in this experiment is called induced current. How does it be set up? There must be an electromotive force that corresponds to the induced current, called the induced electromotive force.

The second experiment: Here we use the apparatus of Fig. 9-5, in which there are two coils A and B with many turns. The coil A is placed near coil B and is connected to a battery and a switch K, and a sensitive galvanometer G is connected to coil B. If we turn on switch K to set up a current in the coil A, the meter deflects momentarily and then return to zero; if we then turn off the switch K to interrupt the current in coil A, the meter deflects again but in the other direction. None of the apparatus is physically moving in this experiment.

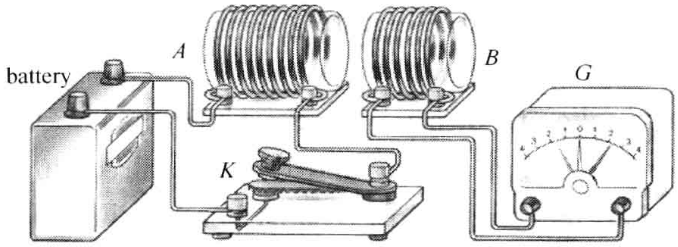


Fig. 9-5 When the switch K is on or off, the galvanometer G deflects, showing that a current has been set up in the loop connected to coil B

Only when the current in the coil A is rising or falling does an induced electromotive force appear in coil B. When there is a steady current in coil A, there is no induced electromotive force, no matter how large that steady current may be.

These two experiments show that an induced electromotive force appears only when something is changing, in Faraday's Language that he might have used:

"An induced electromotive force appears in the coil, only when the magnetic flux that passes through that coil is changing."

9.2.2 Faraday's law of induction

The actual flux of the magnetic field that passes through the coil at any moment is of no concern, but it is the rate at which this number is changing that determines the induced electromotive force.

In the experiment of Fig. 9-4, the number of the magnetic field lines passing through the coil (the magnetic flux) increases when we brought the magnet closer and decreases when we pulled the magnet away. In the experiment of Fig. 9-5, the magnetic flux is associated with the current in coil A. The magnetic flux through the coil B increases (from zero) when we close switch K and decreases (back to zero) when we then open this switch.

Quantitatively, the Faraday's law of induction can be stated as follows: "The induced emf in a circuit is equal (except for a minus sign) to the rate at which the magnetic flux through that circuit is changing with time." Mathematically, this law can be written as

$$\mathscr{E} = -\frac{\mathrm{d}\Phi_\mathrm{m}}{\mathrm{d}t} \tag{9-9}$$

where Φ_m is the magnetic flux passing through the surface bounded by the closed circuit, $\mathscr{E}$ is the induced emf appearing in this closed circuit. Eq. (9-9) is known as Faraday's law.

If the magnetic flux through a coil of N turns, an induced emf appears in every turn and these emfs are to be added-like those of batteries connected in series. If the coil is so tightly wounded that the flux Φ_m through each turn will be the same, such as the case of ideal solenoid or toroid, the total flux through the coil is thus $N\Phi_m$. The product $N\Phi_m$ is called the number of flux linkages. The induced electromotive force in such arrangements is then

$$\mathscr{E}=-\frac{d(N\Phi_m)}{dt}=-N\frac{d\Phi_m}{dt} \qquad (9\text{-}10)$$

The minus signs in Eq. (9-9) and Eq. (9-10) refer to the direction of induced emf. Although we shall rely on Lenz's law (described lately) to give us this information, we can also use the following way to determine its direction.

Firstly, we arbitrarily take one direction of the loop which is boundary of the surface of area as reference direction, as in Fig. 9-6(a) and then the direction of area vector $\boldsymbol{n}$ is determined by the right-hand rule. Secondly, in Fig. 9-6(b), we show a uniform magnetic field $\boldsymbol{B}$ enclosed by the loop, with an angle θ with the area vector $\boldsymbol{n}$ where $\theta< 90°$, and then the flux Φ_m is positive ($\Phi_m= BS\cos\theta > 0$) that is $\Phi_m> 0$. Thirdly, if now we allow the magnitude of the magnetic field to increase with time as in Fig. 9-6(c), the flux through the loop will also increase with time, that is $\frac{d\Phi_m}{dt}$ is positive. Lastly, from Faraday's law, the minus sign requires that the induced emf is negative, so the direction of emf is opposite to the direction of reference direction of the loop, that is in the direction of counterclockwise.

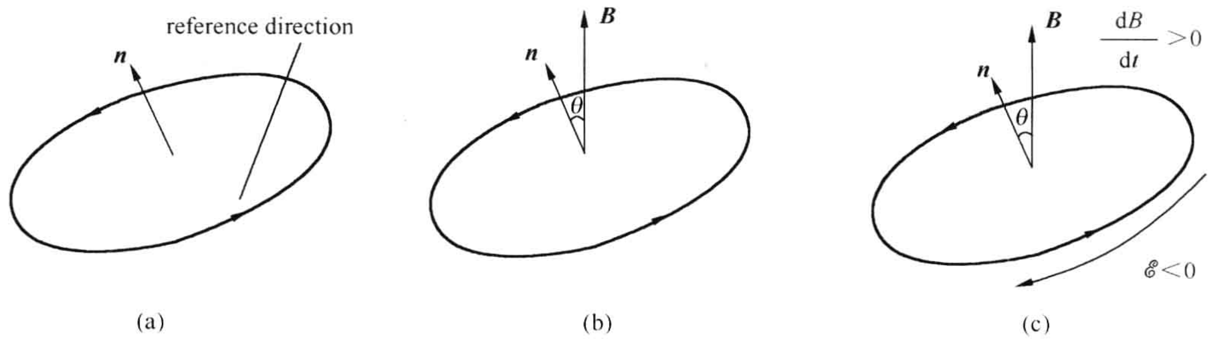


Fig. 9-6 (a) The positive reference direction of circulation around the loop is related to the area vector $\boldsymbol{n}$ by the right-hand rule. (b) The flux $\Phi_m=BS\cos\theta>0$. (c) When the magnitude of $\boldsymbol{B}$ increases with the time, the induced emf is in the negative direction of reference direction

Lenz's Law: In 1834, just 3 years after Faraday put forward his law of induction, Herinch Friedrich Lenz gave his rule (known as Lenz's law) for determining the direction of an induced current in a closed conduction loop:

"The induced emf and induced current in a closed conducting loop are in such a direction as to oppose the change that produces it."

If the loop is not closed, we can usually think in terms of that would happen if the loop is closed and in this way find the direction of the induced emf.

Lenz's law is consistent with the principle of conservation of energy. When you push a magnet bar toward a loop, the magnetic field set up by the induced current in the loop exerts a resisting force on the magnet, so you have to do work, and this work must be exactly equal to the electric energy that appears in the loop.

Example 9-1 Fig. 9-7(a) shows a rectangular conductor loop *ABCDA* of width L, one end of which is in a uniform external magnetic field $\boldsymbol{B}$ directing into the plane of the loop at right angle. This field may be produced, for example, by a large electromagnet. If this loop is pulled to the right at a constant speed v, the resistor of the loop is R.

(1) What is the induced emf in the loop?

(2) Find out the rate at which the external force does work in pulling the loop through the magnetic field?

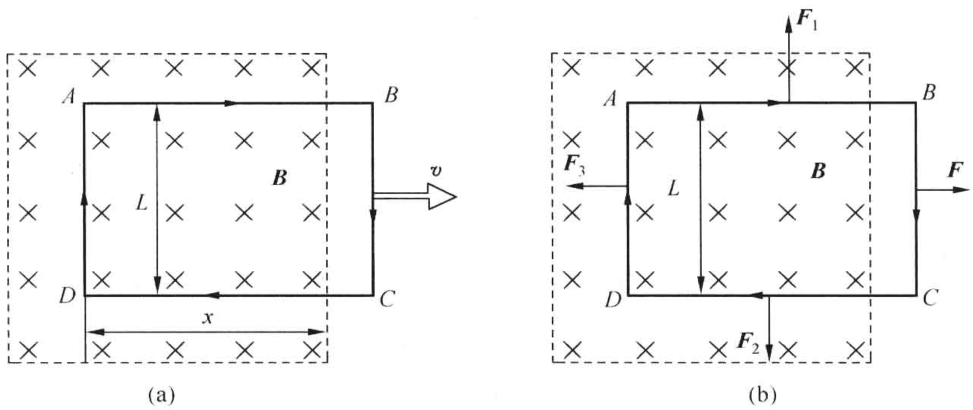


Fig. 9-7 For Example 9-2

Solution (1) We can use Faraday's law to find the induced emf in loop *ABCDA*. Supposing the positive reference direction of the loop is clockwise, according to right-hand rule, the direction of area vector $\boldsymbol{n}$ points perpendicularly into the page. The value of flux Φ_m enclosed by the rectangular loop is

$$\Phi_m = \boldsymbol{B} \cdot \boldsymbol{S} = BLx$$

where x is the length of the loop in the magnetic field. From Faraday's law, we have

$$\mathscr{E} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}(BLx) = -BL\frac{dx}{dt}$$

Considering the x decrease with time t, so $\frac{dx}{dt} < 0$, thus $\frac{dx}{dt} = -v$. Finally we have

$$\mathscr{E} = BLv > 0$$

Since $\mathscr{E} > 0$, the direction of emf is the same as the direction of reference direction, that is clockwise. The direction of emf can also be determined by the Lenz's law. The current in the loop sets up a magnetic field to oppose the "change" of flux Φ_m which decreases caused by pulling the loop out of the field, so the direction of induced current is in clockwise to set up a magnetic field that increases the Φ_m.

(2) Since there is a current in the loop *ABCDA*, which is placed in magnetic field, Ampere forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ will act on three sides of the conductors that make up the loop. As Fig. 9-7(b) shows, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are equal and opposite, and cancel out each other. $\boldsymbol{F}_3$ is the force that opposes external effort to move the loop, so the external force $\boldsymbol{F}$ is then given in magnitude by

$$F = F_3 = ILB$$

The induced current in the loop is

$$I = \frac{\mathscr{E}}{R} = \frac{BLv}{R}$$

And the rate at which that external force does work is then

$$P = Fv = \frac{B^2 L^2 v^2}{R}$$

If we calculate the rate at which energy appears in the loop, we find

$$P_t = I^2 R = \frac{B^2 L^2 v^2}{R}$$

which is exactly equal to the rate at which external force does work on the loop. This means that the moving conductor can be viewed as a source, where mechanical energy is transformed into electric energy.

Example 9-2 A conductor ring of radius R is placed stationary in a uniform external magnetic field, as shown in Fig. 9-8. Suppose that the magnitude of $\boldsymbol{B}$ is changing with time at a steady rate $\frac{\mathrm{d}B}{\mathrm{d}t}\left(\frac{\mathrm{d}B}{\mathrm{d}t} > 0\right)$. Find out the induced emf appearing in the ring.

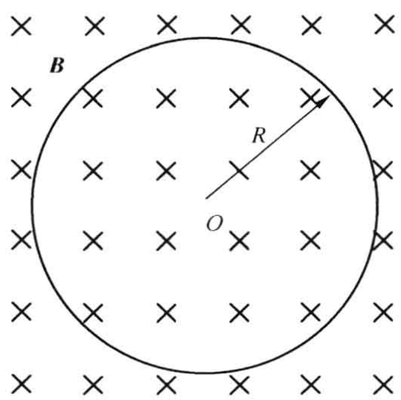


Fig. 9-8 For Example 9-2

Solution Assuming that the reference direction is clockwise, by the right-hand rule, the direction of area vector $\boldsymbol{n}$ is the same as the direction of magnetic field $\boldsymbol{B}$, so we have

$$\Phi_m = \boldsymbol{B} \cdot \boldsymbol{S} = \pi R^2 B(t)$$

The induced emf is then given by

$$\mathscr{E} = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -\pi R^2 \frac{\mathrm{d}B}{\mathrm{d}t}$$

Since $\frac{\mathrm{d}B}{\mathrm{d}t} > 0$ and then $\mathscr{E} < 0$, so the direction of induced emf is in the opposite direction of reference direction, that is counterclockwise.

As we have seen, the induced emf is produced in Examples 9-1 and 9-2. From the point of view of origin of emf, there are two different kinds of nonelectrostatic forces associated in these two examples, respectively. We shall discuss the origin of nonelectrostatic forces in the next two sections.

9.3 Motional Electromotive Force

Fig. 9-9 represents a conductor rod of length L in a uniform magnetic field $\boldsymbol{B}$, which directs perpendicularly into the page. If the conductor is set in motion towards the right at a velocity $\boldsymbol{v}$, perpendicular both to its own length and to the magnetic field $\boldsymbol{B}$, a positive charge q (really an electron) within it is moving with the conductor rod at the same time and is acted on by Lorentz force $\boldsymbol{F}_n = q\boldsymbol{v} \times \boldsymbol{B}$ directing from B toward A. Therefore, the nonelectrostatic force for this case is the Lorentz force and its nonelectrostatic field is

$$\boldsymbol{E}_n = \frac{\boldsymbol{F}_n}{q} = \boldsymbol{v} \times \boldsymbol{B} \tag{9-11}$$

By the definition of emf, Eq. (9-2), the emf in the conductor rod is

$$\mathscr{E}_{BA} = \int_B^A \boldsymbol{E}_n \cdot d\boldsymbol{l} = \int_B^A (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \tag{9-12}$$

The direction of $(\boldsymbol{v} \times \boldsymbol{B})$ can be determined by right-hand rule, and $d\boldsymbol{l}$ is the direction of line integral. As shown in Fig. 9-9, $(\boldsymbol{v}\times\boldsymbol{B}) \cdot d\boldsymbol{l} = vBdl$ and we have

$$\mathscr{E}_{BA} = \int_B^A vBdl = BLv$$

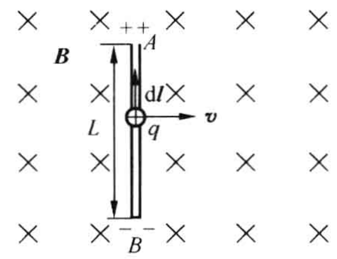


Fig. 9-9 A positive charge q within a moving conductor is acted by Lorentz force $\boldsymbol{F}_n$

Eq. (9-12) is the general expression for the emf induced by moving conductor in magnetic field. Hence, emf is called a motional electromotive force, and the Lorentz force is the non-electrostatic force corresponding to the motional electromotive force. Because of the Lorentz force, free electrons in the conductor rod in Fig. 9-9 move downward, producing a net negative charge at the bottom and leaving a net positive charge at the top. The electrons continue to move down until the electric field produced by the separated charges exerts an upward force on the electrons that balances the Lorentz force. Therefore, the potential difference across the rod is

$$V_{AB} = \mathscr{E}_{BA} = BLv$$

By applying Eq. (9-12) to Example 9-1, we can obtain the same result. For conductors AB and DC, since vector $(\boldsymbol{v} \times \boldsymbol{B})$ is at right angle with $d\boldsymbol{l}$, so that $(\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = 0$; for conductor AD, $\mathscr{E}_{AD} = \int_A^D (\boldsymbol{v}\times\boldsymbol{B}) \cdot d\boldsymbol{l} = BLv$, so the conductor AD is the "source" in Example 9-1. This also leads us to a useful statement: only when the moving conductor cuts lines of magnetic field, the motional electromotive force can be produced.

If we examine the solution process of Example 9-1 carefully, we could find out that the expression $\frac{d\Phi_m}{dt}$ means the "change" of magnetic lines through the loop per unit time interval, which is exactly equal to the number of magnetic lines of magnetic field cut by conductor AD, and the motional emf can be calculated by calculating the number of magnetic field lines cut by moving conductor per second. It provides another method to determine the motional emf produced by an unclosed moving conductor. The motional emf produced by an unclosed moving conductor can also be calculated by Faraday's law, by adding some stationary conductors to form a closed loop, so that the emf calculated by Faraday's law is just equal to the motional emf produced by the moving conductor.

Example 9-3 A copper rod of length L rotates at angular speed ω in a uniform magnetic $\boldsymbol{B}$, as shown in Fig. 9-10. Find the motional emf developed between the two ends of the rod.

Solution The copper rod in Fig. 9-10 may be divided into many length elements. One of them has length dl, moving at velocity $\boldsymbol{v}$ at right angle to the field $\boldsymbol{B}$, so that the direction of

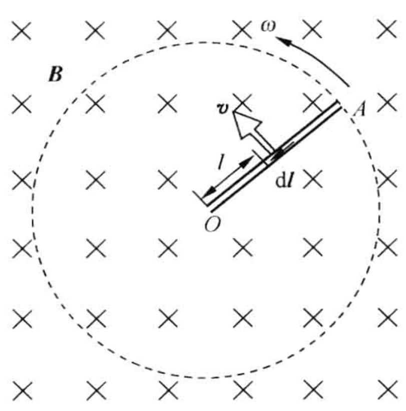


Fig. 9-10 For Example 9-3

vector ($\boldsymbol{v}\times\boldsymbol{B}$) points from A toward O. We take the $\mathrm{d}\boldsymbol{l}$ pointing from A toward O and from Eq. (9-12), we have

$$\mathscr{E}_{AO}=\int_A^O(\boldsymbol{v}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l}=\int_A^O vB\,\mathrm{d}l=\int_0^L \omega Bl\,\mathrm{d}l=\frac{1}{2}B\omega L^2$$

The motional emf in this example can also be calculated in this way: in one second, the conductor turns $\omega/2\pi$ circles. For each circle, the number of magnetic field lines cut by the copper rod is $\pi L^2 B$, thus motional emf has the value

$$\mathscr{E}_{AO}=\frac{\omega}{2\pi}\pi L^2 B=\frac{1}{2}B\omega L^2$$

Example 9-4 In Fig. 9-11, a wire perpendicular to a long straight current-carrying wire is moving at a speed $v=10\mathrm{m/s}$ parallel to the direction of the current. The current is 10 A.

(1) What is the magnitude of the electromotive force between the two ends of the moving wire AB?

(2) What is the potential difference between the ends of the moving wire?

Solution (1) The notations are indicated in Fig. 9-11. According to the right-hand rule, the magnetic field $\boldsymbol{B}$ due to the current-carrying wire is directed into the plane of the page at the right side of the straight current-carrying wire. Along the wire segment AB, the magnitude of the $\boldsymbol{B}$ is $B=\mu_0 I/2\pi r$. Since $\boldsymbol{v}$ is perpendicular to $\boldsymbol{B}$, the cross product $\boldsymbol{v}\times\boldsymbol{B}$ has a magnitude of Bv and directs from B toward A, so the direction of emf points from B toward A. From Eq. (9-12), we have

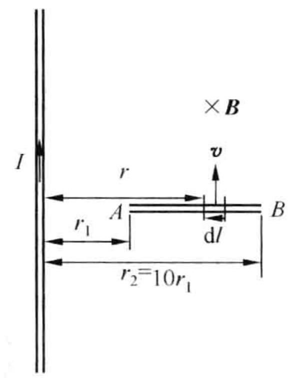


Fig. 9-11 For Example 9-4

$$\mathscr{E}_{BA}=\int_B^A(\boldsymbol{v}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l}=\int_B^A vB(-\mathrm{d}r)=-\int_{r_2}^{r_1}\frac{\mu_0 Iv}{2\pi r}\mathrm{d}r=\frac{\mu_0 Iv}{2\pi}\ln\frac{r_2}{r_1}$$

Using $I=10$ A, $v=10$ m/s and $\frac{r_2}{r_1}=10$, we find that

$$\mathscr{E}_{BA}=46\ \mu\mathrm{V}$$

(2) Because the moving wire AB can be regarded as a source, in an open circuit, using Eq. (9-8), we have

$$V_{AB}=V_A-V_B=\mathscr{E}_{BA}=46\ \mu\mathrm{V}$$

Since the direction of emf is from B toward A, the potential at end A is higher than that of end B.

It should be pointed out that in our discussion given above, only the effect of one component of Lorentz force is concerned. As shown in Fig. 9-12(a), the velocity of the positive charges in moving conductor relative to the magnetic field is $\boldsymbol{u}+\boldsymbol{v}$, where $\boldsymbol{u}$ is the velocity of positive charges relative to conductor, and $\boldsymbol{v}$ is the velocity of conductor relative to magnetic field. Suppose the magnetic field $\boldsymbol{B}$ is points perpendicularly out of the page, the total Lorentz force $\boldsymbol{f}=q(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{B}$, which is perpendicular to vector $\boldsymbol{u}+\boldsymbol{v}$. The total Lorentz force $\boldsymbol{f}$ can be resolved into two components $\boldsymbol{f}_1=q(\boldsymbol{v}\times\boldsymbol{B})$ and $\boldsymbol{f}_2=q(\boldsymbol{u}\times\boldsymbol{B})$. The component $\boldsymbol{f}_1$ performing positive work on positive charge, is the nonelectrostatic force responsible to the motional emf, and the other component $\boldsymbol{f}_2$, which is equal to the external force, does negative work on positive charges.

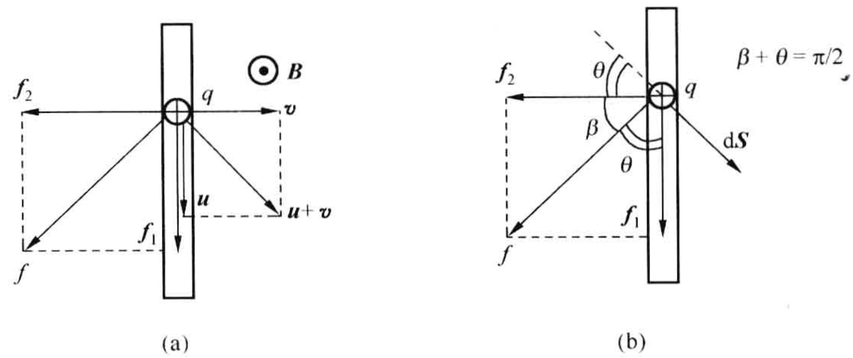


Fig. 9-12 (a) The total Lorentz $\boldsymbol{f}$ can be resolved into two components. (b) For an element displacement d$\boldsymbol{S}$, the total work performed by $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is equal to zero.

For an element displacement d$\boldsymbol{s}$ (Fig. 9-12(b)), the element work performed by $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are $\mathrm{d}W_1$ and $\mathrm{d}W_2$. Since $\mathrm{d}W_1 = qvB\mathrm{d}s\cos\beta = qvB\sin\theta\mathrm{d}s$, and $\mathrm{d}W_2 = -quB\cos\theta\mathrm{d}s$, from Fig. 9-12(b), we have $u/v=\tan\theta$, so that $u\cos\theta = v\sin\theta$, thus $\mathrm{d}W_1+\mathrm{d}W_2= 0$. It means that the work performed by total Lorentz force $\boldsymbol{f}$ is zero. In fact, the work done by component $\boldsymbol{f}_1$ is exactly equal to the work done by external force, so the Lorentz force just play the role of energy transmitter.

9.4 Induced Electric Fields

In the preceding section, we have discussed the emf induced by the relative motion of magnetic field and conductors. But as we have known that according to Faraday's law of induction, the induced emf can also occur in stationary conductors if the magnetic field is changing with time, such as that of the Example 9-2. Since an emf occurs in the ring, a nonelectrostatic force must present within the ring.

We have previously associated electromotive force with an equivalent nonelectrostatic field $\boldsymbol{E}_\mathrm{n}$ (see Eq. (9-1) and Eq. (9-2)). The emf appearing in the ring is the line integral of $\boldsymbol{E}_\mathrm{n}$ around the ring (see Eq. (9-3))

$$\mathscr{E} = \oint_l \boldsymbol{E}_\mathrm{n} \cdot \mathrm{d}\boldsymbol{l} \tag{9-13}$$

The $\boldsymbol{E}_\mathrm{n}$ exerts a force of nonelectrostatic character on the charges and plays the role of nonelectrostatic force which corresponds to the emf induced by the changing of magnetic field. $\boldsymbol{E}_\mathrm{n}$ therefore is called **induced electric field**, which is one of Maxwell's famous hypothesis.

If we combine Eq. (9-13) with Eq. (9-9) we can write Faraday's law of induction as

$$\mathscr{E} = \oint_l \boldsymbol{E}_\mathrm{n} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi_\mathrm{m}}{\mathrm{d}t} \tag{9-14}$$

Eq. (9-14) is another form of Faraday's law of induction, but it shows that **"a changing magnetic field produces an electric field", which is called the induced electric field.** The changing magnetic field appears on the right side of this equation, and the electric field in the integration.

There are two aspects in which Eq. (9-9) and Eq. (9-14) differs from each other:

(1) the closed loop of Eq. (9-9) may be a moving one or a still one, but the closed loop of Eq. (9-14) is a still one;

(2) Eq. (9-9) involves both the motional emf and the emf produced by induced electric field (see Example 9-1 and Example 9-2), Eq. (9-14), however, is suited for all kinds of closed loop, even though it is not a real one.

The induced electric fields are not associated with static charges but with a changing magnetic field. Although both static field (set up by static charges) and induced electric field exert forces on test charges, there are some major differences between them:

(1) Since the electrostatic field is set up by static charges, the lines of the field originate on positive charges and terminate on negative charges, and are never closed; the induced electric field, however, is produced by the changing of magnetic field, and the lines of induced electric field are closed, having neither starting points nor ending points as the lines of magnetic field, so that, induced electric field is a vortex field;

(2) Electrostatic field is a conserved field, that is the line integration of electrostatic field over any closed path has the value of zero ($\oint_l \boldsymbol{E}_e \cdot d\boldsymbol{l} = 0$), but the integration of induced electric field over a closed loop is not equal to zero ($\oint_l \boldsymbol{E}_n \cdot d\boldsymbol{l} = -\frac{d\Phi_m}{dt} \neq 0$), so the induced electric field is not a conserved field.

It should point out that the emf of any loop appears only when the magnetic field flux of the loop is changing and it has nothing to do with what the loop is made up, even though the loop is not exist.

Example 9-5 In Fig. 9-13(a), assume that a uniform magnetic field fills a cylindrical volume of radius R. The magnetic field points into the page at right angles. Suppose the strength of this field is decreasing at a steady rate $(dB/dt) < 0$.

(1) What is the magnitude of induced electric field $\boldsymbol{E}_n$ for $r < R$?

(2) What is the magnitude of induced electric field $\boldsymbol{E}_n$ for $r > R$?

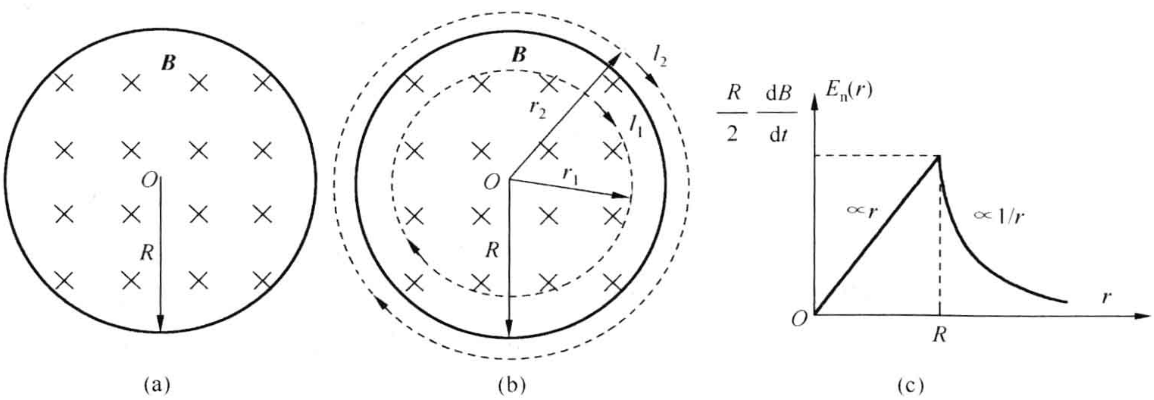


Fig. 9-13 For Example 9-5

Solution (1) From Lenz's law and symmetry we can deduce that the induced electric field $\boldsymbol{E}_n$ at all points around a circular path must be tangent to the circle, and its magnitude is equal. Thus the electric lines of force set up by the changing magnetic field are in this case a set of concentric circles. You should have noticed that the lines of the induced electric field are closed. If $dB/dt<0$, the direction of line of $\boldsymbol{E}_n$ is counterclockwise as shown in Fig. 9-13(b).

For the closed integral path $l_1(r=r_1<R)$, take the direction of counterclockwise as the reference direction (Fig. 9-13(b)). From Faraday's law of induction Eq. (9-14) we have

$$\oint_{l_1} \boldsymbol{E}_n \cdot d\boldsymbol{l} = \oint_{l_1} E_n dl = E_n 2\pi r = -\frac{d\Phi_m}{dt}$$

We note that for $r<R$, the magnetic flux through the surface bounded by l_1 is $\Phi_m = B\pi r^2$, so that

$$E_n 2\pi r = -\pi r^2 \frac{dB}{dt}$$

Solving for E_n, we find

$$E_n = -\frac{r}{2}\frac{dB}{dt} \quad (r<R) \tag{9-15}$$

Note that the induced electric field depends on dB/dt, but not on $\boldsymbol{B}$.

(2) For the closed path $l_2(r=r_2>R)$ (Fig. 9-14(b)), we have

$$\oint_{l_2} \boldsymbol{E} \cdot d\boldsymbol{l} = 2\pi r E_n = -\frac{d\Phi_m}{dt}$$

The flux Φ_m through the surface bounded by l_2 is $\Phi_m = B\pi R^2$, and then we have

$$E_n = -\frac{1}{2}\frac{dB}{dt}\frac{R^2}{r} \quad (r>R) \tag{9-16}$$

Interestingly an electric field is induced in this case even at points that are well outside the changing magnetic field. A plot of the induced electric field $E_n(r)$ is shown in Fig. 9-13(c).

In the following, we take the betatron as an example to introduce the application of the induced electric field. The betatron is a device used to accelerate electrons to high energies by induced electric field, which was invented in 1941 by Donald W. Kerst. Although it is not widely used today, it affords an excellent reality of the induced electric field produced by a varying magnetic field.

Fig. 9-14 shows a cross section of a betatron in a plane that contains its vertical symmetry axis. The alternating magnetic field is produced by means of an alternating current in coils (not shown) around the iron pole pieces. The object marked D is an circular vacuum tube inside of which the electrons circulate are accelerated. The orbit is a circle of radius R, its plane being at right angles to the page. In the Fig. 9-15 the electrons are circulating counterclockwise as viewed from above. Thus, we see them emerging from the figure on the left and entering it on the right. At this instant the distributions of induced electric field $\boldsymbol{E}_n$ and the magnetic field $\boldsymbol{B}$ are shown in Fig. 9-15.

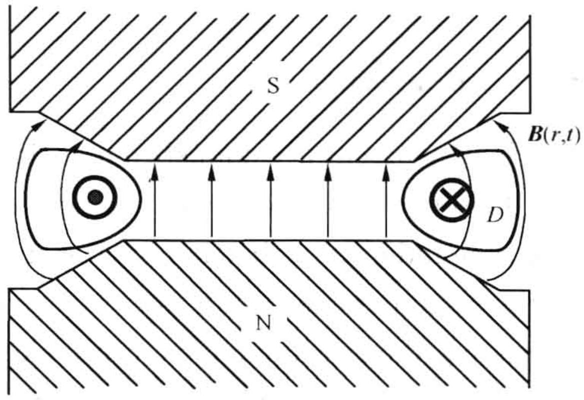


Fig. 9-14 A cross section of a betatron in a plane that contains its vertical symmetry axis

In order to accelerate electrons to high energies, the electrons should be restricted circling in the orbit of tube. The electrons are therefore required to be acted on by a centripetal force: $e(\boldsymbol{v} \times \boldsymbol{B})(e = -1.6\times 10^{-19}$ C). Since magnetic field $\boldsymbol{B}$ is out the plane of the diagram and the lines of induced electric field should be clockwise as shown in Fig. 9-15, then the magnitude of $\boldsymbol{B}$ should be increased that is $\frac{dB}{dt} > 0$. If the magnetic field points out off plane of the diagram when the current in the coil around the iron pole is positive as shown in Fig. 9-16, only in the first 1/4 cycle the condition required is satisfied and in this special interval available the electrons have turned several hundred thousand rounds and the acceleration process is finished.

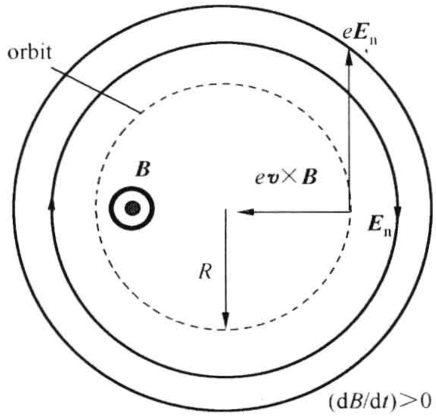


Fig. 9-15 The distribution of electric field $\boldsymbol{E}_n$ and magnetic field $\boldsymbol{B}$ of a betatron at a instant

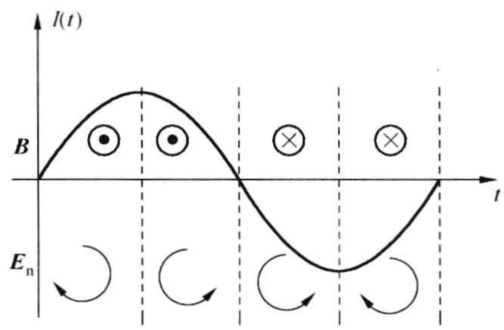


Fig. 9-16 Only in the first 1/4 cycle, is the condition required satisfied

As the electron is accelerated by the induced electric field, its speed v increases and a larger $\boldsymbol{B}$ field is required to provide the necessary centripetal force. The increase of current is one factor and the non-uniform distribution of magnetic field is another factor. We will not discuss the detail (about this) here.

9.5 Self-induction and Mutual-induction Phenomena

9.5.1 Self-induction

When a current is established in a closed circuit, it will set up a magnetic flux through the circuit loop itself. If the current is now changing with time, according to Faraday's law, an emf will be induced in the circuit. For instance, in Fig. 9-17(a), the B_1 and B_2 are two bulbs with the same parameters, L is a coil winding with iron core and R is an adjustable resistor. By adjusting the resistor R, we can make the resistances in two branches of the circuit have the same value. If we switch the battery on, bulb B_2 becomes bright immediately, while bulb B_1 lights up slowly. In Fig. 9-17(b), if we switch off the battery, the bulb doesn't die out immediately, but flashes momentarily; this shows that the current in the circuit doesn't stop as soon as the battery is cut off.

These phenomena can be interpreted as follows. In Fig. 9-17(a), when the switch S is on, because the current passing through L increases from zero, the flux of magnetic through L

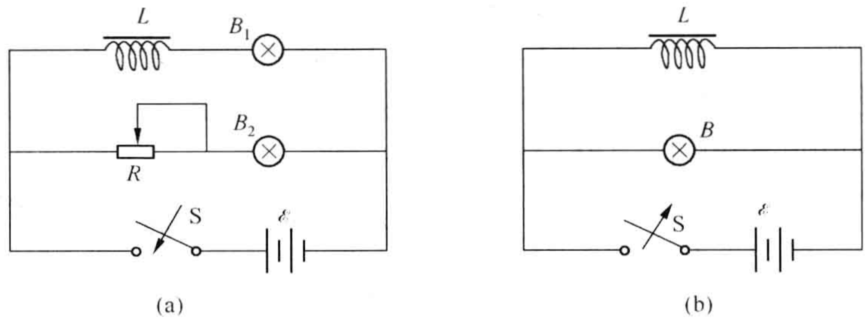


Fig. 9-17 (a)(b) Two demonstrational experiments of self-induction

is increased with the time so that an emf is then induced in the winding. According to Lenz's law, the induced electromotive force within L opposes the current increasing, so when the switch is turned on, the current passing through B_1 increases more slowly than that of B_2. In Fig. 9-17(b), when the switch S is turned off, the current passing through L has the tendency to decrease, an emf is then set up to oppose this decreasing, and this emf will cause the bulb flashing momentarily while the battery is off.

Since the emfs in these circuits are produced by their own current changing, the phenomenon is called self-induction and the emf induced in such way is called a self-induced emf.

Suppose that the magnetic flux through the L in Fig. 9-17(a) is Φ_m, which is proportional to the current in the circuit. So the Φ_m can be written as

$$\Phi_m = LI \tag{9-17}$$

in which L is a constant determined by the size, the shape, and the number of turns as well as the magnetic properties of the material around the circuit. If no ferromagnetic is present, L has no relation with the current in the circuit. Faraday's law of induction tells us that

$$\mathscr{E} = -\frac{d\Phi_m}{dt} = -\left(I\frac{dL}{dt} + L\frac{dI}{dt}\right) \tag{9-18}$$

If the shape of circuit and the magnetic medium around the circuit don't vary with time, and the ferromagnetic material is not present, we have

$$\frac{dL}{dt} = 0$$

Eq. (9-18) becomes

$$\mathscr{E} = -L\frac{dI}{dt} \tag{9-19}$$

From Eq. (9-19) we know that the self-induced emf is proportional to L. Here L is called self inductance. The SI unit of L is henry (abbr. H). From Eq. (9-17) and Eq. (9-19) we have

$$1\ \text{H} = 1\ \text{W/A} \quad \text{or} \quad 1\ \text{H} = 1\ \text{V}\cdot\text{s/A}$$

Both Eq. (9-17) and Eq. (9-19) can be used to determine L.

A circuit or part of a circuit which has self-induction is called an inductor.

Example 9-6 A long and thin straight solenoid with length l and cross sectional area S consists of N turns distributed uniformly over its length. A uniform magnetic medium with permeability μ is filled in the solenoid, as shown in Fig. 9-18. Find out the self inductance of the solenoid.

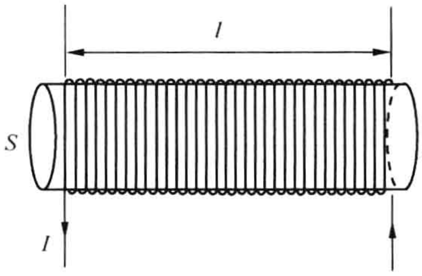


Fig. 9-18 For Example 9-6

Solution If a current I is established in the windings of the solenoid, the magnetic field $\boldsymbol{B}$ in the solenoid has a value of $\mu NI/l$. The total magnetic flux which is called the number of flux linkages equals

$$\Phi = N\Phi_m = NBS = \frac{\mu N^2 IS}{l}$$

By the definition of inductance (Eq. (9-17)), the self inductance L is given by

$$L = \frac{\Phi}{I} = \frac{\mu N^2 S}{l} = \frac{\mu N^2 S \times l}{l \times l} = \mu n^2 V$$

where $V = Sl$ is the volume of the solenoid, and $n = N/l$ is the turns of wire per unit length.

Like the capacitance C, the self inductance L depends on geometrical factors and on medium properties as well.

Since the solenoid is long and tightly wounded, the spreading of magnetic field lines near the edges of the solenoid can be neglected, so that the expression $L = \frac{\mu N^2 S}{l} = \mu n^2 V$ is a good approximation, just as the parallel plate capacitor formula $\left(C = \frac{\varepsilon_r \varepsilon_0 S}{d}\right)$. For a real solenoid, its self inductance is slightly smaller than $L = \frac{\mu N^2 S}{l} = \mu n^2 V$. If there is not magnetic medium in the solenoid, the inductance is $L = \mu_0 n^2 V$.

Example 9-7 Derive an expression for the self inductance of a toroid of rectangular cross section as shown in Fig. 9-19. Parameters are given in the figure. Evaluate the self inductance for $N = 1000$, $a = 5.0$ cm, $b = 10.0$ cm and $h = 1.0$ cm.

Solution Since the magnetic field lines for the toroid are concentric circles, applying Ampere's law $\oint_l \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \sum I_0$ to a circular path of radius r yields

$$2\pi r B = \mu_0 NI$$

where N is the number of turns and I is the current in the toriod windings. Solving B yields

$$B = \frac{\mu_0 NI}{2\pi r}$$

The flux Φ_m for one turn of windings is

$$\Phi_m = \int_S \boldsymbol{B} \cdot d\boldsymbol{S} = \int_a^b Bh\,dr = \int_a^b \frac{\mu_0 NIh}{2\pi r} dr = \frac{\mu_0 NIh}{2\pi} \ln \frac{b}{a}$$

where $dS = h\,dr$ is the area of the elementary strip shown in the Fig. 9-19. The total flux of the toroid is

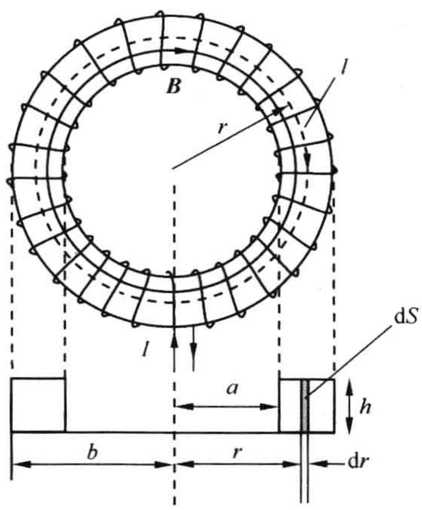


Fig. 9-19 For Example 9-7

$$\Phi = N\Phi_m = \frac{\mu_0 N^2 Ih}{2\pi} \ln \frac{b}{a}$$

From Eq. (9-16), the self inductance is given by

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$$

Substituting numerical values yields

$$L = \frac{(4\pi \times 10^{-7}) \times (10^3)^2 \times (1.0 \times 10^{-2})}{2\pi} \ln \frac{10 \times 10^{-2}}{5 \times 10^{-2}} = 1.4(\mathrm{mH})$$

The result of this example shows again that the self inductance L depends on geometrical factors, turns and magnetic medium properties as well, just as the capacitance C.

9.5.2 Mutual induction

If two coils are placed together, as in Fig. 9-20, the current in coil 1 will set up a magnetic flux through coil 2. While the current in coil 1 changes with time, by Faraday's law of induction, an emf will appear in coil 2. We could have better call this mutual-induction phenomenon, and the emf induced is called mutual induced emf .

Similar to self-induction, the magnetic flux through coil 2 by the current in coil 1 is proportional to the current I_1 in coil 1 and depends on the relative position of coil 1 and coil 2. The flux can be written as

$$\Phi_{21} = M_{21} I_1 \tag{9-20}$$

in which M_{21} is determined by the shapes of coil 1 and coil 2 and relative positions of them.

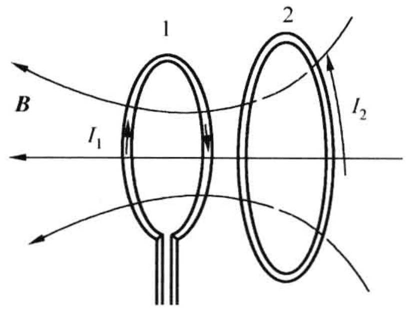


Fig. 9-20 If the current in coil 1 changes with time, an induced emf will appear in coil 2

The mutual induced emf appearing in coil 2 is given by Faraday's law as follows

$$\mathscr{E}_2 = -\frac{d\Phi_{21}}{dt} = -M_{21}\frac{dI_1}{dt} - I_1 \frac{dM_{21}}{dt} \tag{9-21}$$

If there is no ferromagnetic material around coil 1 and coil 2, Eq. (9-21) becomes

$$\mathscr{E}_2 = -M_{21}\frac{dI_1}{dt} \tag{9-22}$$

Let us now interchange the roles of coil 1 and coil 2 in Fig. 9-20, that is, we set up a current I_2 in coil 2, which produces a magnetic flux Φ_{12} passing through coil 1. By the same argument given above, we have

$$\mathscr{E}_1 = -M_{12}\frac{dI_2}{dt} \tag{9-23}$$

We can see that the mutual induced emf in either coil is proportional to the rate of changing of current in the other coil. The constants M_{12} and M_{21} seem to be different, but we assert, without proof, that they are in fact the same in value. So we have

$$M_{21} = M_{12} = M$$

And we can rewrite Eq. (9-20), Eq. (9-22) and Eq. (9-23) as

$$\Phi_{21} = MI_1 \tag{9-24}$$

$$\mathscr{E}_2 = -M\frac{dI_1}{dt} \tag{9-25}$$

$$\mathscr{E}_1 = -M\frac{\mathrm{d}I_2}{\mathrm{d}t} \qquad (9\text{-}26)$$

Analogy to the self inductance, the M is called mutual inductance. The SI unit for M is henry, the same as that of L.

Like that of L, the value of M depends on the geometry shapes of the two coils, the relative positions and the magnetic material around the coils as well.

Example 9-8 Fig. 9-21 shows two circular closely packed coils, having N_1 and N_2 turns, respectively. The smaller(radius R_2) is coaxial with the larger (radius R_1), and they are on the same plane. Assuming $R_1 \gg R_2$, find the expression for the mutual inductance M for this arrangement.

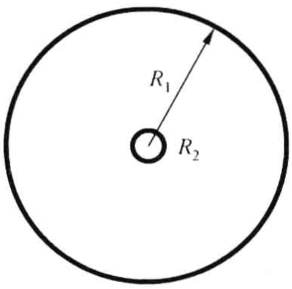


Fig. 9-21 For Example 9-8

Solution Supposing the current in the large coil is I_1, the magnetic field B_1 produced by I_1 at the center of smaller coil is

$$B_1 = \frac{N_1\mu_0 I_1}{2R_1}$$

Note that $R_1 \gg R_2$, so we may consider that the magnetic field in the area of smaller coil is a constant which equals $B_1 = \frac{N_1\mu_0 I_1}{2R_1}$. The magnetic flux through the smaller coil is

$$\Phi_{21} = N_2 B_1 S_2 = \frac{N_1 N_2 \mu_0 I_1 \pi R_2^2}{2R_1}$$

From Eq. (9-24), we then have

$$M = \frac{\Phi_{21}}{I_1} = \frac{N_1 N_2 \mu_0 \pi R_2^2}{2R_1}$$

This result shows that the mutual inductance depends on the geometrical factors (R_1, R_2), turns (N_1, N_2) and on the properties of magnetic medium (μ_0) as well.

9.6 Energy of the Magnetic Field

As we have known that there is electric energy stored in the electric field, the magnetic field also has energy stored in it. To understand this, let's see a demonstrational experiment. In Fig. 9-22, L is an inductor, and B is a bulb. The switch K is formally closed on point a, and the current in the circuit becomes steady. When the switch K is thrown to point b, the bulb does not die out immediately, but flashes brighter momentarily. As the battery is cut off, the only energy supply to the bulb's lighting must be the energy stored in the inductor L. Since the current through the inductor establishes a magnetic field, as long as the current is maintained, so we can conclude that the magnetic field stores a kind of energy—magnetic energy. The energy stored in magnetic field is also demonstrated in experiment of Fig. 9-17(b). To derive a quantitative expression of the energy stored in the magnetic field, suppose the resistance of the bulb B is R, the resistance of source is r, the self inductance of the inductor is L and the emf of source is $\mathscr{E}$. Considering that the switch is just on a, the current increases from zero, that is, when $t=0$, $I=0$ and when $t=t_0$, $I=I_0$. According to the complete circuit law, we have

$$\mathscr{E} - L\frac{\mathrm{d}I}{\mathrm{d}t} = I(R+r) \qquad (9\text{-}27)$$

Multiplying each side of Eq. (9-27) by $I\mathrm{d}t$ and reordering the equation, we obtain

$$\mathscr{E}\, I\mathrm{d}t = I^2(R+r)\mathrm{d}t + LI\mathrm{d}I$$

Integrating yields

$$\int_0^{t_0} \mathscr{E}\, I\mathrm{d}t = \int_0^{t_0} I^2(R+r)\mathrm{d}t + \int_0^{I_0} LI\mathrm{d}I \qquad (9\text{-}28)$$

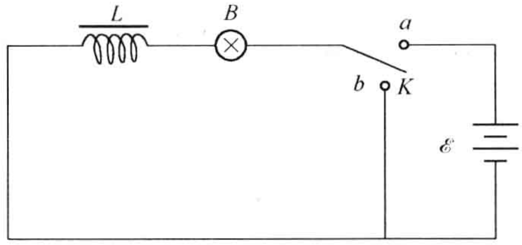


Fig. 9-22 A demonstration of energy stored in magnetic field

Eq. (9-28) suggests that the energy $\int_0^{t_0} \mathscr{E}\, I\mathrm{d}t$, supplied by the source from time $t = 0$ to $t = t_0$, is partly transferred in Joule thermal energy $\left(\int_0^{t_0} I^2(R+r)\mathrm{d}t\right)$ and the rest is used to do the work to establish the magnetic field $\left(\int_0^{I_0} LI\mathrm{d}I\right)$. So the energy stored in magnetic field of the inductor when $I = I_0$ equals to

$$W_{\mathrm{m}} = \int_0^{I_0} LI\mathrm{d}I = \frac{1}{2}LI_0{}^2 \qquad (9\text{-}29)$$

This equation is similar to the expression for the electric energy stored in a charged capacitor, namely: $W_{\mathrm{e}} = \frac{Q^2}{2C}$. Comparing the expression W_{m} and W_{e}, one could easily see that the roles of quantities L and C and I and Q are much similar in their own fields.

The density of magnetic energy w_{m} is an important and useful concept for describing and calculating the energy stored in the magnetic field. For simplicity, consider a very long solenoid of length l and the cross area S, and Sl is the volume V associated with this length. The magnetic energy is stored entirely within the solenoid, because the magnetic field outside the solenoid is approximately zero. Moreover, the stored energy is uniformly distributed throughout the volume of the solenoid, because the magnetic field is uniform everywhere inside. Since $B = \mu_0 nI$ and $L = \mu_0 n^2 V$ (see Example 9-6), thus the energy stored in the volume $V=Sl$ is

$$W_{\mathrm{m}} = \frac{1}{2}LI_0^2 = \frac{1}{2}\mu_0 n^2 V\left(\frac{B}{\mu_0 n}\right)^2 = \frac{1}{2}\frac{B^2}{\mu_0}V$$

Using $B = \mu_0 H$, we have

$$w_{\mathrm{m}} = \frac{W_{\mathrm{m}}}{V} = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\frac{B\times\mu_0 H}{\mu_0} = \frac{1}{2}BH \qquad (9\text{-}30)$$

It is obvious that the density of magnetic energy w_{m} has the formula similar to that of density of electric energy, which is $w_{\mathrm{e}} = \frac{1}{2}ED$. The quantities $\boldsymbol{B}$ and $\boldsymbol{H}$ in the magnetic field are corresponding to $\boldsymbol{E}$ and $\boldsymbol{D}$ in electric field, respectively. For more information, see Table 9-1, in which the corresponding quantities of electric field and magnetic field are listed.

It should point out that even though Eq. (9-30) was derived from a special case (uniform magnetic field), it is valid for any magnetic field. For any magnetic field, the total magnetic energy W_{m} stored in the field can be calculated by integration.

$$W_m = \int_V w_m dV = \int_V \frac{1}{2} BH dV \tag{9-31}$$

where V is the region of space in which a magnetic field exists.

Table 9-1 Some corresponding quantities between electrical field and magnetic field

Capacitor C, capacitance $C=q/V$ depends on geometry factors and medium properties	Inductor L, inductance $L=\Phi/I$ depends on geometry factors and medium properties
Parallel plane capacitor sets up a uniform electric field and is used to derive the density of electric energy	Long thin solenoid sets up a uniform magnetic field and is used to derive the density of magnetic energy
Constant ε, ε_r, and $\varepsilon=\varepsilon_r\varepsilon_0$	Constant μ, μ_r, and, $\mu=\mu_r\mu_0$
Energy storage: $W_e=\dfrac{Q^2}{2C}$ Energy density: $w_e=\dfrac{1}{2}ED$	Energy storage: $W_m=\dfrac{1}{2}LI^2$ Energy density: $w_m=\dfrac{1}{2}BH$

Example 9-9 A long coaxial cable consists of two thin-walled concentric conducting cylinders with radii a and b as shown in Fig. 9-23. Its central conductor carries a steady current I, the out cylinder providing the return path for that current. Calculate the energy stored per unit length.

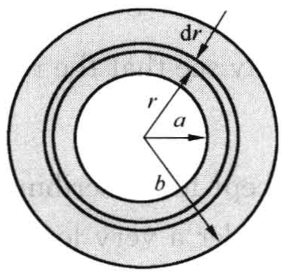


Fig. 9-23 For Example 9-9

Solution The magnetic field outside the cable is zero. So the magnetic energy is stored in the space between the two conductors.

In the space between the two conductors, by using Ampere's law $\oint_l \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \sum I$, leads to

$$B = \frac{\mu_0 I}{2\pi r}$$

The density of magnetic energy is then

$$w_m = \frac{1}{2} BH = \frac{\mu_0 I^2}{8\pi^2 r^2}$$

Consider an element volume dV consisting of a cylindrical shell which has inner radius r and our radius $r+dr$ and unit length $(=1\text{ m})$. Thus the volume of the shell is $2\pi r dr$. The energy dW_m stored within this shell is

$$dW = w_m dV = \frac{\mu_0 I^2}{8\pi^2 r^2} 2\pi r dr = \frac{\mu_0 I^2}{4\pi r} dr$$

The total energy is then

$$W_m = \int dW = \int w_m dV = \int_a^b \frac{\mu_0 I^2}{4\pi r} dr = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}$$

9.7 Displacement Current and Complete Current Law

In the section 9-4, we have already known that a changing magnetic field induces an electric field. In this section, we will show that the converse is also true, that is, a changing electric field induces a magnetic field. The law describing this induction effect of electric field was established by James Clerk Maxwell, the great Scottish physicist, who thereby achieved a wide-ranging unification of the laws of electricity and magnetism, known as Maxwell's equations. The laws discussed from chapter 6 to chapter 8, concluded from electrostatic field and the magnetic field produced by a steady current, are included in Maxwell's equations as the special cases.

The significant achievement of Maxwell's equations is that it predicts the existence of electromagnetic waves in the space and moving with the speed of light! This has deeply influenced our understanding about the nature of light; led to the discovery of radio wave, became the basis of the principles of radio broadcast, television transmitters and receivers, telephones, electromagnets, radars as well as microwave ovens etc..

9.7.1 Displacement current

We have seen how the principle of symmetry permeates throughout the world of physics and how it has often led to new insights or discoveries. It is as though nature were hinting and guiding physicists in their explorations. Maxwell's production that a changing magnetic field produces an electric field successfully interests the generation of induced electromotive force. It was the consideration of symmetry coupled with the analogy method that promoted Maxwell to boldly produce the existence of the converse phenomenon of electromagnetic induction, that is, a changing electric field must produce a magnetic field. In respect of producing a magnetic field, a changing electric field is equivalent to a current which is called as a displacement current.

Here we discuss in detail the evidence for the supposition mentioned above. First, let us examine the application of Ampere's law

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \sum I \tag{9-32}$$

which we became acquainted with in chapter 8, involving a changing electric field. In Fig. 9-24a, parallel plates capacitor is connected to long straight wires, suppose that the wires carry a constant current I which steadily charges up the capacitor. Draw a closed loop L around the wire, choose two surfaces, the flat surface S_1 and the baglike surface S_2, the intersection of S_1 and S_2 is loop L. Let us now apply Ampere's law to S_1 and S_2, respectively. Because S_1 intercepts current I, we have

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I \tag{9-33a}$$

but the baglike surface S_2 passes between the capacitor plates and intercepts no current, hence the right side of Eq. (9-32) is zero, that is

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{9-33b}$$

Note that the left side of above equations is absolutely the same, so that Eq. (9-33a) and Eq. (9-33b) show that Ampere's law fails for the choice of surface.

To resolve this contradiction, the question is then: can we modify Ampere's law so that it gives the right answer for the both surface. To discover the required modification we note that although S_2 does not intercept current in the region between the capacitor plates, it does intercept electric flux as shown in Fig. 9-24 the field lines going from the positive plate to the negative plate cross the baglike surface S_2. This suggests that the required correction to Ampere's law involves the electric flux or electric displacement flux Φ_D.

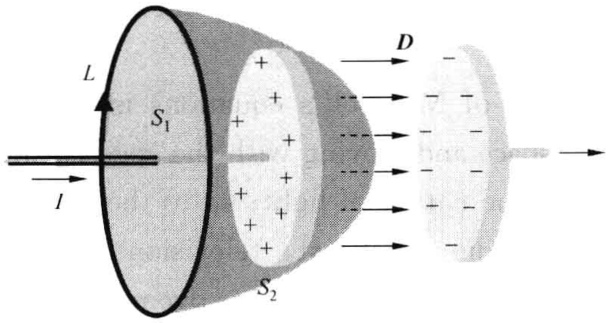


Fig. 9-24 Surface S_1 intercepts current I, the baglike surface S_2 passes the gap between the plates of a capacitor, and intercepts no current. The baglike surface S_2 intercepts electric flux

Second, let us apply Gauss' law to the closed curved surface formed by S_1 and S_2 in Fig. 9-24, that is

$$\Phi_D = \oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = q \tag{9-34}$$

or

$$\Phi_D = \int_{S_1} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} + \int_{S_2} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_{S_2} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = q$$

The rate of change of the displacement flux is then

$$\frac{\mathrm{d}\Phi_D}{\mathrm{d}t} = \int_{S_2} \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}q}{\mathrm{d}t} = I \tag{9-35}$$

where $\frac{\mathrm{d}q}{\mathrm{d}t} = I$ is the magnitude of conductive current.

The quantity of $\frac{\mathrm{d}\Phi_D}{\mathrm{d}t}$ is defined as displacement current I_d,

$$I_d = \frac{\mathrm{d}\Phi_D}{\mathrm{d}t} \tag{9-36}$$

This quantity is of course not a true current, but Maxwell proposed that displacement current has the same effect as conductive current in respect of producing a magnetic field. And the direction of I_d is from the positive plate pointing to the negative plate passing through the gap between them.

By generalizing the definition of current, we can hold on to the notion that current is continuous. In Fig. 9-24, a current I enters the positive plate and leaves the negative plate. The conduction current is not continuous across the capacitor gap because no charge is transported across this gap. However, the displacement current I_d in the gap is exactly equal to the conduction current in magnitude, note that the displacement current has the same direction as the ordinary current arriving at the plates thus retaining the continuity of current.

9.7.2 Maxwell's modification of Ampere's law

According to Maxwell's supposition, there are at least two ways to set up a magnetic field:

(1) by a conduction current and (2) by a displacement current—a changing electric field, from this point of view, considering both possibilities, Maxwell generalized **the Ampere's law** Eq. (9-33) **in its complete form**

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I + \frac{\mathrm{d}\Phi_D}{\mathrm{d}t} \tag{9-37}$$

Eq. (9-37) is Maxwell's modification of Ampere's Law. Because its right side includes two kinds of "currents" intercepted by whatever surfaces whose bound is the Amperian loop L around which the line integral is taken, Eq. (9-37) is called as the complete current law or Ampere-Maxwell law. The great importance of this equation lies in its general validity-Maxwell boldly proposed system of electric fields, currents and magnetic field.

Eq. (9-37) can be rewritten as

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I + \frac{\mathrm{d}\Phi_D}{\mathrm{d}t} = \int_S \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} + \int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} \tag{9-38}$$

in which $\boldsymbol{j}$ is conduction current density vector, that is the direction of I. Similarly, the direction of the displacement current $\frac{\mathrm{d}\Phi_D}{\mathrm{d}t}$ is that of the displacement current density vector $\boldsymbol{j}_\mathrm{d}$ which is defined as

$$\boldsymbol{j}_\mathrm{d} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{9-39}$$

The right-hand rule applied to $\boldsymbol{j}_\mathrm{d}$ gives the direction of the associated magnetic field, just as it does for the conduction current density $\boldsymbol{j}$. The surface integral on the right side of Eq. (9-38) is taken along the surface S whose bound is the closed loop L around which the line integral on the left side is taken.

Now, the time has come to resolve the contradiction between Eq. (9-33a) and Eq. (9-33b). Apply Eq. (9-37) to surface S_1 and surface S_2 who have the same bound-closed loop L in Fig. 9-24, the complete current intercepted by S_1 is conduction current I (no displacement current intercepted by S_1); while the complete current intercepted by S_2 is displacement current $\frac{\mathrm{d}\Phi_D}{\mathrm{d}t}$ (no conduction current intercepted by S_2), we get

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I, \quad \oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}\Phi_D}{\mathrm{d}t}$$

From Eq. (9-35), $\frac{\mathrm{d}\Phi_D}{\mathrm{d}t} = I$ in magnitude, the results are the same whatever the surface chosen.

Apart from the term I, the Maxwell's law, the modification of Ampere's Law, Eq. (9-38), is analogous to the Faraday's law, Eq. (9-9). The latter relates the line integral of the electric fields to the rate of change of magnetic flux. Hence these laws indicate a certain symmetry in the effects of electric and magnetic field on one another. Faraday's law means that a changing magnetic field produces an electric field while the Maxwell's proposition of displacement current means a changing electric field produces a magnetic field, and these two kinds of changing fields depend on one another for their productions, this is the central and vital contribution of Maxwell to the electromagnetic field theory.

We must point out that displacement current and conduction current are equivalent only in respect that both of them produce magnetic fields. They are distinguished from each other in the respects that the conduction current is of charges in motion but there is no motion of charges involved in the displacement current. Therefore, according to Joule's law, for a conduction current I, in an unit time, an amount of energy given by I^2R will appear in the conductor of resistance R as thermal energy, while there is no thermal energy effect at all in a conductor with displacement current.

Example 9-10 Fig. 9-25 shows a uniform electric field $\boldsymbol{E}$ filling a cylindrical region of radius R. It might be produced by a circular parallel-plate capacitor, as suggested in Fig. 9-25, and we assume that E is decreasing at a steady rate $\mathrm{d}E/\mathrm{d}t$. Find the magnetic field in both magnitude and direction at the point apart from the axis of the capacitor by r, (1) for $r<R$; (2) for $r>R$.

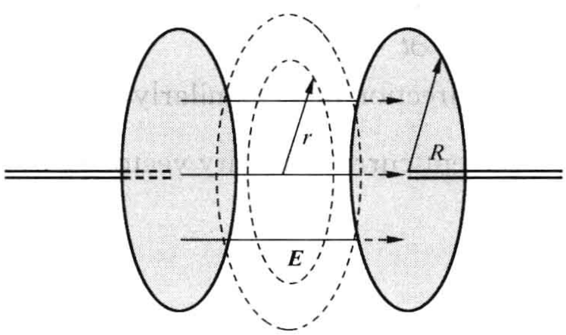


Fig. 9-25 For Example 9-10

Solution From the Maxwell's modification of Ampere's law and with symmetry in mind, we can deduce that the induced magnetic field at all points around a circular path must be tangent to the circle, and the magnitude of $\boldsymbol{B}$ is equal, as shown in Fig. 9-25. Thus, the magnetic field lines must be a set of concentric circles, because this is the only pattern of field lines consistent with the rotational symmetry of the electric field shown in the Fig. 9-25. If $\mathrm{d}E/\mathrm{d}t<0$, the direction of the line of induced magnetic field is clockwise determined by vector $\mathrm{d}\boldsymbol{E}/\mathrm{d}t$ with right-hand rule. By the way, letting us compare Fig. 9-25 with Fig. 9-13 in Example 9-5, we can see the electromagnetic symmetries more clearly. Fig. 9-25 shows a magnetic field produced by a changing electric field while Fig. 9-13 shows the converse. In each figure the appropriate flux, Φ_D or Φ_m is decreasing. However, the lines of $\boldsymbol{B}$ in Fig. 9-25 are clockwise whereas those of $\boldsymbol{E}$ in Fig. 9-13 are counterclockwise; one is determined by right-hand rule, the other by left-hand rule. This is because Ampere's law and Faraday's law differ by a minus sign.

Back to the calculation, (1) inside the capacitor ($r<R$), the electric field is uniform. To find the induced magnetic field, consider a circle L of radius r, the displacement flux through this circle is

$$\Phi_D = \pi r^2 D = \pi r^2 \varepsilon_0 E$$

According to Eq. (9-38), we then have

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}\Phi_D}{\mathrm{d}t} = \pi r^2 \varepsilon_0 \frac{\mathrm{d}E}{\mathrm{d}t}$$

Since the path of integration coincides with a field line, we can immediately evaluate the left side of the above equation.

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 2\pi r H$$

Combining the above two equations, we then obtain

$$2\pi r H = \pi r^2 \varepsilon_0 \frac{\mathrm{d}E}{\mathrm{d}t}$$

so that

$$H = \frac{r}{2}\varepsilon_0 \frac{\mathrm{d}E}{\mathrm{d}t}$$

or

$$B = \frac{r}{2}\varepsilon_0\mu_0 \frac{\mathrm{d}E}{\mathrm{d}t} \quad (r < R)$$

(2) Outside the capacitor ($r>R$), the electric field is zero. Hence the displacement flux through a circle of radius r is

$$\Phi_D = \pi R^2 D = \pi R^2 \varepsilon_0 E$$

the left side of Eq. (9-38) is the same,

$$\oint_L \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 2\pi r H$$

We can then obtain the magnetic field by going through the same steps as above, with the result

$$H = \frac{R^2}{2r}\varepsilon_0 \frac{\mathrm{d}E}{\mathrm{d}t}$$

or

$$B = \frac{R^2}{2r}\varepsilon_0\mu_0 \frac{\mathrm{d}E}{\mathrm{d}t} \quad (r > R)$$

The results of Example 9-10 are obviously analogous to the results of Example 9-5. Note that the magnetic field given by $\frac{R^2}{2r}\varepsilon_0\mu_0 \frac{\mathrm{d}E}{\mathrm{d}t}$ outside the capacitor is the same as that of a current in a long straight wire $\left(B \propto \frac{1}{r}\right)$. As far as this magnetic field is concerned, the removal of a short piece of wire and its replacement by a capacitor makes no difference at all. However, if we take into account the fringing of the electric field at the edges of the capacitor plates, a more precise calculation shows that near the plates the magnetic field differs slightly from that of a current in the long straight wire.

9.8 Maxwell's Equations

Eq. (9-38) or Eq. (9-9) is the last of the fundamental laws that we need to completely describe the behavior of electric and magnetic fields. There are four fundamental laws known as Maxwell's equations, it is the time to make conclusion in the following.

9.8.1 Faraday's law of induction

Eq. (9-9), Faraday's Law of induction discussed in section 2, is given by

$$\mathscr{E} = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t}$$

And Maxwell suggested that a changing magnetic field produce a non-electrostatic field $\boldsymbol{E}_n$, the $\mathscr{E}$ equals to the line integral of $\boldsymbol{E}_n$ around the loop that the changing Φ_m goes through, that is

$$\mathscr{E} = \oint_L \boldsymbol{E}_n \cdot \mathrm{d}\boldsymbol{l}$$

Combining the above two equations, we have

$$\oint_L \boldsymbol{E}_n \cdot d\boldsymbol{l} = -\frac{d\Phi_m}{dt} \tag{9-40}$$

Substitute

$$\Phi_m = \oint_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

into Eq. (9-40)

$$\oint_L \boldsymbol{E}_n \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S}$$

obtained.

In general case, electric field includes both electrostatic field (a conservative field) and non-electrostatic field (a non-conservative vortex field), the total electric field is

$$\boldsymbol{E} = \boldsymbol{E}_e + \boldsymbol{E}_n$$

Considering of $\oint_L \boldsymbol{E}_e \cdot d\boldsymbol{l} = 0$, we have

$$\oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = \oint_L \boldsymbol{E}_n \cdot d\boldsymbol{l}$$

So that, the **generalized Faraday's law** can be written as

$$\oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} \tag{9-41}$$

Note that, the negative sign indicates that the direction of the induced electric field lines is determined by $\frac{\partial \boldsymbol{B}}{\partial t}$ with left-hand rule, as shown in Fig. 9-26(a). And in Eq. (9-41), the area S on the right side is the area around by the loop L on the left side.

9.8.2 Ampere-Maxwell law

According to the discussion in the last section, the **complete form of generalized Ampere's law**, Eq. (9-38), is

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = I + \frac{d\Phi_D}{dt} = \int_S \boldsymbol{j} \cdot d\boldsymbol{S} + \int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot d\boldsymbol{S} \tag{9-42}$$

The magnetic field $\boldsymbol{H}$ on the left side is the total magnetic field produced by both conductive current I and changing electric field $\frac{d\Phi_D}{dt}$. The positive sign on the right side shows that the direction of the magnetic field lines is determined by I and $\frac{\partial \boldsymbol{D}}{\partial t}$ with right hand-rule, as shown in Fig. 9-26(b).

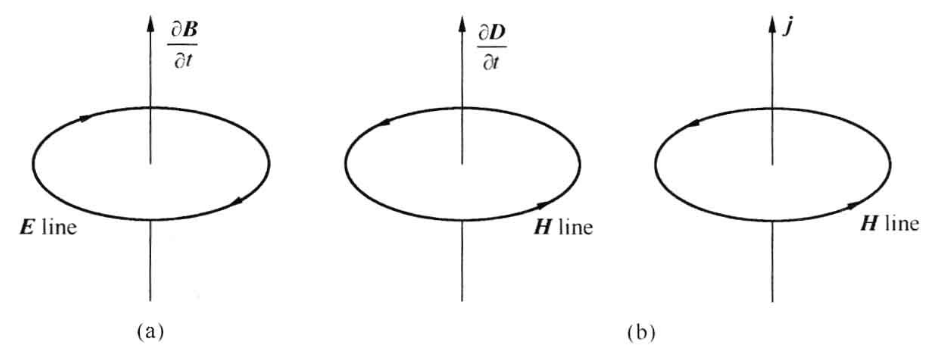


Fig 9-26 The left-hand rule and the right hand-rule to determine the directions of the fields

9.8.3 Gauss' law for electric field

Gauss' law for electric field investigated in chapter 6 and chapter 7, is given by

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \sum q = \int_V \rho dV \tag{9-43}$$

The $\boldsymbol{D}$ on the left side represents the total electric displacement

$$\boldsymbol{D} = \boldsymbol{D}_e + \boldsymbol{D}_n$$

in which $\boldsymbol{D}_e$ is the conservative field produced by the static electric charge and $\oint_S \boldsymbol{D}_e \cdot d\boldsymbol{S} = \sum q$; while $\boldsymbol{D}_n$ is the non-conservative field induced by the changing magnetic field and $\oint_S \boldsymbol{D}_n \cdot d\boldsymbol{S} = 0$. So that Eq. (9-43) holds for the general case of electric fields.

9.8.4 Gauss' law for magnetic field

Gauss' law for magnetic field discussed in section 2 of the last chapter, is written as

$$\oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0 \tag{9-44}$$

The $\boldsymbol{B}$ on the left side represents the total magnetic field produced by either conductive current or changing electric field, or by both. All magnetic fields are vortex fields, so that Eq. (9-44) holds for the general case of magnetic field.

Taken as a whole, the set of above four laws (Eq. (9-41)~Eq. (9-44)) are known as Maxwell's equations which provide a complete description of the interactions among charges, currents, electric fields, and magnetic fields. All the properties of the fields can be deduced by mathematical manipulation of these equations. If the distributions of charges and currents are given, then these equations uniquely determine the corresponding fields. Starting from a given initial condition of the fields, Maxwell's equation can uniquely determine the time evolution of the fields. Thus these equations accomplish for the dynamics of electromagnetic fields just as Newton's equations of motion accomplish for the dynamics of particles.

Even more important, although the empirical foundations on which we based, the development of Maxwell's equations were restricted to charges at rest or charges in uniform motion. These equations, also govern the field of accelerated charges and the fields of light and radio wave, that is, Maxwell's equations predict the existence of electromagnetic wave. From Maxwell's equations, the theoretical expression for the speed of propagation of an electromagnetic wave in vacuum is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{9-45}$$

whose numerical value is just 3.00×10^8 m/s, it agrees with the experimental value for the speed of light. This is one of the great and early triumphs of Maxwell's electromagnetic theory of light.

Maxwell established this theory in the 60s of the 19th century. About 20 years later, Heinrich Hertz, guided by Maxwell's theory, discovered radio waves by experiments in his laboratory, and gave a strong evidence of the truth of Maxwell's theory.

Questions

9-1 Determine the direction of the current through bulb B in Fig. 9-27 at the moment:

(1) Switch S is off;

(2) Coil 2 is brought closer to coil 1;

(3) The iron core in coil 2 is withdrawn;

(4) The resistance of R is increased.

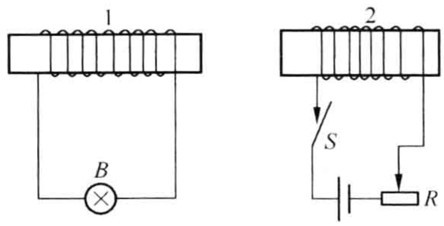


Fig. 9-27 For question 9-1

9-2 Four conductor loops are moving at the same speed v, as shown in Fig. 9-28.

(1) Determine in what loop the emf be set up?

(2) Which loop produces the maximum emf at the moment shown is figure? Why?

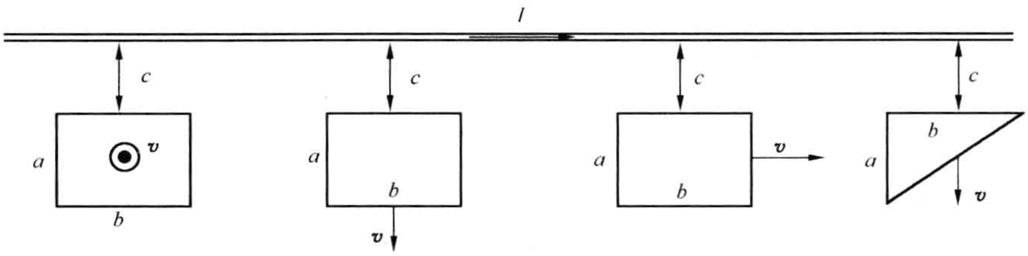


Fig. 9-28 For question 9-2

9-3 Consider a conducting sheet lying in a plane perpendicular to a magnetic field $\boldsymbol{B}$, as shown in Fig. 9-29.

(1) If $\boldsymbol{B}$ suddenly changes, the full change of $\boldsymbol{B}$ is not immediately detected in the region near P. Explain.

(2) If the resistivity of the sheet is zero, the change is never detected at P. Explain.

(3) If $\boldsymbol{B}$ changes periodically at high frequency, and the sheet is made of a material of low resistivity, the region near P is almost completely shielded from the changes in flux.

(4) Why is such a conductor not useful as a shield from static magnetic field?

9-4 A strip of copper is mounted as a pendulum about O in Fig. 9-30. It is free to swing through a magnetic field normal to the page. If the strip has slots cut in it as shown, it can swing freely through the field. If a strip without slots is substituted, the vibratory motion is strongly damped. Explain.

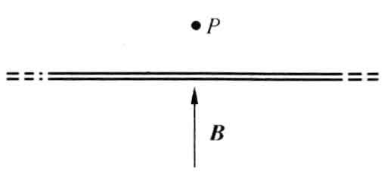


Fig. 9-29 For question 9-3

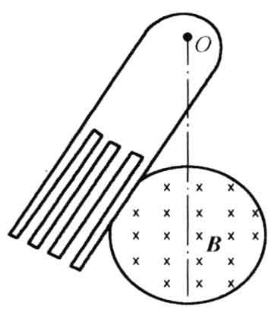


Fig. 9-30 For question 9-4

9-5 A copper ring and a wooden ring of the same dimensions are placed so that there is the same changing magnetic flux through each. How do the induced electric fields in each ring compare?

9-6 How does induced electric field be produced? Compare difference between static electric field and induced electric field.

9-7 A smooth magnet bar is failing in a frictionless copper tube vertically placed, even though the air friction is negligible, the magnet bar will reach a steady speed. Explain.

9-8 Why can a betatron be used for acceleration only during one-quarter of a cycle?

9-9 If a solenoid is divided into two parts of equal length, what is the self inductance of each part? Is it equal to one half of that of the original solenoid?

9-10 You want to wind a coil so that it has resistance R but essentially no inductance. How would you do it?

9-11 Compare the inductances of wires of the same length but in different shapes as shown in Fig. 9-31.

9-12 In Fig. 9-32, A and B are bulbs with the resistance of r, and $r \gg R$. L is an inductor with the resistance equal to R. When the switch is turned on or off, what phenomenon will happen respectively?

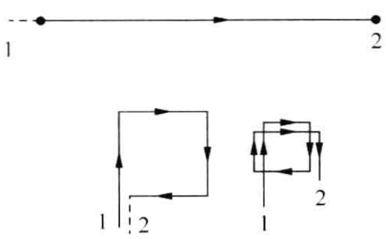


Fig. 9-31 For question 9-11

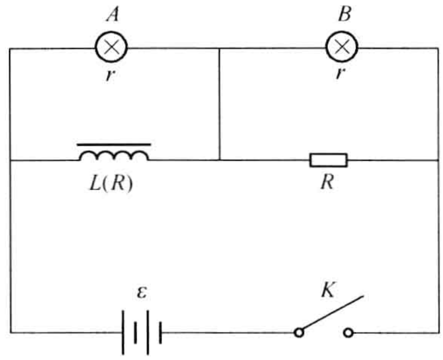


Fig. 9-32 For question 9-12

9-13 If you have two plat coils, how to place them so that the system has maximum and minimum mutual inductance. Assume the distance between them is fixed.

9-14 Fig. 9-33 shows the direction of the conduction current in the wire. Draw the direction of the displacement current between the two plates for the two cases:

(1) The capacitor is being charged;

(2) The capacitor is being discharged $\left(\text{hint: make sure about the sign of } \frac{dE}{dt}\right)$.

9-15 Can a displacement current be measured with an ampere-meter? Explain.

9-16 Find the direction of $\boldsymbol{B}$ lines between the two plates of the capacitor for the two cases described in question 9-14.

9-17 What is the similarity (common point/character) between the conductive current and the displacement current? And what is the intrinsic distinction between them?

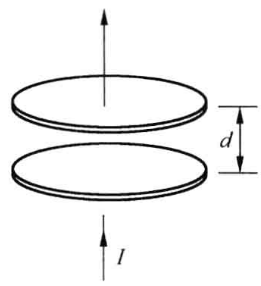


Fig. 9-33 For question 9-14

9-18 The displacement current between the plates of a capacitor has the same magnitude as the conduction current in the wires connected to the capacitor, and yet the magnetic field produced by the former current near and within the capacitor is much weaker than that produced by the latter current near and within the wire. Why?

9-19 In a cylindrical conductor with uniform resistance, as the conduction current is increasing, is there any displacement current? If so, whether or not the directions of two kinds of current are the same? If the conduction current is decreasing, what is the answer?

9-20 A parallel-plate capacitor has circular plates of area A separated by a distance d. A thin straight wire of length d lies along the axis of the capacitor and connects the two plates, and this wire has a resistance R. The exterior terminals of the plates are connected to a source of alternating emf with a voltage $V = V_0 \sin \omega t$. Make a sketch and qualitatively describe.

(1) What is the conductive current in the thin wire?

(2) How about the displacement current through the capacitor?

(3) How about the current arriving at the outside terminals of the capacitor?

(4) How about the magnetic field in the gap between the plates and in the space outside of it?

9-21 Consider the electric field of a single positive electric charge moving at constant velocity, What is the direction of the displacement current intercepted by a circular area perpendicular to the velocity.

(1) in front of the charge;

(2) behind the charge?

9-22 Which of the Maxwell's equations permit us to deduce the electric coulomb field of a static charge? Which of Maxwell's equations permit us to deduce the magnetic field of a charge moving with uniform velocity?

9-23 Metal container isn't allowed to be used in microwave oven. Explain?

Problems

9-1 In a uniform magnetic field, a crooked metal rod is caused to move by an external force at constant speed v = 10.0m/s as shown in Fig. 9-34. Assuming $AB = BC$ = 10.0 cm and B =0.1 T.

(1) What is the emf produced in the rod and in which direction?

(2) What is the potential difference between point A and C, and determine which point on the rod is of the highest potential?

9-2 Fig. 9-35 shows a conductor rod moving up parallel to a long straight wire at speed v. Assuming the current carried by the wire is I ($dI/dt = 0$), the rod is of length l and the angle between the rod and horizontal direction is θ. Calculate the emf induced in the rod.

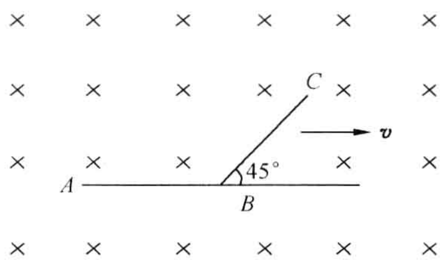


Fig. 9-34 For problem 9-1

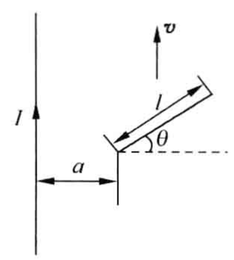


Fig. 9-35 For problem 9-2

9-3 A conductor rod of length l rotating around point O with constant angular speed ω perpendicular to magnetic field, as shown in Fig. 9-36. Suppose the strength of magnetic field is B, AO = (1/3) l and BO = (2/3) l. Calculate the induced emf in the rod, tell which point is of the highest potential.

9-4 A rectangular conductor loop of width a and length b is placed stationary near a long straight wire which carries a current $I = 2t + 1$ (SI). Suppose the wire is in the same plane of the loop and the distance between them is c, as shown in Fig. 9-37. What is the emf induced in the loop at time t ? Tell the direction of induced current in the loop.

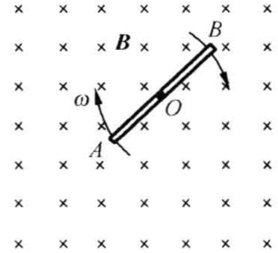


Fig. 9-36 For problem 9-3

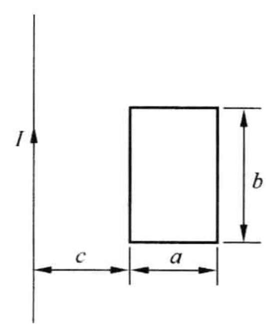


Fig. 9-37 For problem 9-4

9-5 As shown in Fig. 9-38, a rectangular conductor loop is in the plane of two parallel long straight wires. Suppose the currents in two wires are in the opposite direction and of the same value $I = 2t + 1$(SI), the geometry parameters are shown in figure. Calculate the induced emf in two cases:

(1) The loop is motionless.

(2) The loop is moving at constant speed v at right angles with two wires (emf at time $t=0$).

9-6 A triangular conductor coil of N turns is moving with speed v perpendicular to a straight wire, when the distance between point A and wire is b, refer to Fig. 9-39. Calculate the emf set up in the coil. Assume $AB = a$, $\angle CAB = \theta$, the current is $I = I(t)$.

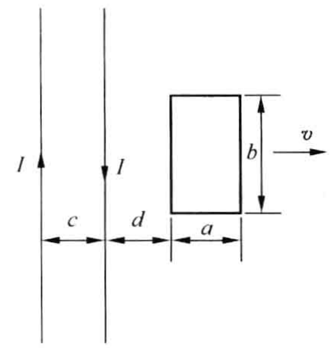


Fig. 9-38 For problem 9-5

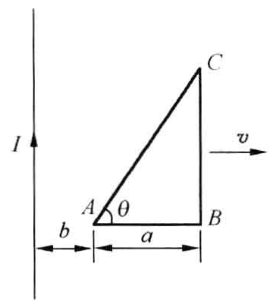


Fig. 9-39 For problem 9-6

9-7 A rectangular conductor coil of N turns is caused to rotate about axis OO' with constant angular speed ω in a uniform magnetic field. Assume the magnitude of $\boldsymbol{B}$ is B. When $t = 0$, the coil is just in the position as Fig. 9-40 shows. Calculate the emf appearing in coil at time t and draw the curve of emf as a function of time t.

9-8 A useful experimental method of measuring magnetic strength uses a ballistic galvanometer connected to the terminals of a small closely wound coil of N turns called a search coil. Assume the coil is placed at point to be tested with its plane normal to the magnetic field. If the coil is quickly given a turn about 90°, the amount of charge measured by the ballistic galvanometer is q. Assume the section of search coil is S and the resistance of coil is R, try to express the magnitude of $\boldsymbol{B}$ at that point with parameters N, S, q and R.

9-9 In Fig. 9-41, a loop of radius $r = 5.0$cm is placed in a uniform magnetic field which directs perpendicularly into the plane of the loop. If the magnitude of $\boldsymbol{B}$ is varying with time according to the relation $B = 2.0t^2 - 3.0t + 5.0$ (SI) and the total resistance of the circuit is $R = 10.0\ \Omega$, what is the magnitude of the induced current in the loop? In what direction does the current move?

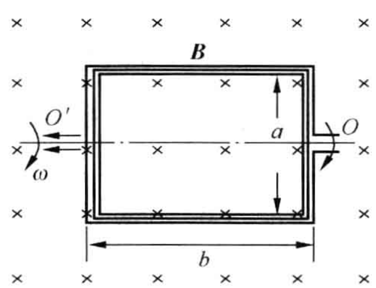


Fig. 9-40 For problem 9-7

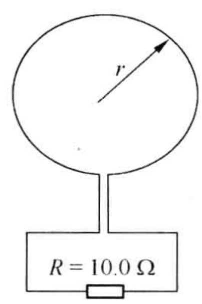


Fig. 9-41 For problem 9-9

9-10 A metal rod is pulled to right on a triangular conductor frame with constant velocity v, as shown in Fig. 9-42. The magnetic field $\boldsymbol{B}$ is perpendicular to the plane of frame. When time $t = t$, the $x = vt$.

(1) If the magnetic field $\boldsymbol{B}$ is uniform and of the magnitude B, calculate the emf induced in the loop.

(2) If the magnetic field $\boldsymbol{B}$ is non-uniformly distributed and following the distribution $B = x+1$ (T), calculate the emf set up in the loop.

9-11 Fig. 9-43 shows a copper rod moving on conducting rails with velocity $\boldsymbol{v}$ parallel to a long straight wire which carries a current I. The geometrical parameters are given in the figure. Find out

(1) What is the rate of thermal energy being generated in the rod?

(2) What is the force must be applied to the rod by an external agent to maintain its motion at speed v ?

(3) What is the rate of this external agent do work on the rod?

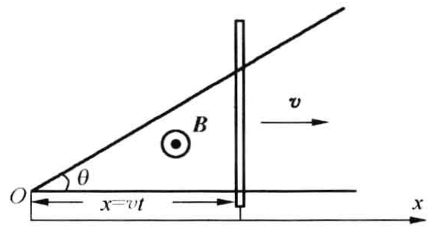


Fig. 9-42 For problem 9-10

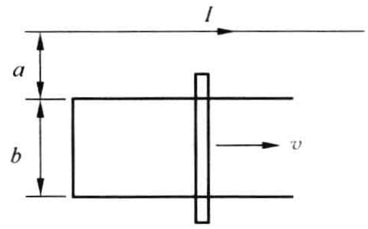


Fig. 9-43 For problem 9-11

9-12 As shown in Fig. 9-44, a metal rod of length $l = 50.0$ cm makes contact with a partial circuit and completes the circuit. The circuit area is perpendicular to a uniform magnetic field $\boldsymbol{B}$, if the resistance of the total circuit is $R = 0.5\ \Omega$, calculate

(1) How large a force is needed to move the rod with a constant speed v ?

(2) What is the magnitude and direction of the emf induced in the rod when it is moved?

(3) What is the rate of development of thermal energy in the circuit?

9-13 A wire of length l, mass m and resistance R slides without friction down parallel conducting rails of negligible resistance, as Fig. 9-45 shows. The raids are connected to each other at the bottom by a resistance less raid, so that the wire and rails form a closed rectangular conducting loop. The plane of the rails makes an angle θ with the horizontal plane, and a uniform magnetic field $\boldsymbol{B}$ is perpendicular to the horizontal plane and exists throughout the plane of rails.

(1) Show that if the raid is long enough the wire acquires a steady state velocity of magnitude

$$v = \frac{mgR\sin\theta}{B^2 l^2 \cos^2\theta}$$

(2) Prove that the result is consistent with the principle of energy conservation.

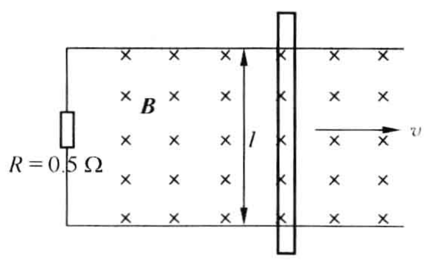


Fig. 9-44 For problem 9-12

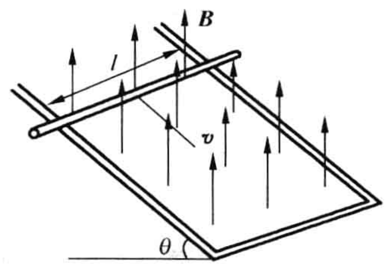


Fig. 9-45 For problem 9-13

9-14 A uniform magnetic field $\boldsymbol{B}$ fills a cylindrical volume of radius R. A metal rod of length l is placed as shown in Fig. 9-46. If the magnitude of $\boldsymbol{B}$ is changing at the rate $\mathrm{d}B/\mathrm{d}t > 0$, show that the potential difference between the ends of the rod is given by

$$\mathscr{E} = \frac{l}{2}\frac{\mathrm{d}B}{\mathrm{d}t}\sqrt{R^2 - \left(\frac{l}{2}\right)^2}$$

(hint: (1) make up a closed loop by adding some path and use Faraday's law of induction; (2) using: $\mathscr{E} = \int_A^B \boldsymbol{E}_n \cdot \mathrm{d}\boldsymbol{l}$).

9-15 Fig. 9-47 shows a uniform magnetic field $\boldsymbol{B}$ confined to a volume of a cylinder of radius R, a infinite long straight wire is placed as shown in figure. Suppose the distance between the center of cylinder and the wire is a, the magnitude or $\boldsymbol{B}$ is changing at the rate $\mathrm{d}B/\mathrm{d}t < 0$. What is the emf set up in the wire? In what direction?

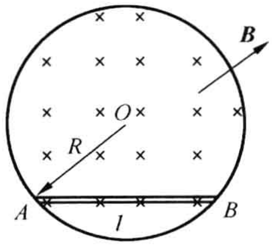


Fig. 9-46 For problem 9-14

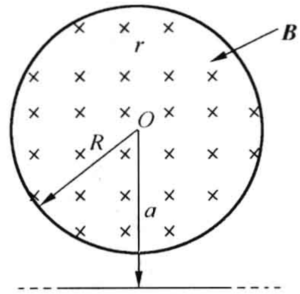


Fig. 9-47 For problem 9-15

9-16 Two long parallel wires whose centers are a distance d apart carry equal currents in opposite directions as shown in Fig. 9-48, and the radii of the wires are a. Calculate the self inductance of unit length of such a pair of wires (neglecting the flux within the wires themselves).

9-17 In Fig. 9-49, a rectangular loop of wire is placed stationary near a long straight wire which is in the plane of the loop. The geometrical parameters are given in the figure.

(1) What is the mutual inductance of this system?

(2) If the current in the wire is changing at the rate $dI/dt = 0.50$ (A/s), calculate the mutual-emf induced in the rectangular loop.

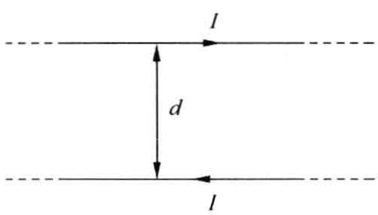


Fig. 9-48 For problem 9-16

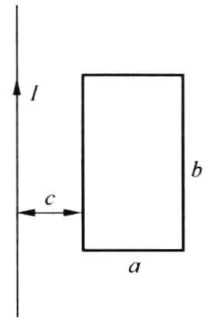


Fig. 9-49 For problem 9-17

9-18 A N_1 turns loop solenoid is wound uniformly on a long iron rod with length L and cross section A, the relative permeability of the iron is μ_m. Another N_2 turns loop coil which is used as a secondary is wound on top of it. Find the mutual inductance of the system.

9-19 An inductor of inductance 5 H carries a current that varies with time according to $I = -0.02t$ (SI). Find the induced emf and the energy stored in the magnetic field at time $t = 1$ s.

9-20 A thin toroid with average radius $r = 20.0$ cm and the cross sectional area $S = 1.00\ \text{cm}^2$ is shown in Fig. 9-50 with $N = 1000$ turns of wire wound uniformly on the core. Suppose the permeability of the core is $\mu = 1.0 \times 10^{-4}$.

(1) Find the self inductance of this toroid.

(2) If the current in the wire is $I = 5.00$ A, what is the magnetic energy stored in this field?

(3) If the current is changing at the rate $dI/dt = 10$(A/s), calculate the emf induced.

9-21 Two coils are connected as shown in Fig. 9-51. The coils have inductances L_1 and L_2, respectively. The coefficient of mutual inductance is M.

(1) Show that this combination can be replaced by a single coil of equivalent inductance given by

$$L_{eq} = L_1 + L_2 + 2M$$

(2) How could the coils in the figure been connected to yield an equivalent inductance of

$$L_{eq} = L_1 + L_2 - 2M$$

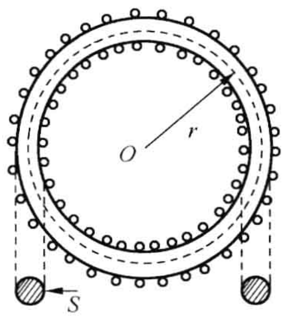


Fig. 9-50 For problem 9-20

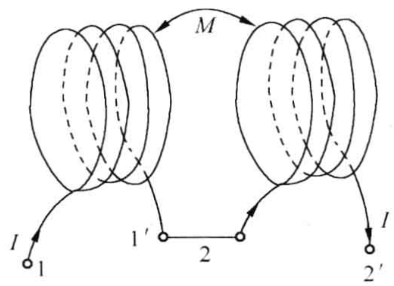


Fig. 9-51 For problem 9-21

9-22 Prove that the displacement current in a capacitor can be written as

$$I_d = C\frac{dV}{dt}$$

9-23 A parallel-plate capacitor is being charged with a conduction current I. Find the complete currents intercepted by the baglike surface S in the two cases shown in Fig. 9-52(a) and Fig. 9-52(b), respectively.

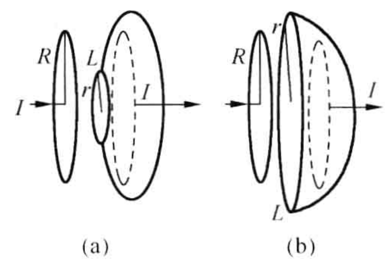


Fig. 9-52 For problem 9-23

9-24 A parallel-plate capacitor consists of circular plates of radius 0.30 m separated by a distance of 0.20 cm. The voltage applied to the capacitor is made to increase at a steady rate of 2.0×10^3 V/s. Assume that the electric charge distributes itself uniformly over the plates, and ignore the fringing effects.

(1) What is the rate of increase of the electric field between the two plates?

(2) What is the magnetic field between the plates at a radius of 0.15 m? At 0.30 m?

9-25 In example 9-10, show that the expressions derived for $B(r)$ can be written as

$$B(r) = \frac{\mu_0 I_d}{2\pi r} \quad (r \geqslant R)$$

$$B(r) = \frac{\mu_0 I_d r}{2\pi R^2} \quad (r \leqslant R)$$

Compare these expressions with those derived in Example 8-2 and problem 8-12.

9-26 A capacitors consisting of two parallel circular plates with radius R=18cm is connected to a source of emf $\mathscr{E} = \mathscr{E}_m \sin\omega t$, where $\mathscr{E}_m$ = 220 V and ω=130 rad/s. The maximum value of the displacement current is I_d=7.6 μA. Neglect fringing effects:

(1) What is the maximum value of the current?

(2) What is the maximum value of $d\Phi_e/dt$? Here Φ_e is the electric flux through the region between the plates.

(3) What is the separation d between plates?

(4) Find the maximum value of the magnitude of $\boldsymbol{B}$ between the plates at a distance r =11cm from the center.

9-27 A parallel-plate capacitor has square plates 1.0 m on a side as in Fig. 9-53. There is a charging current of 2.0 A flowing into (and out of) the capacitor.

(1) What is the displacement current through the region between the plates?

(2) What is $d\boldsymbol{E}/dt$ in this region?

(3) What is the displacement current through the dashed square path between the plates?

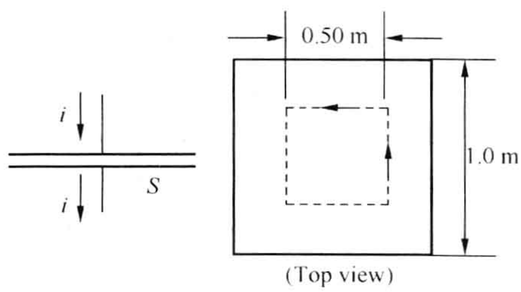


Fig. 9-53 For problem 9-27

(4) What is $\oint_l \boldsymbol{B} \cdot d\boldsymbol{l}$ around this dashed square path?

9-28 The space between the plates of a leaky capacitor is filled with a material of resistance $5.0\times10^5\ \Omega$. The capacitor has a capacitance of 2.0×10^{-6} F, its plates are circular, with a radius of 30 cm, and its electric field is uniform at $t=0$, the initial voltage across the capacitor is zero.

(1) What is the displacement current if we increase the voltage at the steady rate of 1.0×10^3 V/s?

(2) At what time will the real current leaking through the capacitor equal the displacement current?

(3) What is the magnitude of the magnetic field between the plates at radius $r=20$ cm at $t=0$ and at $t=1$ s?

9-29 A long cylindrical conducting rod with radius R is centered on the x axis as shown in Fig. 9-54. A narrow saw cuts is made in the rod at $x=b$. A conduction current I, increasing with time in a way given by $I=\alpha t$, flows toward the right in the rod; α is a positive proportionality constant. At $t=0$ there is no charge on the cut faces near $x=b$.

(1) Find the magnitude of the charge on these faces, as a function of time.

(2) Use Gauss' law for electricity to find E in the gap as a function of time.

(3) Sketch the lines of $\boldsymbol{B}$ for $r<R$, where r is the distance from the x axis.

(4) Use Ampere-Maxwell law to find the $B(r)$ in the gap for $r<R$.

(5) Compare the above answers with $B(r)$ in the rod $r<R$.

9-30 In Fig. 9-55, a point charge $+q$ is moving with a constant velocity $\boldsymbol{v}$ ($v\ll c$) toward to point O.

(1) Calculate the displacement current through the circular area perpendicular to the velocity, with the center O and radius R.

(2) Using the complete current law (Ampere -Maxwell law), find the magnetic field of the point at the rim, apart from the charge by r. Compare this result with that obtained by using the equation of $\boldsymbol{B}=\frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\boldsymbol{r}}{r^3}$.

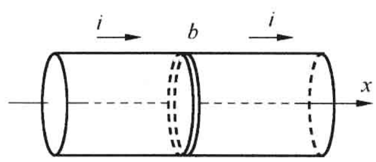


Fig. 9-54 For problem 9-29

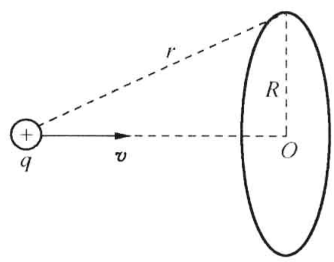


Fig. 9-55 For problem 9-30

Appendix 1
The International System of Units

1. The SI Base Units

Quantity	Name	Symbol	Definition
length	meter	m	It is defined as the length of the path travelled by light in vacuum in 1/299792458 of a second(1983)
mass	kilogram	kg	The kilogram is defined as being equal to the mass of the International Prototype Kilogram(a certain platinum-iridium cylinder, 1889)
time	second	s	The second has been defined to be the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom(1967)
electric current	Ampere	A	The constant current that will produce an attractive force of 2×10^{-7} Newton per meter of length between two straight, parallel conductors of infinite length and negligible circular cross section placed one metre apart in a vacuum(1946)
thermodynamic temperature	Kelvin	K	The kelvin is defined as the fraction 1/273.16 of the thermodynamic temperature of the triple point of water(1967)
Amount of substance	mole	mol	The mole is defined as an amount of a substance that contains as many elementary entities as there are atoms in 12 grams of pure carbon-12 (1971)
luminous intensity	candela	cd	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} Hertz and that has a radiant intensity in that direction of 1/683 Watt per steradian(1979)

2. Some SI Derived Units

Quantity	Name of unit	Symbol
area	square meter	m^2
volume	cubic meter	m^3

table continued

Quantity	Name of unit	Symbol
frequency	Hertz	Hz s^{-1}
mass density	kilogram per cubic meter	kg/m^3
speed, velocity	meter per second	m/s
angular velocity	radian per second	rad/s
acceleration	meter per second squared	m/s^2
angular acceleration	radian per second squared	rad/s^2
force	Newton	N $kg \cdot m/s^2$
pressure	Pascal	Pa N/m^2
work, energy, heat	Joule	J $N \cdot m$
power	Watt	W J/s
quantity of electricity	Coulomb	C $A \cdot s$
potential difference electromotive force	Volt	V W/A
electric field strength	Volt per meter	V/m
electric resistance	Ohm	Ω V/A
capacitance	Farad	F $A \cdot s/V$
magnetic flux	Weber	Wb $V \cdot s$
inductance	Henry	H $V \cdot s/A$
magnetic field	tesla	T Wb/m^2
magnetic field strength	ampere per meter	A/m
entropy	joule per kelvin	J/K
specific heat capacity	joule per kilogram kelvin	$J/(kg \cdot K)$
thermal conductivity	watt per meter kelvin	$J/(m \cdot K)$
radiant intensity	watt per steradian	W/sr

Appendix 2

Some Fundamental Constants of Physics*

Constant	Symbol	Values and units
speed of light in a vacuum	c	299 792 458 m/s
vacuum permeability	μ_0	12.56637061×10^{-7} N/A^2
vacuum permittivity	ε_0	$8.854\ 187\ 817\times10^{-12}$ F/m
gravitational constant	G	$6.67384(80)\times10^{-11}$ m^3/(kg · s^2)
Planck constant	h	$6.626\ 069\ 57(29)\times10^{-34}$ J · s
reduced Planck constant	$\hbar$	$1.054\ 571\ 726(47)\times10^{-34}$ J · s
elementary charge	e	$1.602\ 176\ 565(35)\times10^{-19}$ C
electron mass	m_e	$9.109\ 382\ 91(40)\times10^{-31}$ kg
proton mass	m_p	$1.672\ 621\ 777(74)\times10^{-27}$ kg
proton-electron mass ratio	m_p/m_e	1836.152 672 45(75)
neutron mass	m_n	$1.674\ 927\ 351(74)\times10^{-27}$ kg
Avogadro constant	N_A	$6.02214129(27)\times10^{23}$ mol^{-1}
universal gas constant	R	8.314 4621(75) J/(K · mol)
Boltzmann constant	k	$1.380\ 6488(13)\times10^{-23}$ J/K
atomic mass unit	u	$1.660\ 538\ 921(73)\times10^{-27}$ kg
Rydberg constant	R	10 973 731.568 539(55) m^{-1}

* The 2010 CODATA recommended values.

Words

Chapter 1

acceleration	加速度	horizontally	水平的
algebraic	代数的	illustrate	描述，插图，说明
analogy	类比，类推，相似	impact	冲击，冲量，着陆
angular variables	角变量	initial	初始的，开始的
approximately	近似的，大约的	instantaneous	瞬时的，即时的
arc	弧，弓形，弧光	integral	积分，积分的
artificial	人造的	integration	积分法，求积
assume	假定，假设	intersect	相交，交叉，横断
axis axes (*pl.*)	轴，坐标轴，中心线	kinematics	运动学
centripetal	向心的	latitude	纬度，纬线
chord	弦，和谐	launch	发射，使升空，使下水
circumstance	情况，形势，环境	magnitude	大小，数量，量值
cliff	悬崖，峭壁	mutually	相互的
coincide	一致，重合，相符合	neutron	中子
component	分量，组分，元件	normal	法线的，法向的，法线
concavity	凹形，凹面	nucleus	原子核
constant	常数，不变的，恒定的	orientation	取向，方位
coordinate system	坐标系	parabola	抛物线
corresponding to	对应于，符合的	parallel	平行的，类似的，并联
curvilinear	曲线的	path	路径，路里，轨道
derivation	求导，偏差，推导	perpendicular	垂直的
derivative	导数，微商，被导出的	prelude	序言，序曲，序幕
dimension	维数，量纲	projectile	抛物体
displacement	位移	project	发射，投影，项目
dynamics	动力学	proportional	成正比的，成比例的
equivalent	同等的，等效的，当量	proton	质子
frame of reference	参考系	quadrant	象限，四分之一圆
galaxy	银河系	quantity	(数)量，值，总量
generalize	普遍化，一般化，推广	quark	夸克
geometry	几何学	radian	弧度
grindstone	砂轮，磨石	radius of curvature	曲率半径
head	率领，驶往	radius radii (*pl.*)	半径

rectilinear	直线的	trajectory	轨道、弹道
sequence	次序,顺序,数列	transformation	变换,转化
sign	符号,标记,征兆	translational motion	平动
specification	规定,规格,说明书	triangle	三角形
speed	速率	variable	可变的,变量,变数
stars	恒星	variation	变量,变化
substitute	代换,替换,代入	vector	矢量,向量
superposition principle	叠加原理	velocity	速度
tangential	切线的,正切的,切向的	vertical	垂直的,纵的,竖直的
tangent	正切,切线的	whereas	鉴于,既然,虽然

Chapter 2

a ball bearing	滚珠(轴承)	inertia	惯性
buoyant	浮力	internal	内部的
coefficient	系数,率	intruder	入侵者
compression	压力	medium	介质,媒介,中间
concentric	同心的	missile	导弹
cone, conical	圆锥,圆锥的	mission	使命
coplanar	共面的	oppose	反对
crude	粗糙的	pit	穴,坑
denote	表示,符号	prefixed	词头,前缀
drag	拖曳力,阻力	retard	放慢,使减速
elastic	弹性的	rotor	旋转体
escape	逃逸	simultaneous	同时的
exert	施加,行使	slab	厚板
external	外部的	spontaneous	自发的
Foucault's pendulum	傅科摆	tension	张力
free-body diagram	隔离体图	viscosity	黏滞性
friction	摩擦(力)	viscous	黏滞的
gravitation	引力		

Chapter 3

altitude	高度	corresponding to	对应于
angular momentum	角动量	crate	板条箱,柳条箱
astronaut	宇航员	designate	标示,符号表示
calculus	积分学	diameter	直径
category	积类,范畴	disorder	无序,无规则,混乱
compress	压缩	dissipative	耗散的,消耗,损耗
configuration	位形,相对位置	dot product	点积
conservation	守恒,不灭,保存	element work	元功
conservative force	保守力	elevation	高度,海拔,上升

encounter	遭遇,碰见	scalar product	标量积
enlarge	放大,扩大	scalar	标量
extension	伸长量,伸延	shuttle	往返,穿梭来回
geometrical	几何的	special theory of relativity	特殊相对论
horizontal	水平的	stimulate	刺激,激励
hypothetical	假设的,假想的	stretch	拉伸,延伸
increment	增量	subscript	脚标,下标
isolate	隔离,使绝缘	thermal	热(量)的
momentum	动量	thermodynamics	热力学
oscillate	振动,振荡	tug war	拔河
pendulum	单摆	undergo	经历,经受
potential energy	势能,位能	undesirable	不合意的,不合需要的
preceding	上述的,以前的	universal force of gravitation	万有引力 verge 临界,边缘
prediction	预言,预测	violate	违背,破坏
quantum	量子的	yield, yield to	产生,得到,服从
random	随机的,偶然的		
restoring force	回复力,恢复力		

Chapter 4

abrupt	突然的,陡峭的	incident	入射的,事故
analogous	类似的,相似的	inelastic	非弹性的
aphelion	远日点	isotope	同位素
ballistic	冲击的,弹道的	Kepler	开普勒
Bohr-atom model	玻尔原子模型	label	标记,称为
bullet	子弹	linear impulse	线冲量
cannon	大炮	linear momentum	线动量
canoe	独木舟,划子	massive	大而重的,粗大的
coincident	重合,一致,符合的	merit	值得,长处,优点
collision	碰撞	momentarily	短暂地
combustion chamber	燃烧室	mount	安装,装配
composite	组合的,合成的	muzzle	炮口,喷嘴
cross product	叉乘积,矢量积	nozzle	喷口
deuteron	氘核,重氢核	perihelion	近日点
elliptic	椭圆的	prevail	胜过,占优势的
embrace	包围,抓住,包含	propulsion	推进,推动
emerge	出现,显露	recoil	反冲
fire hose	灭火(水龙)皮带管	simultaneously	同时地
fission	裂变	skater	溜冰者
Haley's comet	哈雷彗星	target	靶,目标
high jump	跳高	torque	力矩
impact force	冲力	whirl	旋转,旋涡

Chapter 5

acrobatic	杂技的
acrobat	杂技演出
arbitrary	任意的,随意的
astrodome	天体
astrophysical	天文(学)的
blade	叶片,刀片,(桨)叶
bounce	弹,跳起
brake staff	刹车(总成)
brake	刹车
category	范畴,积类,类型
clutch	离合器,接合器
collapse	崩坍
counterclockwise	反时针的,逆时针的
cylinder	圆筒,圆柱,汽缸
deformation	变形,畸变,失真
deliver	提供,输送
demonstration	演示,示范,说明
distribute	分布,散布
dive	跳水,潜水,俯冲
drum-wheel	鼓轮
dumb bell	哑铃
erect	直立的
experience	经历,经受
fetal	似胎儿的,胎儿的
flywheel	飞轮,惯性轮
gear	齿轮
gramophone	唱机,留声机
gravitational	引力的,重力的
gymnast	体操运动员
helicopter	直升机
hexagonal	六角形的
inadequate	不充足的,不够的
interior	内部的,内在性质
noticeable	可察觉的
outstretched	伸长的,扩张的
parabolic	抛物线的
phonograph	留声机,唱机
pivot	轴尖,装在枢轴上
propeller	螺旋桨,推进器
pulley	滑轮,滑车,皮带轮
pulsar	脉冲星
radial	径向的,辐射(式)的
reign	统治,支配
reorientation	重定向,重新取向
rigid body	刚体
rigid	刚性的
rim	轮缘,边
rotational inertia	转动惯量
sake	缘故,目的
shaft	(旋转)轴
shrink	收缩,缩紧
sphere	球,球体
spin	自旋,旋转,纺
springboard	跳板
stool	凳子,踏脚凳
subatomic	亚原子的,原子内的
sweep	扫过,掠过
swing	摇摆,打秋千
symmetry	对称(性),均称
tightrope	绷索
translation	平移,平动,翻译
tuck	卷起,折叠,缩拢
turntable	唱片转盘
undershot	下冲,下射的
uniformly	均匀地
wrap	卷,缠绕

Chapter 6

action-at-a-distance	超距作用
algebraic sum	代数和
arrowhead	箭头
assert	断言
attributable	归属
binomial	二项式
cathode ray tube	阴极射线管
circumference	圆周
circumstance	情形环境
closed surface	闭合曲面

compensate	补偿
concentrate	集中
conservative field	保守场
coulomb	库仑
cross-sectional area	横截面积
decay	衰变
deflect	偏转
demonstration	演示
dial	钟表面
differential	微分
dilemma	进退两难
dipole moment	偶极矩
discrete	分立的、离散的
distinguish	区分
disturbance	扰动
electric charge	电荷
electric dipole	电偶极子
electric field line	电力线、电场线
electric field	电场
electric potential	电势
electric quadruple	电四极子
electrify	起电
electron	电子
emerging from	从……中出来
equipotential surface	等势面
equivalent	等效的
fine	细的
fluorescent screen	荧光屏
flux	通量
fringe	边缘
Gaussian surface	高斯面
gradient	梯度
immerse	浸入
imply	意味着，暗示
individual	单独的，合体的
infinitesimal	无穷小的
instantaneous	瞬时的
integration	积分
intermediary	中间的
intersect	相交
lint	软麻布
neutrality	中性
neutron	中子
orientation	方位
originate	起始于……
parentheses	括号
permittivity	电容率、介电常数
perpendicular bisector	垂直平分线
point charge	点电荷
proton	质子
quantization	量子化
quantum	量子
quark	夸克
semicircle	半圆
sheet	薄板
simultaneously	同时地
sketch	素描
straightforward	直截了当地
subatomic	亚原子的
sufficiently	充分地
superposition principle	叠加原理
tangent	切线
terminate	终至于……
torsion balance	扭秤

Chapter 7

agitation	搅动、鼓动
alignment	排列成行
blunt	钝的
bound charge	束缚电荷
breakdown	击穿
capacitance	电容
circuit	电路
conductor	导体
deficiency	不足，欠缺
dielectric	电介质
electric displacement	电位移
electrolyte	电解液
electrostatic equilibrium	静电平衡
electrostatic generator	静电起电机
electrostatic induction	静电感应
electrostatic shielding	静电屏蔽
expenditure	消耗，支出
flash	闪光

induced charge	感应电荷	sheath	鞘、外皮、外壳
ionize	电离	slab	厚板
ionosphere	电离层	speculate	推测
ion	离子	squat	矮胖的，蹲下的
leak	泄漏	store	存贮
lightning rod	避雷针	stretch	拉伸
mica	云母	subsidiary	辅助的
nitrogen	氮	surplus	剩余物，过剩的
overlap	重叠	threefold	三倍，三重的
parallel	并联	tilt	倾斜
pierce	刺穿，突破	tip discharge	尖端放电
polarization	极化	tolerate	容许，忍受
reciprocal	倒数	vacancy	空，空位
reservoir	储存器，容器	wax	蜡
series	串联	withdraw	撤回

Chapter 8

alignment	直线排列，准线	diametrically	沿直径方向地
Amperian loop	安培环路	doughnut	环形物，面包圈
analogous to	与……相似	drift	漂移
Antarctica	南极洲	elegance	优美
appropriately	适当地	eliminate	消去，删去
arbitrary	任意的	emerge	露出
arc	弧	equidistantly	等距离地
bound	边界	evaluate	估计，估价
cable	电缆	even	偶数
carrier	载体	evidently	明显地
classification	分类	explicitly	明白，显然地
cobalt-nickel ferrite	钴-镍铁氧体	extension	延伸
coercive force	矫顽力	ferromagnetic	铁磁(性)的
coercivity	矫顽磁力	ferromagnetism	铁磁性
coil	线圈	fraction	分数，小数
compass needle	罗盘指针	frustum	截留物，截体
compensate	补偿	galvanometer	检流计
cone	锥形	geometrically	几何地
configuration	组态，配合	grasp	抓住
confine	限制	Hall effect	霍尔效应
curl	卷曲	helical	螺旋形的
cyclotron	回旋加速器	helix	螺旋线
dee	D形电极	hemisphere	半球
derivation	求导，推导	hinge	铰接，铰链
diamagnetism	抗磁性	horizontal	水平的
diamagnet	抗磁体	horseshoe	蹄形物

hypothesis	假说
hysteresis loop	磁滞回线
hysteresis	磁滞(现象)
iron	铁
linearly	线性地
Lorentz force	洛伦兹力
magnetic domains	磁畴
magnetic field	磁场
magnetic permeability	磁导率
magnetization history	磁化历史
magnet	磁铁
mass spectrometer	质谱仪
media (*pl.*)	介质,媒介物
medium	介质,媒介物
mutual	相互的,共同的
neighborhood	邻居
odd	奇数
omit	略去
ore	矿石
paramagnetism	顺磁性
paramagnet	顺磁体
parameter	参数
permeability	磁导率
polarity	极性
randomly	随机地
remanence	剩磁
retentivity	剩磁,顽磁性
revolution	圈,转圈
ribbon	飘带
solenoid	螺线管
spacing	间距
spiral	螺旋(形)的,盘旋
standardization	标准化
strip	条
terminate	终点
toroid	螺绕环
trace	踪迹、追溯
transverse	横向的
traversal	横过的
voltmeter	伏特计
vortex	蜗旋
wander	漫游,漂移
wholly	统统,完全地
winding	绕组,线圈,一转

Chapter 9

alternating	交变的
betatron	电子感应加速器
closed packed coil	密绕线圈
coaxial	共轴的
complete circuit ohm law	全电路欧姆定律
concentric	同心的
continuity	连续性
contradictory	矛盾,对立
cylindrical shell	圆柱形壳
deflect	偏转
demonstrational experiment	演示实验
displacement current	位移电流
dry cell	干电池
electromagneticinduction	电磁感应
electromotive force (emf)	电动势
electrostatic force	静电力
empirical	经验的
evidence	证据,证明
flux linkage	磁通链
galvanometer	电流计
generalize	推广,普遍化
induced emf	感应电动势
insulator loop	绝缘线圈
intercept	拦截,相交
invert	反转
motional emf	动生电动势
mutual-induction	互感
non-conservative	非保守的
non-electrostatic force	非静电力
numerical value	数值
self-induction	自感
source	电源
stationary	静止不动的
strength	强度
symmetry	对称性
validity	有效,正确
vital	重大的,有生命力的
vortex field	蜗旋场

Answers to Problems

Chapter 1

1-1 (1) $\boldsymbol{v}=8t\boldsymbol{j}+\boldsymbol{k}$ (SI) (2) $\boldsymbol{a}=8\boldsymbol{j}$ (SI)

1-2 $v=\dfrac{v_0^2t}{\sqrt{(H-h)^2+(v_0t)^2}}$, $a=\dfrac{(H-h)^2v_0^2}{[(H-h)^2+(_0t)^2]^{\frac{3}{2}}}$

1-3 (1) $y=2-\dfrac{x^2}{4}$

(2) $t=0$: $\boldsymbol{r}_0=2\boldsymbol{j}$m, $|\boldsymbol{r}_0|=2$ m, $\alpha_0=90°$

$t=2.0$ s: $\boldsymbol{r}_2=(4\boldsymbol{i}-2\boldsymbol{j})$ m, $|\boldsymbol{r}_2|=4.47$ m, $\alpha_2=-26°32'$

(3) $\Delta\boldsymbol{r}=(4\boldsymbol{i}-4\boldsymbol{j})$ m, $|\Delta\boldsymbol{r}|=4\sqrt{2}=5.6$ m, $\alpha=-45°$

1-4 (1) $\Delta\boldsymbol{r}=(6\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k})$m; $|\Delta\boldsymbol{r}|=10.5$ m; $\alpha=55°09'$, $\beta=61°34'$, $\gamma=131°49'$

(2) $\bar{\boldsymbol{v}}=(6\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k})$m/s, $|\bar{\boldsymbol{v}}|=10.5$ m/s

(3) $\boldsymbol{v}_1=(4i+5j-3k)$m/s; $|v_1|=7.07$ m/s; $\alpha_1=55°33'$, $\beta_1=45°$, $\gamma_1=115°06'$

$\boldsymbol{v}_2=(8\boldsymbol{i}+5j-12\boldsymbol{k})$m/s; $|\boldsymbol{v}_2|=15.3$ m/s; $\alpha_2=58°28'$, $\beta_2=70°56'$, $\gamma_2=141°39'$

1-5 (1) $\dfrac{x^2}{3}+y^2=1$, the path is an ellipse

(2) $\begin{cases} v_x=\dfrac{dx}{dt}=-\dfrac{\sqrt{3}}{4}\pi\sin\dfrac{\pi}{4}t, \\ v_y=\dfrac{dy}{dt}=\dfrac{\pi}{4}\cos\dfrac{\pi}{4}t, \end{cases}$ $\begin{cases} a_x=\dfrac{dv_x}{dt}=-\dfrac{\sqrt{3}}{16}\pi^2\cos\dfrac{\pi}{4}t \\ a_y=\dfrac{dv_y}{dt}=-\dfrac{\pi^2}{16}\sin\dfrac{\pi}{4}t \end{cases}$

(3) $t=1.0$ s: $\boldsymbol{r}_1=\left(\dfrac{\sqrt{6}}{2}\boldsymbol{i}+\dfrac{\sqrt{2}}{2}\boldsymbol{j}\right)$m, $|\boldsymbol{r}_1|=1.41$ m, $\alpha=\arctan\dfrac{\sqrt{3}}{3}=\dfrac{\pi}{6}$;

$\boldsymbol{v}_1=\left(-\dfrac{\sqrt{3}}{8}\pi\boldsymbol{i}+\dfrac{\sqrt{2}}{8}\pi\boldsymbol{j}\right)$m/s, $|\boldsymbol{v}_1|=1.11$ m/s, $\theta=\arctan\left(-\dfrac{\sqrt{3}}{3}\right)=\dfrac{5\pi}{6}$;

$\boldsymbol{a}_1=\left(-\dfrac{\sqrt{6}}{32}\pi^2\boldsymbol{i}-\dfrac{\sqrt{2}}{8}\pi^2\boldsymbol{j}\right)$ m/s^2, $|\boldsymbol{a}_1|=0.87$ m/s^2, $\phi=\dfrac{7}{6}\pi$

1-6 (1) $a=3t^2-4t$ (m/s^2), $x=0.25t^4-0.67t^3+t+1.42$ (m)

(2) $t=2.0$ s: $a_2=4.0$ m/s^2; $v_2=1.0$ m/s; $x_2=2.06$ m

1-7 $v=\sqrt{x^2-x+25}$ m/s

1-8 $\begin{cases} x=5\sin t\text{(SI)}, \\ y=-5\cos t+10\text{(SI)} \end{cases}$, $x^2+(10-y)^2=25$(SI), the path is a circle

1-9 $v=\sqrt{\dfrac{k}{m}(A^2-x^2)}$ (SI)

1-10 (1) $\boldsymbol{r}=\dfrac{bv_0}{2}t^2\boldsymbol{i}+v_0t\boldsymbol{j}$ (2) $x=\dfrac{b}{2v_0}y^2$ (3) $a_t=\dfrac{b^2v_0y}{\sqrt{b^2y^2+v_0^2}}$, $a_n=\dfrac{bv_0^2}{\sqrt{b^2y^2+v_0^2}}$

1-11 (1) $\boldsymbol{v}=(6t\boldsymbol{i}+4t\boldsymbol{j})$m/s, $\boldsymbol{r}=[(10+3t^2)\boldsymbol{i}+2t^2\boldsymbol{j}]$m

(2) $y=\dfrac{1}{3}(2x-20)$ m, $\alpha=33°41'$, the path is a straight line

1-12 (1) $t=0.705$ s (2) 0.715 m

1-13 $v_{x3}=24.5$ m/s, $v_{y3}=-8.8$ m, $v_3=26.1$ m/s; $x_3=73.5$ m

$y_3=17.7$ m; $R=103$ m, $T=4.2$ s

1-14 (1) The width $x=420$ m (2) $t=7.09$ s

(3) $v_x=60.1$ m/s; $v_y=-44.6$ m/s; $v=74.8$ m/s; $\theta=-36.6°$

1-15 $62.8°\leqslant\theta_1\leqslant64.2°$ or $25.8°\leqslant\theta_2\leqslant31.1°$ **1-16** (1) $t=13$ s (2) $t=32$ s

1-17 (1) $\omega=7.27\times10^{-5}$ rad/s; for Beijing: $v_B=365$ m/s; $a_{nB}=2.58\times10^{-2}$ m/s^2

(2) for Guangzhou: $v_G=428$ m/s; $a_{nG}=3.10\times10^{-2}$ m/s^2

1-18 (1) $t_1=2.0$ s (2) $t_2=2.83$ s

1-19 (1) $\omega=\pi$ rad/s (2) $a_n=10\pi^2$ m/s^2, $a_t=10\pi$ m/s^2

(3) $a=\sqrt{a_n^2+a_t^2}=10\pi\times\sqrt{\pi^2+1}$ m/s^2 $=103.5$ m/s^2, $\phi=\arctan\dfrac{a_n}{a_t}=72°20'$

1-20 $a_t=5.36$ m/s^2, $a_n=8.2$ m/s^2

1-21 (1) $a_t=c$, $a_n=\dfrac{(b+ct)^2}{R}$ (2) $t=\sqrt{\dfrac{R}{c}}-\dfrac{b}{c}$ (3) $n=\dfrac{b+c}{\pi R}$

1-22 $v_{RE}=80$ m/s **1-23** (1) 66.7 s (2) 0.75 m/s (3) 1.68m/s

1-24 (1) -20.0 km/h (2) -180 km/h (3) 91.6 km/h, $\alpha=131°$

1-25 (1) $v_{PG}=225$ km/h, $\alpha=16.8°$ (2) $v_{PG}=205$km/h, $\theta=17.6°$

Chapter 2

2-1 (1) $a_b=6.1$ m/s^2 (2) $a_s=0.98$ m/s^2, both relative to the floor and on the left

2-2 (1) 708 N (2) 939 N (3) 477 N (4) 0 (5) -159 N

2-3 (A) (1) $f_1=100$ N, in same direction as $\boldsymbol{v}$; (2) $f_2=-80$ N, in direction opposite to $\boldsymbol{v}$

(B) (1) $f_1=110$ N, $\alpha_1=24.2°$between $\boldsymbol{f}_1$ and $\boldsymbol{v}$; (2) $f_2=92$ N, $\alpha_2=151°$between $\boldsymbol{f}_2$ and $\boldsymbol{v}$

2-4 $v_0=\sqrt{Rg\tan\theta}$; if $v>v_0$, the friction will supply in centripetal direction; if $v<v_0$, the friction will be opposite to centripetal direction

2-5 (1) 1.05 N (2) 3.62 m/s^2 (3) $a'=a$, $T'=-T$ becoming push instead pull

2-6 $v=(6t^2+4t+6)$m/s, $x=(2t^3+2t^2+6t+5)$m **2-7** $\tau=2\pi\sqrt{\dfrac{L\cos\theta}{g}}$

2-8 (1) the upward static frictional force by the wall (2) the minimum $n=0.54$ r/s

2-9 (1) $F=\dfrac{m_1+m_2+M}{\sqrt{m_1^2-m_2^2}}m_2g$, (2) $N=(M+m_1+m_2)g$

2-10 (1) $f_k=\mu_k m\dfrac{v^2}{R}$, $a_\tau=-\mu_k\dfrac{v^2}{R}$ (2) $t=\dfrac{2R}{\mu_k v_0}$

2-11 (1) 1.47 m/s^2 (2) 3.36 s (3) 4.94 m/s

2-12 (1) 0.942 m/s (2) 109 N, 46.6 N (3) $n=\dfrac{1}{2\pi}\sqrt{\dfrac{g}{L\cos\theta}}=26.2$ r/min

2-13 $\omega=\sqrt{2\dfrac{g}{r}\sin\alpha}$, $N=3mg\sin\alpha$

2-14 (1) $N=mg\left(1+\dfrac{2}{gr}\right)=3.7$ mg (2) $N=mg\left(-1+\dfrac{v^2}{gr}\right)=1.7$ mg (3) up position

2-15 $h=3.56\times10^4$ km **2-16** 11.2×10^3 m/s

2-17 $a_1=\dfrac{4m_2m_3}{m_1m_2+m_1m_3+4m_2m_3}g$, $a_2=\dfrac{m_1m_3-m_1m_2-4m_2m_3}{m_1m_2+m_1m_3+4m_2m_3}g$

$a_3=\dfrac{m_1m_3-m_1m_2+4m_2m_3}{m_1m_2+m_1m_3+4m_2m_3}g$, $T_2=\dfrac{2m_1m_2m_3g}{m_1m_2+m_1m_3+4m_2m_3}$

$T_1=2T_2$

2-18 $a_{BA}=-1.06$ m/s^2 **2-19** (1) $v=\sqrt{\dfrac{g}{l}\sin\theta(l^2-a^2)}$ (2) $t=\sqrt{\dfrac{l}{g\sin\theta}}\ln\dfrac{l+\sqrt{l^2-a^2}}{a}$

2-20 $v_0=\dfrac{mg}{k}$, $t=0.69\dfrac{m}{k}$ **2-21** same as 2-2 **2-22** $a=g\tan\theta$

2-23 (1) $a'=0.6$ g, (2)$T=0.6$ mg **2-24** $\omega_{max}=\sqrt{\dfrac{g\mu_s}{r}}$

Chapter 3

3-1 (1) $-\pi\mu_k mgR$ (2) $-2\mu_k mgR$. The work done by frictional fore is depends on the path

3-2 (1) 30.1 J (2) −30.1 J (3) 0 (4) 0.225 **3-3** $p = 4.92\times10^7$ W **3-4** (1) 109 W (2) 191 W
3-5 3.83×10^4 W **3-7** $W = F\cdot(\sqrt{H^2+x_1^2} - \sqrt{H^2+x_2^2})$
3-8 (1) 36 J (2) 72 W **3-9** (1) 12 J; −16 J (2) 1.55 m/s; 0
3-10 $v=\sqrt{2gl\sin\theta}$ **3-11** (1) 5.3×10^2 J (2) 12 W; 912 W **3-12** $v=\sqrt{\frac{g}{l}\sin\alpha(l^2-a^2)}$
3-14 -3.34×10^9 J **3-16** $U=U_0\left[\left(\frac{\sigma}{r}\right)^{12}-\left(\frac{\sigma}{r}\right)^6\right], U_0 = 5.60\times10^{-21}$ J
3-17 (1) 19.3J (2) 22.5 m/s **3-18** $\Delta E'_P = \frac{1}{2}k(x_2^2-x_1^2) = \Delta E_P$
3-19 The total work done by a pair of force depends on the force and the relative displacement.
3-20 (1) 39.5 cm (2) 3.65 cm **3-21** (1) 2.1 m (2) 262 N/m
3-22 246 J; 94.6 N **3-23** (1) 8 mg (2) $\frac{5}{2}R$ (3) $\frac{5}{3}R$, 1.85R
3-24 29.4 N **3-25** $F=(m_1+m_2)g$ **3-26** (1) $x_1=\frac{2mg\sin\alpha}{k_1+k_2}$ (2) $v_m = g\sin\alpha\sqrt{\frac{m}{k_1+k_2}}$
3-27 $\sqrt{\frac{GM}{3R_e}}; \frac{GMm}{6R_e}; -\frac{GMm}{3R_e}; -\frac{GMm}{6R_e}$ **3-28** (1) $-\frac{\mu mg}{2l}(l-a)^2$ (2) $\sqrt{g/l}\,[(l^2-a^2)-\mu(l-a)^2]^{1/2}$

Chapter 4

4-1 (1) 776 N (2) 2466 N **4-2** (1) 11.2 kg m/s (2) 9.33×10^3 N (3) 6.67×10^4 m/s^2
4-3 (1) 6.14×10^{-2} N·s (2) 6.14 N, 6.84° with normal **4-4** 1.80×10^3 W
4-5 (1) −2.2 m/s (2) 47.8 m/s **4-6** (1) 0.2 m/s (2) 1.8 m/s (3) −1.4 m/s
4-7 $v'=v+\frac{m}{M}u$ **4-8** 1.88 m **4-9** $\frac{mu}{M+m}\cdot\frac{v_0\sin\theta}{g}$ **4-10** $\frac{d_2}{d_1}=1-\frac{m}{m+M}$
4-11 2.2×10^7 m/s; 23°; 52° **4-12** −2792 N, 0.2 m **4-13** 319 m/s **4-14** 0.2
4-15 (1) −400 J (2) 2 J (3) −398 J **4-16** $(M+m)g/k+(mg/k)\sqrt{1+\frac{2kh}{(M+m)g}}$
4-17 (1) mgd (2) $mgtd$ **4-18** $L=mRv$ **4-19** $\frac{E_k}{E_{ki}}=\frac{h^2}{l^2}$ **4-20** 82 r/min
4-21 1.07×10^{-34} kg·m^2/s **4-22** $x_c=0$, $y_c=6.8\times10^{-12}$ m **4-23** $x_c=0$, $y_c=\frac{4R}{3\pi}$
4-24 $x_2 = 24.4\times10^3$ m, $y_2 = -30$ m **4-25** 1.88 m **4-26** $\frac{m}{l}(v_0^2+yg)$

Chapter 5

5-1 (1) 1.25 rad/s^2 (2) 5.5 s (3) $a_t = 0.5$ m/s^2, $a_n = 2.5$ m/s^2, $a = 2.55$ m/s^2, $\alpha = 63.4°$
5-2 1719 r/min **5-3** (1) 9.42 rad/s^2 (2) 10 s (3) 75 r
5-4 (1) 7.5 md^2 (2) 12 md^2 **5-5** $\frac{1}{4}\lambda l^3$ **5-6** $\frac{1}{2}mR^2$
5-7 (1) 3.92 m/s^2, 19.6 rad.s^2, 5.88 N (2) 1.24 s
5-8 1.89×10^{-2} kg·m^2, 0.226 N **5-9** 1.96 m/s^2, 5.88 m/s^2
5-10 $\omega_0 = 28.6$ rad/s **5-11** $\omega=\sqrt{3g/l}$ **5-12** 5.03 s, 3.14 s
5-13 (1) $\beta=\frac{m_2 gR - m_1 gr\sin\varphi}{I+m_2R^2+m_1r^2}$ (2) I=4.44 kg·m^2
5-14 6.58×10^{-2} J, 3092×10^{-2} W **5-15** (1) −12324 J, (2) 5.2 min
5-17 7.67 m/s **5-18** (1) 1.8×10^3 J (2) 2.7 kW **5-19** (1) 0.49 m (2) 0.245 m, 1.31 m/s
5-20 The men and the stool begin to rotate with $\omega = 1.6$ r/s
5-21 $\frac{4m_1+2m_2}{m_1+2m_2}\omega$ **5-22** (1) $\left(\frac{1}{2}MR^2-\Delta mR^2\right)\omega$ (2) $\omega R\sqrt{\frac{2h}{g}}$
5-23 (1) $\frac{2m}{M+2m}\cdot 2\pi$ (2) $\frac{2m}{M}\cdot 2\pi$ **5-24** 1.69 kg·m^2, 2.53 kg·m^2
5-25 $\frac{M-3m}{M+3m}\cdot v_0$, $\frac{6m}{M+3m}\cdot\frac{v_0}{l}$

Chapter 6

6-1 (2)$\pm 2.38\times10^{-8}$ C **6-2** 1.01×10^{5} N/C **6-4** $\pm\frac{\sqrt{2}R}{2}$

6-5 $\frac{\lambda}{4\pi\varepsilon_0(R^2+x^2)^{\frac{3}{2}}}[x(2\pi R-d)\boldsymbol{i}+Rd\boldsymbol{j}]$ **6-6** $-\frac{\lambda R^2}{\pi\varepsilon_0(R^2+x^2)^{\frac{3}{2}}}$

6-7 $\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{a}{x\sqrt{x^2+a^2}}\boldsymbol{i}+\left(\frac{1}{\sqrt{x^2+a^2}}-\frac{1}{x}\right)\boldsymbol{j}\right]$ **6-8** $\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{R^2+x^2}}\right)+\frac{\lambda L}{4\pi\varepsilon_0 x(x-L)}$

6-9 $\frac{Q}{\pi^2\varepsilon_0 r^2}$, vertically downward **6-10** $\frac{Q}{4\pi^2\varepsilon_0 l}\left(\frac{1}{R}-\frac{1}{\sqrt{R^2+l^2}}\right)$

6-11 $\frac{q}{6\varepsilon_0}$ **6-12** $\frac{\lambda r}{2\pi\varepsilon_0 R^2}(r\leqslant R)$; $\frac{\lambda}{2\pi\varepsilon_0 r}(r\geqslant R)$

6-13 $0(r\leqslant R)$; $\frac{\lambda(r^2-a^2)}{2\pi\varepsilon_0 r(b^2-a^2)}(a<r<b)$; $\frac{\lambda}{2\pi\varepsilon_0 r}(r>b)$

6-14 $\frac{\sigma z}{2\varepsilon_0\sqrt{R^2+z^2}}$ **6-15** (1) $\frac{\rho x}{\varepsilon_0}$ (2) $\frac{\rho d}{2\varepsilon_0}$ **6-16** (1) 0 (2) $\rho\frac{(r^3-a^3)}{3\varepsilon_0 r^2}$ (3) $\rho\frac{(b^3-a^3)}{3\varepsilon_0 r^2}$

6-18 (1) -6.76×10^{5} V (2) -1.93 J **6-19** -1.79×10^{-10} J

6-20 (1) 0 $(r\leqslant R)$; (2) $\frac{\sigma R}{\varepsilon_0}\ln\frac{R}{r}$

6-21 (1) $U=-\frac{qr^2}{8\pi\varepsilon_0 R^3}(r<R)$, $U=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{3}{2R}\right)$ $(r>R)$

(2) $U=-\frac{q}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right)(r<R)$, $U=\frac{q}{4\pi\varepsilon_0 r}(r>R)$

6-22 $x_c=\frac{aq_1^2}{q_1^2-q_2^2}$; $R=\frac{aq_1q_2}{q_1^2-q_2^2}$ **6-23** $\frac{\lambda}{2\varepsilon_0}\left(1+\frac{1}{\pi}\ln\frac{R_2}{R_1}\right)$

6-24 $-8\boldsymbol{i}-4\boldsymbol{j}$ **6-25** (1) $\frac{\lambda}{4\pi\varepsilon_0}\ln\frac{L+x}{x}$ (2) $\frac{\lambda L}{4\pi\varepsilon_0 x(L+x)}$

6-26 (1) $\frac{k}{4\pi\varepsilon_0}(\sqrt{L^2+y^2}-y)$ (2) $\frac{k}{4\pi\varepsilon_0}\left(1-\frac{y}{\sqrt{L^2+y^2}}\right)$

Chapter 7

7-1 $\varepsilon_0 E$ **7-2** $Q_1=1.1\times10^{-9}$ C; $Q_2=2.8\times10^{-10}$ C

7-3 (1)8.99×10^{6} km (2) 9.84×10^{-22} C/m^2

7-4 (1) 0 (2) $\frac{q}{4\pi\varepsilon_0 r^2}$ (3) 0 (4) 0 (5) $-q$, 0

7-5 $\frac{r(R_1Q_2+R_2Q_1)}{R_2(R_1+r)}$

7-6 (1) $\sigma_1=\frac{q_1+q_2}{2S}$, $\sigma_2=\frac{q_1-q_2}{2S}$, $\sigma_3=\frac{q_2-q_1}{2S}$, $\sigma_4=\frac{q_1+q_2}{2S}$

(2) $E=\frac{q_1-q_2}{2\varepsilon_0 S}$, $V=\frac{(q_1-q_2)d}{2\varepsilon_0 S}$

7-7 $q'=-\frac{q}{3}$ **7-8** $\frac{3\varepsilon_0 S}{d}$

7-9 $\frac{\pi\varepsilon_0 L}{\ln\left(\frac{d}{a}\right)}$, Hint: the area of plate is approximately equal to $4\pi ab$

7-10 (1) 3.75 μF

(2) $q_1=250$ μC, $q_2=125$ μC, $q_3=375$ μC, $V_1=V_2=25$ V, $V_3=75$ V

(3) 500 μC, 100 V

7-11 43 pF **7-12** $\varepsilon_r=4$ **7-13** (1) $\frac{2\varepsilon S}{d}$ (2) $\frac{1}{2}\left(1+\frac{\Delta\varepsilon}{\varepsilon}\right)Q$

7-16 (1) $D=5.0\times10^{-7}$ C/m^2, $E=1.05\times10^{4}$ V/m (2) 5.0×10^{-9} C (3) 4.07×10^{-9} C

7-17 (1) 0, $\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\varepsilon_r}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{1}{R_2}\right]$ (2) $\frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}$, $\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\varepsilon_r}\left(\frac{1}{r}-\frac{1}{R_2}\right)+\frac{1}{R_2}\right]$ (3) $\frac{Q}{4\pi\varepsilon_0 r^2}$, $\frac{Q}{4\pi\varepsilon_0 r}$

7-18 (1) $\frac{4\pi\varepsilon_1\varepsilon_2 rR_1R_2}{(\varepsilon_1-\varepsilon_2)R_1R_2+(\varepsilon_2R_2-\varepsilon_1R_1)r}$

(2) $\frac{\varepsilon_2(\varepsilon_1-\varepsilon_0)rR_2V}{R_1[(\varepsilon_1-\varepsilon_2)R_1R_2+(\varepsilon_2R_2-\varepsilon_1R_1)r]}$, $\frac{\varepsilon_1(\varepsilon_1-\varepsilon_0)rR_1V}{R_2[(\varepsilon_1-\varepsilon_2)R_1R_2+(\varepsilon_2R_2-\varepsilon_1R_1)r]}$,

$\frac{\varepsilon_0(\varepsilon_1-\varepsilon_2)R_1R_2V}{r[(\varepsilon_1-\varepsilon_2)R_1R_2+(\varepsilon_2R_2-\varepsilon_1R_1)r]}$

7-19 (1) -2.57×10^{-6} C/m², 1.71×10^{-22} C/m² (2) 1.61×10^5 V/m, 1.08×10^5 V/m

7-20 150 μC, 200 μC, 350 μC; 1875 J, 2500 J, 30625 J

7-21 parallel: 9.0×10^{-3} C, 1.35×10^{-2} C; 1500 V, 1500 V; 6.75 J, 10.1 J;
series: 5.4×10^{-3} C, 5.4×10^{-3} C; 900 V, 600 V; 2.43 J, 1.62 J

7-22 (1) 2.0 J **7-23** $\frac{A^2\pi R^7}{7\varepsilon_0}$

Chapter 8

8-1 (1) 0.24 Wb (2) 0 (3) 0.24 Wb **8-2** $\phi\left(1-\frac{R_1^2}{R_2^2}\right)$ **8-3** $\frac{\mu_0 I}{\pi a}\left(1-\frac{\sqrt{2}}{2}\right)$

8-4 (1) $\frac{\mu_0 I}{4R}\left(1+\frac{1}{\pi}\right)$ (2) $\frac{\mu_0 I}{4\pi R}$ **8-5** $\frac{2\sqrt{3}\mu_0 I}{3\pi l}$ **8-6** $\frac{\mu_0 I}{2\pi R}-\frac{\mu_0 I}{8R}$

8-7 (1) $\frac{\mu_0 NIR^2}{2}\left\{\frac{1}{\left[R^2+\left(x+\frac{R}{2}\right)^2\right]^{3/2}}+\frac{1}{\left[R^2+\left(x-\frac{R}{2}\right)^2\right]^{3/2}}\right\}$

8-8 $\frac{\mu_0 I}{\pi l}\arctan\left(\frac{l}{2z}\right)$ in the x direction **8-9** $\frac{\mu_0 NI\sin^3\theta}{2(R-r)}\ln\frac{R}{r}$ **8-10** $\frac{\mu_0\omega Q}{8\pi R}$

8-11 $\frac{\mu_0 IL}{2\pi}\ln\frac{b}{a}$ **8-12** $\frac{\mu_0 Ir}{2\pi R^2}(r<R)$, $\frac{\mu_0 I}{2\pi r}(r>R)$ **8-13** $\frac{\mu_0 J_0 r^2}{3a}$ **8-15** 6.3×10^{-3} T

8-16 $\mu_0 nIk+\frac{\mu_0 I'}{2\pi r}(-\sin\theta\boldsymbol{i}+\cos\theta\boldsymbol{j})$

The equation of the magnetic field lines are:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \quad \text{being a helix} \\ z = \frac{2\pi r^2 nI}{I'}\theta \end{cases}$$

8-17 (1) $\frac{\mu_0 NI}{2\pi r}(R_1<r<R_2)$ (2) $\phi=\frac{\mu_0 NIh}{2\pi}\ln\frac{R_2}{R_1}$

8-18 (1) $\frac{\mu_0 Iq}{2\pi d}$, opposed to the current (2) $\frac{\mu_0 Iq}{2\pi d}$, in the same direction of the current

8-19 (1) 3.4×10^{-4} m (2) 2.6×10^3 eV **8-20** (1) 45° (2) 4.8×10^{-3} m, 3.0×10^{-2} m

8-21 $\frac{\pi m v r}{e}=2.14\times10^{-7}$ Wb **8-23** (1) 8.4×10^{-4} m/s (2) 1.3×10^{-3} V/m (3) 2.5×10^{-5} V

8-24 3.2×10^{-3} N in upward direction **8-25** $I_3/I_1=2$, $F_3/F_1=4$ **8-26** BIR

8-27 4.3×10^{-3} Nm, in downward direction **8-28** $\frac{\pi IB}{2}(R_1^2-R_2^2)$ **8-29** 28.9 **8-30** 0.3T

8-31 (1) around the rim of the plate (2) 7.96 $\times10^3$ A (3) 2.30×10^{-4} T

8-32 (1) $\frac{I_0}{2\pi r}$, $\frac{\mu_0\mu_r I_0}{2\pi r}$ (2) $(\mu_r-1)I_0$

Chapter 9

9-1 (1) 0.071 V, form point B to point C (2) -0.071 V, point C

9-2 $\frac{\mu_0 Iv}{2\pi}\ln\frac{a+l\cos\theta}{a}$, form point B to point A **9-3** $\frac{1}{6}B\omega L^2$, point B

9-4 $-\frac{b\mu_0}{\pi}\ln\frac{a+c}{a}$, counterclockwise

9-5 (1) $\frac{\mu_0 bt}{\pi}\ln\frac{(a+d)(d+c)}{a(a+d+c)}$, clockwise

(2) $\frac{\mu_0 bt}{\pi}\ln\frac{(a+d)(d+c)}{a(a+d+c)}+\frac{\mu_0 b}{2\pi}\left(\frac{1}{d}-\frac{1}{a+d}-\frac{1}{d+c}+\frac{1}{a+d+c}\right)$

9-6 $\frac{\mu_0 NIv\tan\theta}{2\pi}\left(\ln\frac{a+b}{b}-\frac{a}{a+b}\right)$ **9-7** $NBab\omega\sin(\omega t)$ **9-8** $\frac{Rq}{NS}$

9-9 $-0.4\pi r^2 t+3\pi r^2=7.8\times10^{-3}(3-4t)$

$t=0.75$ s, $I=0$; $t<0.75$ s, $I>0$, counterclockwise; $t>0.75$ s, $I<0$, clockwise

9-10 (1) $-v^2 tB\tan\theta$ (2) $-v^2 t(1+vt)\tan\theta$

9-11 (1) $\frac{\mu_0^2 i^2 v^2}{4\pi^2 R}\left(\ln\frac{a+b}{a}\right)^2$ (2) $\frac{\mu_0^2 i^2 v}{4\pi^2 R}\left(\ln\frac{a+b}{a}\right)^2$

9-12 (1) 0.02 N (2) 0.2 V, counterclockwise (3) 0.08 J

9-15 $-\frac{\pi R^2}{2}\frac{dB}{dt}$, from right to left **9-16** $\frac{\mu_0}{\pi}\ln\frac{2d-a}{a}$

9-17 (1) $\frac{\mu_0 b}{2\pi}\ln\frac{a+c}{c}$ (2) $\frac{\mu_0 b}{4\pi}\ln\frac{a+c}{c}$ **9-18** $\frac{\mu_0\mu_m N_1 N_2 A}{l}$ **9-19** 0.1 V, 1×10^{-3} J

9-20 (1) 7.96 mH (2) 9.95×10^{-2} J (3) 0.0796 V

9-23 (1) $\frac{r^2}{R^2}I$ (2) I **9-24** (1) 1×10^6 V/ms (2) 8.3×10^{-13} T, 1.7×10^{-12} T

9-26 (1) 7.6 μA (2) 859 kV/s (3) 3.39 mm (4) 5.16×10^{-12} T

9-27 (1) 2.0 A (2) 2.3×10^{11} V/m (3) 0.5 A (4) 6.3×10^{-7} T·m

9-28 (1) 2.0×10^{-3} A (2) 1.0 s (3) 0.89×10^{-9} T, 1.8×10^{-9} T

9-29 (1) $\frac{1}{2}\alpha t^2$ (2) $\frac{\alpha t^2}{2\pi R^2\varepsilon_0}$ (3) $\frac{\mu_0 r\alpha t}{2\pi R^2}$ (4) same as (3)

9-30 (1) $\frac{qR^2 v}{2(R^2+x^2)^{3/2}}$ (2) $\frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\boldsymbol{r}}{r^3}$

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