高等学校信息领域全英文课程
“十三五”系列规划教材

Modern Control Theory

现代控制理论

（英文版）

刘向杰　张金芳　编著

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北　京

内 容 简 介

本书是“现代控制理论”双语教学教材，注重培养学生的国际视野和竞争能力。内容主要包括状态变量及状态空间表达式、线性空间变换、线性状态空间方程求解、系统稳定性、系统能控性、能观性以及状态反馈与状态观测。 全书贯穿了 MATLAB 在线性系统中的应用方法。

本书可作为高等院校自动化专业本科生的教材，也可供相关专业的教师参考。

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前 言

教育部在 2001 年提出加强大学本科教学的 12 项措施，其中要求各高校在三年内开设 5%～10%的双语课程，并引进原版教材和提高师资水平, 双语课程随后成为质量工程的重要内容和教学评估的重要依据。作者一直承担着华北电力大学自动化专业本科生“自动控制原理”和“现代控制理论”课程的教学任务，在本科生双语教学方面积累了一定的教学实践经验，但存在一些问题：对于一些重要的基础课或专业理论课，双语教学使学生的学习难度加大，原因在于课程本身就有很大难度，学生的科技英语的单词量不够，程度差异也比较大。

“现代控制理论”是自动化专业的专业基础课。课程设在第六学期，它和第五学期开设的“自动控制原理”一并构成自动化专业的核心理论基础。作为教学对象的三年级下学期本科生已经修完了所有相关的数学课程，具有了较为完善的数学基础知识，并在修完了第五学期开设的“自动控制原理”课后，对自动控制的原理、概念和方法有了一定的了解。此时学生已经具备了学习“现代控制理论”的所有基础知识，而且学生已经在第四学期学完了大学英语，这时开展双语教学可以在继续延续和深化英语学习的同时，使英语学习更加专业化。由此，教学内容适合开展双语教学，课程定位为完成基础英语学习后由学科基础课学习向专业课学习中运用英语的过渡桥梁。自动控制领域的科学研究方法，已经由最早的经典控制中以输入输出模型为主，发展为现今的现代控制中以状态空间模型为主。因而，“现代控制理论”是从事自动化专业必备的知识。采用双语教学有利于自动化专业的学生掌握最新的现代控制理论知识和提高专业英语水平，直接促进学生全面了解本专业的发展动态。

作者自 2006 年在华北电力大学自动化本科专业开设“现代控制理论”双语教学课程以来，至今已经培养了十届本科生。2009 年该课程入选国家级双语教学示范课。多年来的教学实践表明，学生英语实际运用能力得到提高。学生在毕业设计过程中也体现了较好的查阅英文文献的能力。十年来双语教学培养的学生上百次获得国内外大学生数学建模竞赛奖等奖项。此外，学生在参与实验室从国外引进设备的投用和实验开发时也显示了较强的适应能力。双语教学课程的开展也起到了很好的辐射和示范作用。

“现代控制理论”一直是国外大学自动控制专业的主干课程。多年来国内出版了许多相关的教材，其内容都来自国外教材。“现代控制理论”的教学目标是使学

生牢固树立线性系统中状态空间的概念，进一步理解系统稳定性这一控制学科最为重要的概念，掌握能控与能观、状态反馈与状态估计等核心方法。同时作为双语教学，注重培养学生国际视野和竞争能力。然而, 现有的国外教材大部分内容较深,适合研究生采用。“现代控制理论”是自动化专业本科生的专业基础课，共 32 学时，因而需要出版适合我国自动化专业本科生双语教学的现代控制理论教材。本书是作者多年 “现代控制理论”双语教学工作的梳理。在此，谨对那些曾给予作者诸多帮助和建议的学生、老师、教学和科研同行、编辑等深表谢意。

本书共 6 章。第 1 章包括状态变量及状态空间表达式。第 2 章是线性空间变换。第 3 章是线性状态空间方程的解。第 4 章是系统稳定性。第 5 章是能控性、能观性。第 6 章是状态反馈与状态观测。全书贯穿了 MATLAB 在线性系统中的应用方法。本书适合高等院校自动化专业本科生及相关专业的教师和学生。

感谢国家自然科学基金(61273144, 60574051)及北京市教改项目(GJJG201409)对本书编写给予的资助和支持。

由于作者水平有限, 书中难免存在不当之处, 欢迎广大读者批评指正。

刘向杰　张金芳

2015 年 10 月

目　　录

Chapter 1
System Model

1.1 Introduction

In control theory research, the system model should be set up first. In this chapter, different kinds of models are discussed and some examples are given to show how to set up a model of a system. The relationships between these model types are also described.

1.2 Models of Systems

A mathematical expression that appropriately relates the physical system quantities to the system components is called the mathematical model of a system.

There are basically two types of system description; one is the external description, called the input-output description. The other is the internal one, called the state-space description. In the former one, a system in operation involves the following three elements: the system's input (or excitation), the system itself, and the system's output (or response). This description just reveals the casual relationship between the external variables (the input and the output) without characterizing the internal structure. In the later one, the description is a class of mathematic models based on the analysis of the internal structure of the system. It is a classical modern approach of describing a system. Correspondingly, there are two types of mathematical models of the system, which can facilitate the system analysis (it is well known that in order to analyze a system, the mathematical model must be available).

The input-output description of the system will be introduced in Section 1.2.1 and 1.2.2 in the differential equation and the transfer function form; the state-space model will be presented in Section 1.2.3.

1.2.1 Differential Equation

The differential equation is the fundamental mathematical model of a system. This description includes all the linearly independent equations of a system, as well as the appropriate initial conditions. The differential equation method is demonstrated by the following examples.

Example 1.1 Consider the network shown in Fig.1.1, where R ,C and L stand for the resistance, the capacitance and the inductance of the circuit respectively. Derive the network's differential equation mathematical model.

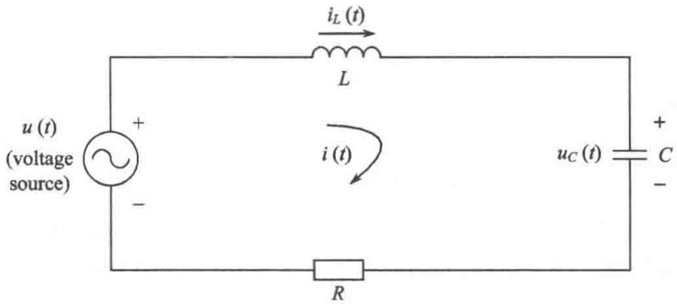


Fig.1.1 RLC network

Solution

Applying the Kirchhoff's voltage law,

$$L\frac{di}{dt}+\frac{1}{C}\int_0^t idt+Ri=v(t) \tag{1-1}$$

The above integro-differential equation constitutes a mathematical description of the network. This model is a second-order differential equation. Two appropriate initial conditions should be given to complete the description. The inductor's current $i_L(t)$ and the capacitor's voltage $v_C(t)$ at the instant that the switch closes (at $t=0$) are adopted as initial conditions:

$$i_L(0)=I_0$$
$$v_C(0)=V_0$$

where I_0 and V_0 are given constants.

The integro-differential equation and the two initial conditions thus constitute a complete description of the network shown in Fig.1.1.

Example 1.2 Consider the network shown in Fig.1.2, where R ,C and L stand for the resistance, the capacitance and the inductance of the circuit respectively. We assign the current of the inductance $L_x(x=1,2)$ as $i_x(x=1,2)$, and the voltage of the capacitance $C_x(x=1,2)$ as $v_x(x=1,2)$. Derive the network's differential equation mathematical model.

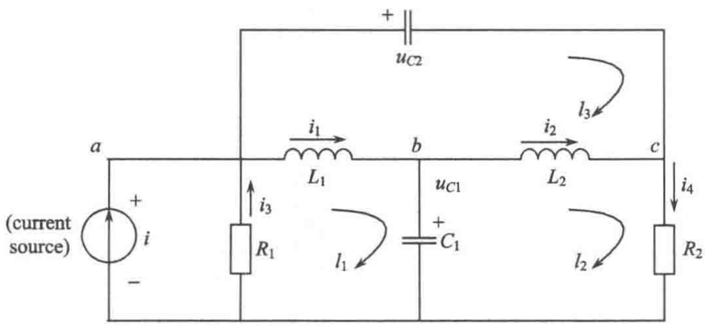


Fig.1.2 A three-loop network

Solution

The differential equation method for describing this network is based on the three differential equations which arise by applying the Kirchhoff's current law. These three loop equations are

$$
\begin{aligned}
&-L_1\frac{di_1}{dt}+u_{C1}+R_1 i_3=0\\
&-u_{C1}+L_2\frac{di_2}{dt}+R_2 i_4=0\\
&L_2\frac{di_2}{dt}-L_1\frac{di_1}{dt}-u_{C2}=0
\end{aligned}
\tag{1-2a}
$$

By applying the Kirchhoff's voltage law,

$$
\begin{aligned}
&i+i_3+i_1-C_2\frac{du_{C2}}{dt}=0\\
&C_1\frac{du_{C1}}{dt}+i_1+i_2=0\\
&C_2\frac{di_2}{dt}+i_2-i_4=0
\end{aligned}
\tag{1-2b}
$$

the initial conditions are $v_{C1}(0)=v_{C10}$, $v_{C2}(0)=v_{C20}$ and $i_{L1}(0)=i_{L10}$, $i_{L1}(0)=i_{L10}$.

Example 1.3 Consider the mechanical system shown in Fig.1.3, where y,K,m and B are the position of the mass, the spring's constant, the mass, and the friction coefficient respectively. Derive the system's differential equation mathematical model.

Solution

By using the d'Alembert's law of forces, the following differential equation is obtained

$$m\frac{d^2y}{dt^2}+B\frac{dy}{dt}+Ky=f(t) \tag{1-3}$$

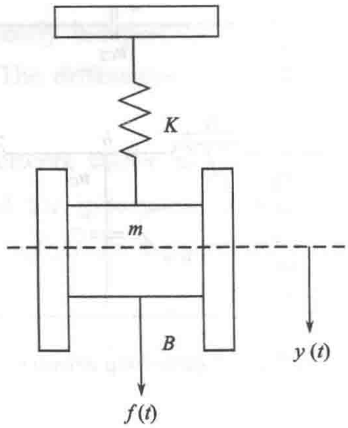


Fig.1.3 A spring and a mass

The initial conditions of the above equation are the distance $y(t)$ and the velocity $v(t)=dy/dt$ at the instant $t=0$, i.e., at the instant when the external force $f(t)$ is applied. Therefore the initial conditions are

$$y(0)=Y_0 \text{ and } v(0)=\left[\frac{dy}{dt}\right]_{t=0}=V_0$$

where Y_0 and V_0 are given constants.

The differential equations and the two initial conditions constitute the complete description of the mechanical system shown in Fig.1.3.

Remark 1.2.1

A differential equation is a description in the time domain which can be applied to many categories of systems, such as linear and nonlinear system, time-invariant and time-variant system, with lumped and distributed parameters, zero and nonzero initial conditions and many others.

1.2.2 Transfer Function

In contrast to the differential equation, which is a description in the time domain, the transfer function model is a description in the Laplace domain and holds only for a restricted category of systems, i.e., for linear time-invariant systems with zero initial conditions. The transfer function is designated by $G(s)$ and is defined as follows.

Definition

The transfer function $G(s)$ of a linear, time-invariant system with zero initial conditions is the ratio of the Laplace transform of the output $y(t)$ to the Laplace transform of the input $u(t)$, i.e.,

$$G(s)=\frac{L\{y(t)\}}{L\{u(t)\}}=\frac{Y(s)}{U(s)} \tag{1-4}$$

The introductory examples used in Section 1.2.1 will also be used for the derivation of their transfer functions.

Example 1.4 Consider the network shown in Fig.1.1. Derive the transfer function $G(s)=I(s)/V(s)$.

Solution

Fig.1.1, in the Laplace domain and with zero initial conditions I_0 and V_0, can be shown in Fig.1.4. From the Kirchhoff's voltage law,

$$LsI(s)+RI(s)+\frac{I(s)}{Cs}=V(s) \tag{1-5}$$

The transfer function is

$$G(s)=\frac{I(s)}{V(s)}=\frac{I(s)}{\left[Ls+R+\frac{1}{Cs}\right]I(s)}=\frac{Cs}{LCs^2+RCs+1} \tag{1-6}$$

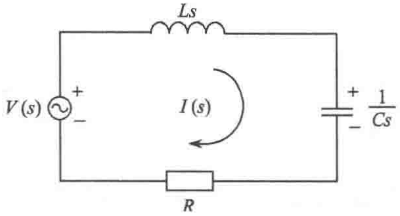


Fig.1.4 RLC circuit

When the voltage $V_R(s)$ across the resistor is chosen as the output, the transfer function becomes

$$G(s)=\frac{V_R(s)}{V(s)}=\frac{RI(s)}{V(s)}=\frac{RCs}{LCs^2+RCs+1} \tag{1-7}$$

Example 1.5 Consider the electrical network shown in Fig.1.2. Determine the transfer function $G(s)=I_2(s)/V(s)$.

Solution

This network, in the Laplace domain and with zero initial conditions, is shown in Fig.1.5. The equations for the two loops can be expressed:

$$\left[R_1+\frac{1}{Cs}\right]I_1(s)-\frac{1}{Cs}I_2(s)=V(s) \tag{1-8a}$$

$$-\frac{1}{Cs}I_1(s)+\left[R_2+Ls+\frac{1}{Cs}\right]I_2(s)=0 \tag{1-8b}$$

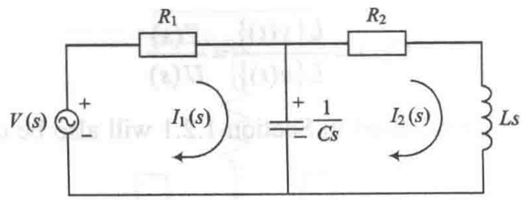


Fig.1.5 A two-loop network

Equation (1-8b) yields

$$I_1(s) = \left[LCs^2 + R_2Cs + 1\right] I_2(s) \tag{1-9}$$

Substituting equation (1-9) into equation(1-8a),

$$(R_1Cs+1)(LCs^2+R_2Cs+1)I_2(s) - I_2(s) = CsV(s) \tag{1-10}$$

Hence

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Cs}{(R_1Cs+1)(LCs^2+R_2Cs+1)-1}$$

$$= \frac{1}{R_1LCs + (R_1R_2C+L)s + R_1 + R_2} \tag{1-11}$$

Example 1.6 Consider the mechanical system shown in Fig.1.3. Determine the transfer function $G(s) = Y(s)/F(s)$.

Solution

This system, in the Laplace domain and with zero initial conditions, is shown in Fig.1.6.

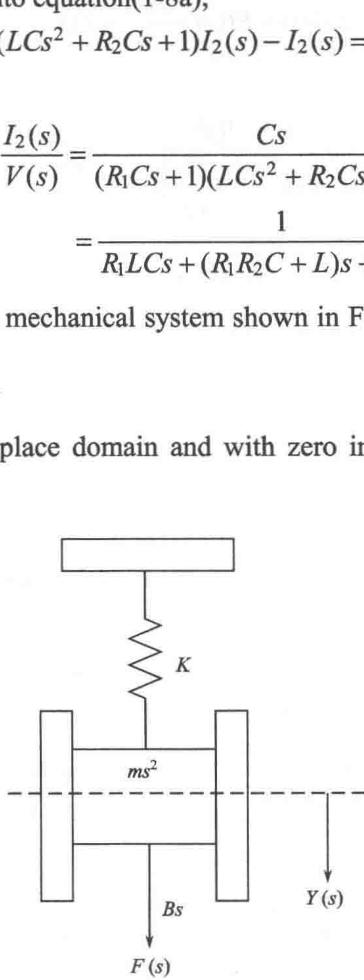


Fig.1.6 A spring and a mass

Using the d'Alembert's law of forces,

$$ms^2Y(s) + BsY(s) + KY(s) = F(s) \tag{1-12}$$

the transfer function is

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + Bs + K} \tag{1-13}$$

Remark 1.2.2

In the above examples, it can be seen that the transfer function $G(s)$ is the ratio of two polynomials in the Laplace domain. In general, $G(s)$ has the following form

$$G(s) = \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} = K\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)} \tag{1-14}$$

where $-p_i$ (i=1,2,⋯,n) are the roots of the denominator which are called the poles of $G(s)$, and $-z_i$ are the roots of the numerator which are called the zeros of $G(s)$. Poles and zeros, particularly the poles, play a significant role in the behavior of a system.

1.2.3 The State-space Model

The state-space model is a description in the time domain which may be applied to a very wide category of systems, such as linear and nonlinear systems, time-invariant and time-variant systems, systems with nonzero initial conditions, etc. The term *state* of a system refers to the past, present, and future of the system. From the mathematical point of view, the *state* of a system is expressed by its state variables. Usually, a system is described by a finite number of state variables, which are designated by $x_1(t), x_2(t), \cdots, x_n(t)$ and are defined as follows.

1. Definition

The state variables $x_1(t), x_2(t), \cdots, x_n(t)$ of a system are defined as a (minimum) number of variables such that if we know

(a) their values at a certain instant t_0;

(b) the input of the system for $t \geqslant t_0$;

(c) the mathematical model which relates the inputs, the state variables, and the system itself;

then the determination of the system's states for $t > t_0$ is guaranteed.

Consider a system with multi-inputs and multi-outputs (MIMO), as shown in Fig.1.7.

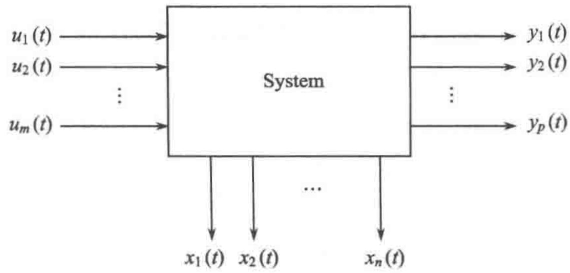


Fig.1.7 System with multi inputs and multi outputs

The input vector is designated by $u(t)$ and has the form

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \tag{1-15}$$

where m is the number of inputs. The output vector is designated by $y(t)$ and has the form

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} \tag{1-16}$$

where p is the number of outputs. The *state vector* $x(t)$ has the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{1-17}$$

where n is the number of state variables.

The *state equations* are a number of n first-order differential equations which relate the input vector $u(t)$ to the state vector $x(t)$ and have the form

$$\dot{x}(t) = f\left[x(t), u(t)\right] \tag{1-18}$$

where $f(\cdot)$ is a column with n elements. The function $f(\cdot)$, in general, is a complex nonlinear function of $x(t)$ and $u(t)$. Note that equation (1-18) is a set of *dynamic equations*.

The output vector $y(t)$ of the system is related to the input vector $u(t)$ and the state vector $x(t)$ as follows:

$$y(t) = g\left[x(t), u(t)\right] \tag{1-19}$$

where $g(\cdot)$ is a column with p elements. Relation (1-19) is called the *output equation.* The function $g(\cdot)$ is generally a complex nonlinear function of $x(t)$ and $u(t)$. Note that equation (1-19) is a set of *algebraic (non-dynamic) equations.*

The initial conditions of the state-space equations (1-18) are the values of the elements of the state vector $x(t)$ for $t = t_0$ and are denoted as

$$x(t_0) = x_0 = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} \tag{1-20}$$

The state-space equations (1-18), the output equation (1-19), and the initial conditions (1-20), i.e., the following equations, constitute the description of a dynamic system in the *state space.*

$$\dot{x}(t) = f[x(t), u(t)] \tag{1-21a}$$

$$y(t) = g[x(t), u(t)] \tag{1-21b}$$

$$x(t_0) = x_0 \tag{1-21c}$$

Since the dynamic state equation (1-21a) plays a dominate role in equations (1-21), all the three equations in (1-21), will be called, for simplicity, state equations.

When the system is a linear stationary one, the state space model is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1-22}$$

where A is the systematic matrix; B is the input/control matrix; C is the output matrix and D is the direct transfer matrix.

The state equations (1-21) are, in the field of automatic control, the modern method of system description. Thus the state-space model relates the following four elements: the input, the system, the state variables, and the output. In contrast, the differential equations and the transfer function relate three elements: the input, the system and the output—wherein the input is related to the output via the system directly (i.e., without giving information about the state of the system). It is exactly for this reason that these two models are called input-output model.

2. The construction of the state space model

There are three ways to set up the state space model which can be based on:

(1) the transform of the block diagram;

(2) the first-principal modeling;

(3) the input-output model.

Each way will be described in detail and examples will be given.

The transform of the block diagram

Example 1.7 The system block diagram is shown in Fig.1.8(a), where u is the input and y is the output. Try to deduce the state-space equations.

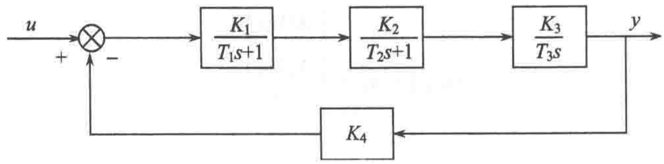


Fig.1.8 (a)

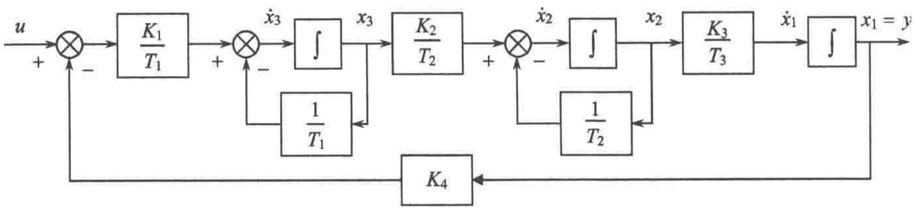


Fig.1.8 (b)

Solution

The structure of each link part is shown in Fig.1.8 (b), thus the relative equations can be derived as follows:

The state equations:
$$\begin{cases} \dot{x}_1 = \dfrac{K_3}{T_3}x_2 \\ \dot{x}_2 = -\dfrac{1}{T_2}x_2 + \dfrac{K_2}{T_2}x_3 \\ \dot{x}_3 = -\dfrac{1}{T_1}x_3 - \dfrac{K_1K_4}{T_1}x_1 + \dfrac{K_1}{T_1}u \end{cases} \tag{1-23}$$

The output equation:
$$y = x_1$$

The above equations can be rewritten in the vector-matrix form,
$$\dot{x} = \begin{bmatrix} 0 & \dfrac{K_3}{T_3} & 0 \\ 0 & -\dfrac{1}{T_2} & \dfrac{K_2}{T_2} \\ -\dfrac{K_1K_4}{T_1} & 0 & -\dfrac{1}{T_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_1}{T_1} \end{bmatrix} u \tag{1-24}$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

The first-principal modelling

When a physical system is given, the mechanism analysis can be carried out, with proper assumption and simplification. The mechanism model can be set up with chosen inputs and outputs. If the middle variables are eliminated, then the differential equation mentioned above can be obtained. If the middle variables are chosen as the state variables, then the state space model can be achieved.

Example 1.8 Consider the system shown in Fig.1.9. The current of $C_{1,2}$ are $C_{1,2}\dot{u}_{C1,2}$ respectively, and the voltage of $L_{1,2}$ are $L_{1,2}\dot{i}_{1,2}$. The input is the current source, and the outputs are the voltages of capacitances C_1 and C_2. Derive the system's state space representation.

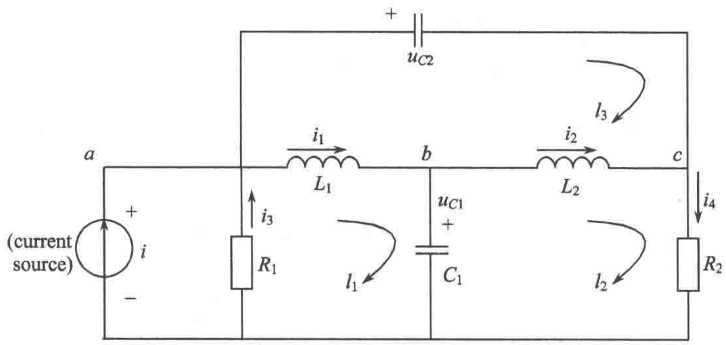


Fig.1.9 A three-loop network

Solution

The differential equations method for describing this network is based on the three differential equations which arise by applying the Kirchhoff's voltage law. These three loop equations are

$$-L_1\frac{di_1}{dt}+u_{C1}+R_1i_3=0$$

$$-u_{C1}+L_2\frac{di_2}{dt}+R_2i_4=0$$

$$L_2\frac{di_2}{dt}-L_1\frac{di_1}{dt}-u_{C2}=0$$

By applying the Kirchhoff's current law,

$$i+i_3+i_1-C_2\frac{du_{C2}}{dt}=0$$

$$C_1\frac{du_{C1}}{dt}+i_1+i_2=0$$

$$C_2 \frac{du_{C2}}{dt} + i_2 - i_4 = 0$$

Define

$$u_{C_1} = x_1, \quad u_{C_2} = x_2$$

$$i_1 = x_3, \quad i_2 = x_4$$

The system's state space equation can be expressed as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{C_1} & -\frac{1}{C_1} \\ 0 & -\frac{1}{C_2(R_1+R_2)} & \frac{R_1}{C_2(R_1+R_2)} & -\frac{R_2}{C_2(R_1+R_2)} \\ \frac{1}{L_1} & -\frac{R_1}{L_1(R_1+R_2)} & -\frac{R_1R_2}{L_1(R_1+R_2)} & -\frac{R_1R_2}{L_1(R_1+R_2)} \\ \frac{1}{L_2} & -\frac{R_2}{L_2(R_1+R_2)} & -\frac{R_1R_2}{L_2(R_1+R_2)} & -\frac{R_1R_2}{L_2(R_1+R_2)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{R_1}{C_2(R_1+R_2)} \\ -\frac{R_1R_2}{L_1(R_1+R_2)} \\ -\frac{R_1R_2}{L_2(R_1+R_2)} \end{pmatrix} i \quad (1\text{-}25)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} u_{C_1} \\ u_{C_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Example 1.9 Consider the mechanical system shown in Fig.1.10, where y, K, m and B are the position of the mass, the spring's constant, the mass, and the friction coefficient respectively. In the role of external forces f, derive the system's state space representation where y_1, y_2 are the outputs.

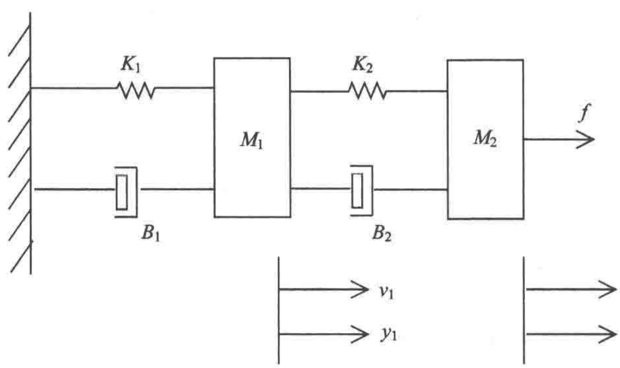


Fig.1.10 The mass-spring-damper system

Solution

Choose the position y_1, y_2 and the velocity v_1, v_2 of the mass M_1, M_2 as the state variables.

$$x_1 = y_1, \qquad x_2 = y_2$$

$$x_3 = v_1 = \frac{dy_1}{dt}, \qquad x_4 = v_2 = \frac{dy_2}{dt}$$

By using the Newton's laws of motion, for M_1:

$$M_1\frac{dv_1}{dt} = K_2\left(y_2 - y_1\right) + B_2\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) - K_1y_1 - B_1\frac{dy_1}{dt}$$

For M_2:

$$M_2\frac{dv_2}{dt} = f - K_2\left(y_2 - y_1\right) - B_2\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right)$$

with u=f, we can get

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{1}{M_1}\left(K_1 + K_2\right)x_1 + \frac{K_2}{M_1}x_2 - \frac{1}{M_1}\left(B_1 + B_2\right)x_3 + \frac{B_2}{M_1}x_4$$

$$\dot{x}_4 = \frac{K_2}{M_2}x_1 - \frac{K_2}{M_2}x_2 + \frac{B_2}{M_2}x_3 - \frac{B_2}{M_2}x_4$$

The above equations can be expressed in a compact form :

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{1}{M_1}(K_1 + K_2) & \frac{K_2}{M_1} & -\frac{1}{M_1}(B_1 + B_2) & \frac{B_2}{M_1}\\ \frac{K_2}{M_2} & -\frac{K_2}{M_2} & \frac{B_2}{M_2} & -\frac{B_2}{M_2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 0\\ \frac{1}{M_2}\end{pmatrix}f \qquad (1\text{-}26)$$

The output equation is

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

Example 1.10 Consider a cart with an inverted pendulum hinged on top of it as shown in Fig.1.11. For simplicity, the cart and the pendulum are assumed to move in only one plane, while the friction, the mass of the stick, and the gust of wind are disregarded. The problem is to maintain the pendulum at the vertical position.

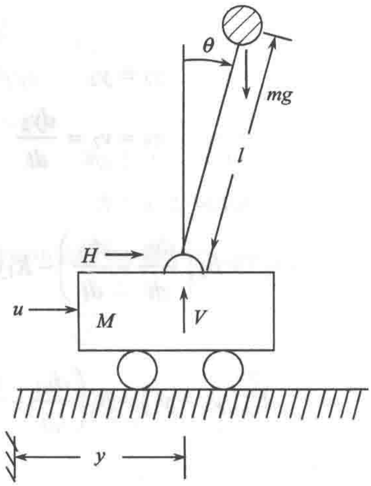


Fig.1.11 The inverted pendulum system

Let H and V be, respectively, the horizontal and vertical forces exerted by the cart on the pendulum as shown. The application of the Newton's law to the linear movements yields

$$M\frac{d^2y}{dt^2} = u - H$$

$$H = m\frac{d^2}{dt^2}(y + l\sin\theta) = m\ddot{y} + ml\ddot{\theta}\cos\theta - ml(\dot{\theta})^2\sin\theta$$

$$mg - V = m\frac{d^2}{dt^2}(l\cos\theta) = ml[-\ddot{\theta}\sin\theta - (\dot{\theta})^2\cos\theta]$$

The application of the Newton's law to the rotational movement of the pendulum around the hinge yields

$$mgl\sin\theta = ml\ddot{\theta}\cdot l + m\ddot{y}l\cos\theta$$

$$\sin\theta = \theta,\quad \cos\theta = 1$$

$$mg = V$$

$$M\ddot{y} = u - m\ddot{y} - ml\ddot{\theta},\qquad g\theta = l\ddot{\theta} + \ddot{y}$$

which imply

$$M\ddot{y} = u - mg\theta$$

$$Ml\ddot{\theta} = (M+m)g\theta - u$$

Define

$$x_1 = y,\ x_2 = \dot{y},\ x_3 = \theta,\ x_4 = \dot{\theta}$$

Then the state space model can be derived as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ \dfrac{-1}{Ml} \end{bmatrix} u \tag{1-27}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example 1.11 Consider a dc separately excited motor system shown in Fig. 1.12. In the diagram, R and L stand for the resistance and inductance of the armature loop respectively. J is the inertia of the rotating part, and B is the viscous friction coefficient. Develop the state space equations when the armature voltage u is chosen as the control variable.

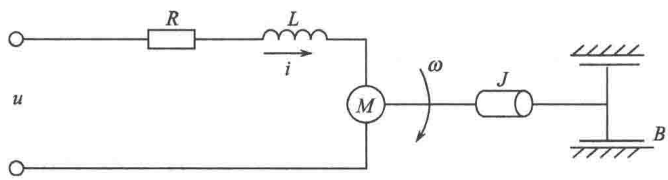


Fig.1.12 The system of dc separately excited motor

Since the inductance L and the rotating inertia J are energy storage elements, their corresponding physical variables, e.g., the current i and the rotating angular speed ω, are independent with each other. They could be chosen as state variables:

$$x_1 = i$$

$$x_2 = \omega$$

then

$$\frac{dx_1}{dt} = \frac{di}{dt}, \qquad \frac{dx_2}{dt} = \frac{d\omega}{dt}$$

Using the circuit equations of the armature circuit,

$$L\frac{di}{dt} + Ri + e = u$$

According to the dynamics equations, there is

$$J\frac{d\omega}{dt} + B\omega = K_a i$$

According to the electromagnetic induction relationship, there is

$$e = K_b \omega$$

where, e is the back electromotive force; K_a, K_b are the torque constant and back electromotive force respectively.

According to the three equations above, the model of the system may be rewritten as:

$$\frac{di}{dt} = -\frac{R}{L}i + -\frac{K_b}{L}\omega + \frac{1}{L}u$$

$$\frac{d\omega}{dt} = \frac{K_a}{J}i - \frac{B}{J}\omega$$

Put $x_1 = i$, $x_2 = \omega$ into the above equations, there is:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{K_b}{L} \\ \frac{K_a}{J} & -\frac{B}{J} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} u$$

If the angle speed ω is chosen to be the output, then

$$y = x_2 = \begin{pmatrix} 0, & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If the angle θ is chosen to be the output, then the above two state variables are not enough to represent the dynamics of system, and another state variable x_3 should be introduced

$$x_3 = \theta$$

So

$$\dot{x}_3 = \dot{\theta} = x_2$$

State equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{K_b}{L} & 0 \\ \frac{K_a}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix} u \tag{1-28}$$

The output equation

$$y = x_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The input-output model

When a system is given by the input-output model, i.e., transfer function or differential equation, the state space model can be derived with the input-output model, this process is called realization. Base on different types of the input-output model, different algorithm can be adopted for realization. The detail procedure can be found in Section1.3.5.

1.3 Transition from One Mathematical Model to Another

As we know, every mathematical model has advantages and disadvantages: to take the advantages of all mathematical models, one must have the flexibility of transition from one model to another. This issue of transition from one mathematical model to another is obviously of great practical and theoretical importance. In the following, we present some of the transition methods.

1.3.1 From Differential Equation to Transfer Function for SISO Systems

Case 1. The right-hand side of the differential equation does not involve derivatives

Consider a SISO system described by the following differential equation

$$y^{(n)}+\alpha_{n-1}y^{(n-1)}+\cdots+\alpha_1 y^{(1)}+\alpha_0 y=\beta_0 u \tag{1-29}$$

where all the system's initial conditions are assumed to be zero, i.e., $y^{(k)}(0)=0$, for $k=1,2,\cdots,n-1$. Applying the Laplace transform to equation (1-29) can result in:

$$s^n Y(s)+\alpha_{n-1}s^{n-1}Y(s)+\cdots+\alpha_1 sY(s)+\alpha_0 Y(s)=\beta_0 U(s)$$

Hence, the transfer function is given by

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+\alpha_0} \tag{1-30}$$

Case 2. The right-hand side of the differential equation involves derivatives

Consider a SISO system described by the differential equation

$$y^{(n)}+\alpha_{n-1}y^{(n-1)}+\cdots+\alpha_1 y^{(1)}+\alpha_0 y=\beta_m u^{(m)}+\cdots+\beta_1 u^{(1)}+\beta_0 u \tag{1-31}$$

where $m<n$ and all initial conditions are assumed to be zero, i.e., $y^{(k)}(0)=0$, for $k=0,1,\cdots,n-1$. We can determine the transfer equation (1-31) as follows: let $z(t)$ be the solution of equation (1-29), with $\beta_0=1$. Then, the superposition principle is used, the solution $y(t)$ of equation (1-31) will be obtained

$$y(t)=\beta_m z^{(m)}+\beta_{m-1}z^{(m-1)}+\cdots+\beta_1 z^{(1)}+\beta_0 z \tag{1-32}$$

Applying the Laplace transformation to equation (1-32), we obtain

$$Y(s)=\beta_m s^m Z(s)+\beta_{m-1}s^{m-1}Z(s)+\cdots+\beta_1 sZ(s)+\beta_0 Z(s) \tag{1-33}$$

Here, we have set $z^{(k)}(0)=0$, for $k=0,1,\cdots,n-1$. When $\beta_0=1$, the solution of equation (1-29) is $z(t)=y(t)$, where it is assumed that all initial conditions of $y(t)$ (and hence of $z(t)$) are zero. In equation (1-30), when $\beta_0=1$, we have

$$Z(s) = \left[\frac{1}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \right] U(s) \tag{1-34}$$

By substituting equation (1-34) into (1-33), the transfer function $G(s)$ of the differential equation (1-31) is obtained as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \tag{1-35}$$

Remark 1.3.1

The transfer function $G(s)$ given by equation (1-35) can be easily derived from equation (1-31) if we set s^k in place of the kth derivative and replace $y(t)$ and $u(t)$ with $Y(s)$ and $U(s)$, respectively, i.e., we can derive equation (1-35) by replacing $y^{(k)}(t)$ with $s^k Y(s)$, and $u^{(k)}(t)$ with $s^k U(s)$ in equation (1-31).

1.3.2 From Transfer Function to Differential Equation for SISO Systems

Let a SISO system be described by equation (1-35). Then, working backwards the method given in Remark 1.3.1, the differential equation (1-31) can be constructed by substituting s^k with the kth derivative and $Y(s)$ and $U(s)$ with $y(t)$ and $u(t)$, respectively.

1.3.3 From $G(s)$ to $g(t)$ and Vice Versa

The matrices $G(s)$ and $g(t)$ are related through the Laplace transform:

$$L\{g(t)\} = G(s) \quad \text{or} \quad g(t) = L^{-1}\{G(s)\} \tag{1-36}$$

1.3.4 From State Equations to Transfer Function Matrix

Consider a system described by the following state equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(t_0) &= x(0) = x_0 \end{aligned} \tag{1-37}$$

Take the Laplace transformation to both sides of equation (1-37)

$$\begin{aligned} &sx(s) - x(0) = Ax(s) + Bu(s) \\ &y(s) = Cx(s) + Du(s) \\ &x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s) \\ &y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}Bu(s) + Du(s) \end{aligned}$$

For zero initial condition, we have

$$y(s) = \left[C(sI - A)^{-1}B + D \right] u(s)$$

Then the system's transfer function matrix $G(s)$ is given by the relation

$$G(s) = C(sI - A)^{-1}B + D \tag{1-38}$$

Example 1.12 Derive the transfer function of the following state-space equation.

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x$$

Solution

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & -2 \end{pmatrix};\quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};\quad D = 0;$$

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s} & 0 & \frac{1}{s(s+1)} & 0 \\ 0 & \frac{1}{s} & 0 & \frac{1}{s(s+1)} \\ 0 & 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & 0 & \frac{1}{s+1} \end{pmatrix}$$

The result can be obtained according to (1-38)

$$G(s) = \begin{bmatrix} \frac{1}{s(s+1)} & \frac{1}{s} \\ \frac{1}{s} & \frac{s-1}{s(s+1)} \end{bmatrix}$$

MATLAB can be adopted to computer this equation. Type

```
a=[0 0 1 0;0 0 0 1;0 0 -1 0;0 0 0 -1];
b=[0 1;1 1;1 0;0 -2];
c=[1 0 0 0;0 1 0 0];
d=[0 0;0 0];
[N1,d1]=ss2tf(a,b,c,d,1)
[N2,d2]=ss2tf(a,b,c,d,2)
which yields
N1 =
  0         0              1.0000         1.0000          0.0000
```

```
    0          1.0000          2.0000          1.0000          0
d1 =
     1    2    1    0    0
N2 =
    0          1.0000          2.0000          1.0000          0
    0          1.0000          0.0000         -1.0000          0.0000
d2 =
     1    2    1    0    0
```

Thus the transfer matrix is

$$G(s)=\begin{bmatrix} \dfrac{s^2+s}{s^4+2s^3+s^2} & \dfrac{s^3+2s^2+s}{s^4+2s^3+s^2} \\ \dfrac{s^3+2s^2+s}{s^4+2s^3+s^2} & \dfrac{s^3-s}{s^4+2s^3+s^2} \end{bmatrix}$$

By simplifying

$$G(s)=\begin{bmatrix} \dfrac{1}{s(s+1)} & \dfrac{1}{s} \\ \dfrac{1}{s} & \dfrac{s-1}{s(s+1)} \end{bmatrix}$$

Here, **[N1,d1]=ss2tf(a,b,c,d,1)** computes the transfer matrix from the first input to all outputs, e.g., the first column of $G(s)$. **N1** is the numerator coefficient of the first column of $G(s)$, **d1** is the denominator coefficient of the first column of $G(s)$. In a similar way, **[N2,d2]=ss2tf(a,b,c,d,2)** computes the transfer matrix from the second input to all outputs.

1.3.5 From Transfer Function Matrix to State Equations for SISO Systems

The transition from $G(s)$ to state equations is the well-known problem of the state-space realization. This is, in general, a difficult problem and has been, and still remains, a topic to study. In the following, we will present some introductory results regarding this problem.

Let a system be described by a scalar transfer function with the form

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\beta_{n-1}s^{n-1}+\beta_{n-1}s^{n-1}+\cdots+\beta_1 s+\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+\alpha_0} \tag{1-39}$$

or, equivalently, by the differential equation

$$y^{(n)}+\alpha_{n-1}y^{(n-1)}+\cdots+\alpha_1 y^{(1)}+\alpha_0 y=\beta_n u^{(n)}+\beta_{n-1}u^{(n-1)}\cdots+\beta_1 u^{(1)}+\beta_0 u \tag{1-40}$$

Equation (1-40) can be expressed in the form of two equations as follows:

$$z^{(n)}+\alpha_{n-1}z^{(n-1)}+\cdots+\alpha_1 z^{(1)}+\alpha_0 z=u \tag{1-41}$$

$$y(t) = \beta_{n-1}z^{(n-1)} + \cdots + \beta_1 z^{(1)} + \beta_0 z \tag{1-42}$$

Let $z(t)$ be the solution of equation (1-41). Then, the solution of equation (1-40) will be given by equation (1-42).

The state variables $x_1, x_2, \cdots, x_n$ are defined as follows:

$$\begin{aligned}
x_1(t) &= z(t) \\
x_2(t) &= z^{(1)}(t) = x_1^{(1)}(t) \\
x_3(t) &= z^{(2)}(t) = x_2^{(1)}(t) \\
&\ \vdots \\
x_n(t) &= z^{(n-1)}(t) = x_{n-1}^{(1)}(t)
\end{aligned} \tag{1-43}$$

Substituting equation (1-42) into equation (1-41) can result in

$$\dot{x}_n(t) = -\alpha_{n-1}x_n(t) - \cdots - \alpha_1 x_2(t) - \alpha_0 x_1(t) + u(t) \tag{1-44}$$

Also, substituting equations (1-43) into equation (1-42) can result in

$$y(t) = \beta_{n-1}x_n(t) + \beta_{n-2}x_{n-1}(t) + \cdots + \beta_1 x_2(t) + \beta_0 x_1(t) \tag{1-45}$$

Equations (1-43)-(1-45) can be expressed in a matrix form:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + bu(t) \\
y(t) &= c^T x(t)
\end{aligned} \tag{1-46}$$

where $x^T = (x_1, x_2, \cdots, x_n)$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{n-1} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{1-47}$$

$$c = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{bmatrix}$$

Hence, equations (1-46) constitute the state equations' description of the transfer function

(1-39).

Due to the special form of matrix A and vector b, we say that the state equations (1-46) are in *phase canonical form*, while the state variables are called *phase variables*. Phase variables are, in general, state variables which are defined according to equations (1-43); i.e., every state variable is the derivative of the previous one. In particular, the special form of matrix A and vector b is characterized by:

If the first column and the last row in matrix A are deleted, then a $(n-1)\times(n-1)$ unit matrix is revealed. Also, the elements of the last row of A are the coefficients of the differential equation (1-40), placed in reverse order and all having negative signs. The vector b has all its elements equal to zero, except for the nth element, which is equal to one.

Example 1.13 The differential equation mathematical model of a system is given as

$$\dddot{y}+6\ddot{y}+41\dot{y}+7y=6u$$

Try to derive the state equation and the output equation.

Solution

We choose $y/6$, $\dot{y}/6$, $\ddot{y}/6$ as the state variables

$$x_1=\frac{y}{6},\quad x_2=\frac{\dot{y}}{6},\quad x_3=\frac{\ddot{y}}{6}$$

Then,

$$\dot{x}_1=\frac{\dot{y}}{6}=x_2$$

$$\dot{x}_2=\frac{\ddot{y}}{6}=x_3$$

$$x_3=\frac{\dddot{y}}{6}=-7x_1-41x_2-6x_3+u$$

The state equation is

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -7 & -41 & -6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}u \tag{1-48}$$

The output equation is

$$y=6x_1=(6\quad 0\quad 0)\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

1.4 Summary

Three kinds of models and their relationship are introduced in this chapter. Each type of model has its merit and shortage. The transfer function is widely used in classical control theory which mainly studies the SISO linear system in Laplace domain. The differential equation is used in time domain, and the state space equation is commonly used in modern control theory in which the MIMO (multi-input multi-output) system is studied in time domain. The following contents of modern control theory which studies the properties of a system and system synthesis are mainly based on the state space description.

Appendix: Three Power Generation Models

Case 1. Thermal Power Generation

The fundamental dynamics of a 160 MW drum-type boiler-turbine-generator plant can be represented by a third order MIMO coupling nonlinear model over a wide operating range. Typically, the coordinated control governs the dominant behaviour of the power unit through the power and steam pressure control loops, as shown in Fig.1.13.

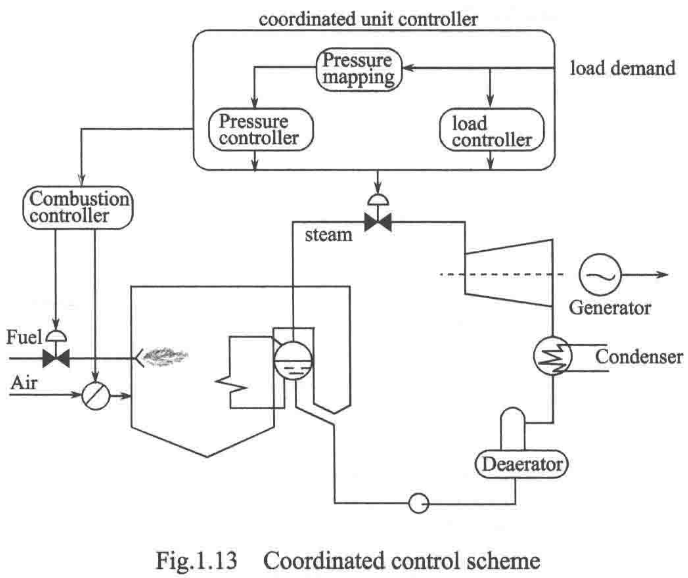


Fig.1.13 Coordinated control scheme

The inputs are positions of valve actuators that control the mass flow rates of fuel (u_1 in pu), steam to the turbine (u_2 in pu), and feedwater to the drum (u_3 in pu). The outputs are electric power (E in MW), drum steam pressure (P in kg/cm^2), and drum water level deviation(L in mm). The state variables are electric power (E), drum steam pressure (P), and fluid (steam-water) density (ρ_f). The state equations are:

$$\begin{aligned}
\frac{dP}{dt} &= 0.9u_1 - 0.0018u_2P^{9/8} - 0.15u_3 \\
\frac{dE}{dt} &= ((0.73u_2 - 0.16)P^{9/8} - E)/10 \\
\frac{d\rho_f}{dt} &= (141u_3 - (1.1u_2 - 0.19)P)/85
\end{aligned} \tag{1-49}$$

The drum water level output is calculated using the following algebraic equations:

$$\begin{aligned}
q_e &= (0.85u_2 - 0.14)P - 45.59u_1 - 2.51u_3 - 2.09 \\
\alpha_s &= (1/\rho_f - 0.0015)/(1/(0.8P - 25.6) - 0.0015) \\
L &= 50(0.13\rho_f + 60\alpha_s + 0.11q_e - 65.5)
\end{aligned} \tag{1-50}$$

where α_s is the steam quality and q_e is the evaporation rate(kg/s).

Define x_1, x_2 and x_3 to be the drum steam pressure ($\mathrm{kg/cm^2}$), the electric power (MW), and the steam-water fluid density in the drum, respectively. The output y_3 is the drum water level (cm) calculated using two algebraic calculations α_{cs} and q_e which are the steam quality and the evaporation rate (kg / s).The input u_1, u_2 and u_3 are normalized positions of valve actuators that control the mass flow rates of fuel, steam to the turbine, and feedwater to the drum ,respectively. Then the state-space equation becomes:

$$\begin{aligned}
\dot{x}_1 &= -0.0018u_2x_1^{9/8} + 0.9u_1 - 0.15u_3 \\
\dot{x}_2 &= \left(0.073u_2 - 0.016\right)x_1^{9/8} - 0.1x_2 \\
\dot{x}_3 &= \left[141u_3 - \left(1.1u_2 - 0.19\right)x_1\right]/85 \\
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= 0.05\left(0.13073x_3 + 100\alpha_{cs} + q_e/9 - 67.975\right) \\
\alpha_{cs} &= \frac{\left(1 - 0.001538x_3\right)\left(0.8x_1 - 25.6\right)}{x_3\left(1.0394 - 0.0012304x_1\right)} \\
q_e &= \left(0.845u_2 - 0.147\right)x_1 + 45.59u_1 - 2.514u_3 - 2.096
\end{aligned} \tag{1-51}$$

Case 2. Nuclear Power Generation

The water level model developed by E.Irving is a fourth-order transfer function representation, with its power level dependent parameters listed in Table 1:

$$Y(s)=\frac{G_1}{s}(Q_w(s)-Q_v(s))-\frac{G_2}{1+\tau_2 s}(Q_w(s)-Q_v(s))+\frac{G_3 s}{\tau_1^{-2}+4\pi^2T^{-2}+2\tau_1^{-2}s+s^2}Q_w(s) \tag{1-52}$$

where $Y(s)$, $Q_w(s)$ and $Q_v(s)$ represent the water level, the feed-water flow-rate and the steam flow-rate respectively. τ_1, τ_2 and T are the damping time constants and the oscillation period.

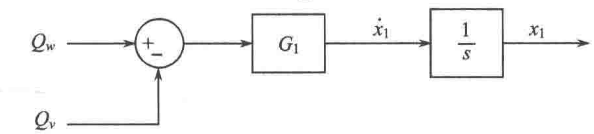
Table1 Parameters of The Steam Generator Model with Respect to Operating Power

P(%power)	5	15	30	50	100
q_v(kg / s)	57.4	180.8	381.7	660	1435
G_1	0.058	0.058	0.058	0.058	0.058
G_2	9.63	4.46	1.83	1.05	0.47
G_3	0.181	0.226	0.310	0.215	0.105
τ_1	41.9	26.3	43.4	34.8	28.6
τ_2	48.4	21.5	4.5	3.6	3.4
T	119.6	60.5	17.7	14.2	11.7

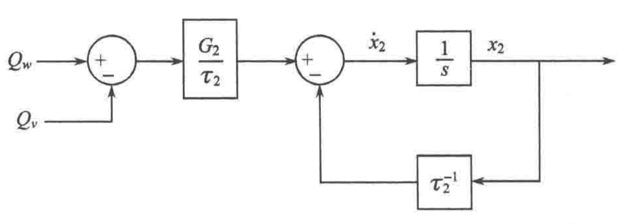
Let

$$\begin{aligned}
Y_1(s)&=\frac{G_1}{s}(Q_w(s)-Q_v(s))\\
Y_2(s)&=-\frac{G_2}{1+\tau_2 s}(Q_w(s)-Q_v(s))\\
Y_3(s)&=\frac{G_3 s}{\tau_1^{-2}+4\pi^2T^{-2}+2\tau_1^{-1}s+s^2}Q_w(s)
\end{aligned} \tag{1-53}$$

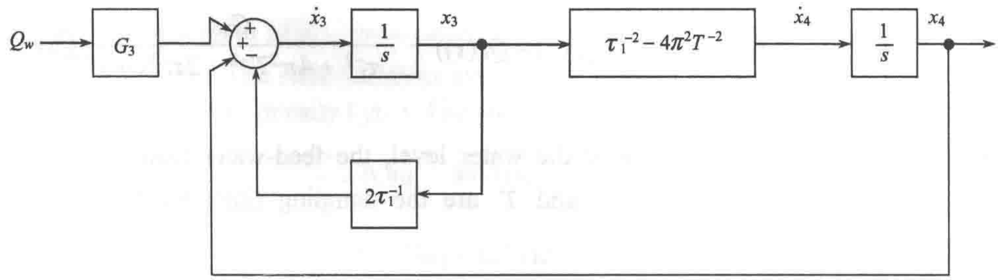
Define the state variables as shown in the Fig.1.14.



(a) $Y_1(s)$



(b) $Y_2(s)$



(c) $Y_3(s)$

Fig.1.14

Then,

$$\begin{aligned}
\dot{x}_1(t) &= G_1(Q_w(t) - Q_v(t)) \\
\dot{x}_2(t) &= -\tau_2{}^{-1}x_2(t) - \frac{G_2}{\tau_2}(Q_w(t) - Q_v(t)) \\
\dot{x}_3(t) &= -2\tau_1{}^{-1}x_3(t) + x_4(t) + G_3 Q_w(t) \\
\dot{x}_4(t) &= -(\tau_1^{-2} + 4\pi^2 T^{-2})x_3(t)
\end{aligned} \tag{1-54}$$

The control variable $u = Q_w$, the disturbance $d = Q_v$, and the water level output $y = x_1 + x_2 + x_3$, equtions. (1-54) can be rearranged in the following state-space equations:

$$\begin{cases} \dot{x}_c(t) = A(\theta)x_c(t) + B(\theta)u_c(t) + W(\theta)d_c(t) \\ y_c(t) = Cx_c(t) \end{cases} \tag{1-55}$$

where

$$x_c = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$A(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & 0 \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix}, \quad W(\theta) = \begin{bmatrix} d_1 \\ d_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}$$

$$a_{22} = -\tau_2^{-1}, \ a_{33} = -2\tau_1^{-1}, \ a_{34} = 1, \ a_{43} = -(\tau_1^{-1} + 4\pi^2 T^{-2})$$

$$b_1 = G_1, \ b_2 = -G_2\tau_2^{-1}, \ b_3 = G_3$$

$$d_1 = -G_1, \ d_2 = \frac{G_2}{\tau_2}$$

Case 3. Wind Power Generation

The doubly fed induction generators (DFIGs) have been widely used in the modern wind energy systems, due to the advantages of variable speed operation and four quadrant active and reactive power capabilities. In DFIG, the stator is directly connected to the power grid, while the rotor is connected to the grid by a bidirectional converter as shown in Fig.1.15. This converter controls active and the reactive power between the stator and ac supply or a standalone grid. The equivalent circuit of a DFIG in an arbitrary reference frame rotating at synchronous angular speed ω_1 is shown in Fig.1.16.

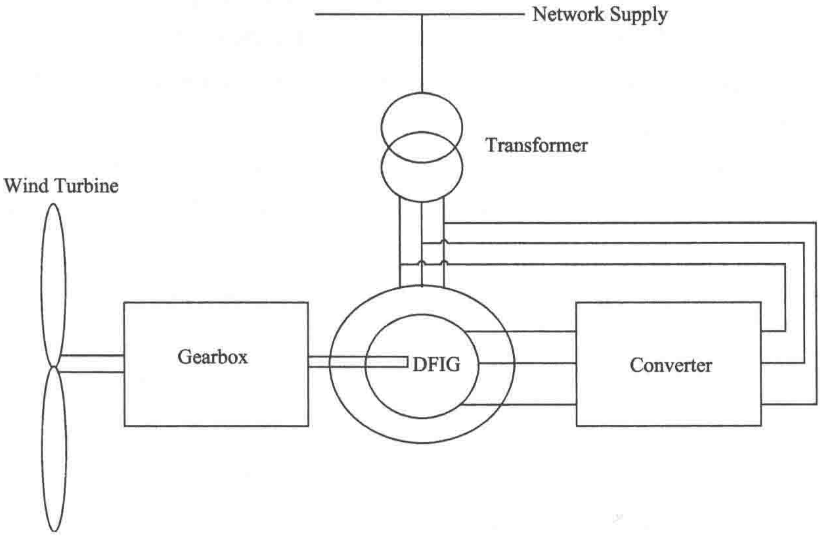


Fig.1.15 The doubly fed induction generator system

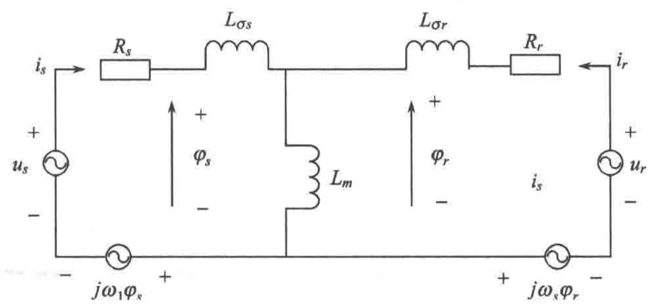


Fig.1.16 The equivalent circuit of a DFIG

The DFIG model in the synchronous reference frame is given by

$$u_s = R_s i_s + d\varphi_s/dt + j\omega_1\varphi_s \tag{1-56}$$

$$u_r = R_r i_r + d\varphi_r/dt + j(\omega_1 - \omega_r)\varphi_r \tag{1-57}$$

where the relationship between fluxes and currents is

$$\varphi_s = L_s i_s + L_m i_r \tag{1-58}$$

$$\varphi_r = L_m i_s + L_r i_r \tag{1-59}$$

and generator active and reactive power are

$$P = \frac{3}{2}\left(u_{sd} i_{sd} + u_{sq} i_{sq}\right) \tag{1-60}$$

$$Q = \frac{3}{2}\left(u_{sq} i_{sd} + u_{sd} i_{sq}\right) \tag{1-61}$$

The subscripts s and r represent the stator and rotor parameters respectively: ω_1 represents the synchronous speed; ω_r represents rotor angular speed; R_s and R_r represent stator and the rotor windings per phase electrical resistance; L_s , L_r and L_m represent the proper and the mutual inductances of the stator and rotor windings; u represents voltage vector.

The DFIG power control aims independent stator active P and reactive power Q control by means of a rotor current regulation. For this purpose, P and Q are represented as functions of each individual rotor current. We use stator flux oriented control, that decouples the dq axis, which means: $\varphi_{sd} = \varphi_s$, $\varphi_{sq} = 0$ .Thus, (1-58) becomes

$$i_{sd} = \frac{\varphi_s}{L_s} - \frac{L_m}{L_s} i_{rd} \tag{1-62}$$

$$i_{sq} = -\frac{L_m}{L_s} i_{rq} \tag{1-63}$$

Similarly, using stator flux orientation, the stator voltage becomes $u_{sd} = 0$ and . $u_{sq} = u_s$ Hence, the active (1-60) and reactive (1-61) power can be calculated by using (1-62) and (1-63)

$$P = -\frac{3}{2} u_s \frac{L_m}{L_s} i_{rq} \tag{1-64}$$

$$Q = \frac{3}{2}\left(\frac{\varphi_s}{L_s} - \frac{L_m}{L_s} i_{rd}\right) u_s \tag{1-65}$$

Thus, rotor currents will reflect on stator current and on stator active and reactive power, respectively. Consequently, this principle can be used on stator active and reactive power control of the DFIG.

The DFIG power control is realized by the rotor currents control using (1-64) and (1-65). Using (1-62) and (1-63), the rotor voltage (1-57), in the synchronous referential frame, becomes

$$u_r = \left(R_r + j\sigma L_r \omega_{sl}\right) i_r + \sigma L_r \frac{di_r}{dt} + j\frac{L_m}{L_s} \omega_{sl} \varphi_s \tag{1-66}$$

then:

$$\frac{di_r}{dt} = \frac{-R_r}{\sigma L_r} i_r - j\omega_{sl} i_r + \frac{1}{\sigma L_r} u_r - j\frac{L_m}{\sigma L_r L_s} \omega_{sl} \varphi_s \tag{1-67}$$

$$
\begin{aligned}
\frac{di_r}{dt} &= -\frac{R_r}{\sigma L_r} i_{rd} + \omega_{sl} i_{rq} + \frac{1}{\sigma L_r} u_{rd} \\
\frac{di_r}{dt} &= -\omega_{sl} i_{rd} - \frac{R_r}{\sigma L_r} i_{rq} + \frac{1}{\sigma L_r} u_{rq} - \frac{\omega_{sl} L_m}{\sigma L_r L_s} \varphi_s
\end{aligned} \tag{1-68}
$$

(1-68) can be expressed as:

$$
\begin{bmatrix} \frac{di_{rd}}{dt} \\ \frac{di_{rq}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_r}{\sigma L_r} & \omega_{sl} \\ -\omega_{sl} & -\frac{R_r}{\sigma L_r} \end{bmatrix} \begin{bmatrix} i_{rd} \\ i_{rq} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma L_r} & 0 \\ 0 & \frac{1}{\sigma L_r} \end{bmatrix} \begin{bmatrix} u_{rd} \\ u_{rq} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{\omega_{sl} L_m}{\sigma L_r L_s} \varphi_s \end{bmatrix} \tag{1-69}
$$

$$
\begin{bmatrix} i_{rd} \\ i_{rq} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{rd} \\ i_{rq} \end{bmatrix}
$$

where $\omega_{sl} = \omega_1 - \omega_r$ and $\sigma = 1 - \frac{L_m^2}{L_s L_r}$.

Assuming $x = [x_1, x_2]^T = [i_{rd}, i_{rq}]^T$, $\dot{x} = [\dot{x}_1, \dot{x}_2]^T = [\dot{i}_{rd}, \dot{i}_{rq}]^T$,

$u = [u_1, u_2]^T = [u_{rd}, u_{rq}]^T$, $y = [y_1, y_2]^T = [i_{rd}, i_{rq}]^T$

(1-69) can be expressed in the standard space state form,

$$
\begin{aligned}
\dot{x} &= Ax + Bu + G\omega \\
y &= Cx
\end{aligned} \tag{1-70}
$$

where $A = \begin{bmatrix} -\frac{R_r}{\sigma L_r} & \omega_{sl} \\ -\omega_{sl} & -\frac{R_r}{\sigma L_r} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\sigma L_r} & 0 \\ 0 & \frac{1}{\sigma L_r} \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $G\omega = \begin{bmatrix} 0 \\ -\frac{\omega_{sl} L_m}{\sigma L_r L_s} \varphi_s \end{bmatrix}$

Exercise

1.1 Consider the network shown in Fig.1.17. Derive the network's differential equation mathematical model.

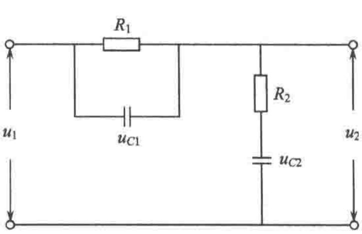


Fig.1.17

1.2 Consider the mechanical system shown in Fig.1.18, where $x_1 = u_{C1}, x_2 = u_{C2}$, K, B, M are the spring's constant, the friction coefficient and the mass, respectively. Derive the state-space equations.

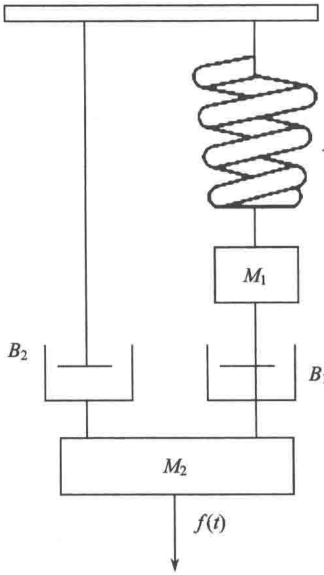


Fig.1.18

1.3 Consider the network shown in Fig.1.17. Derive the network's state-space equations.

1.4 Consider the network shown in Fig.1.19. Derive the network's transfer function $\dfrac{U_C(s)}{U(s)}$.

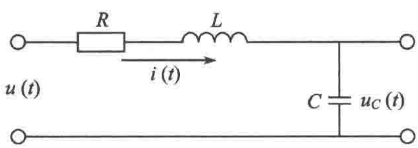


Fig.1.19

1.5 Consider the network shown in Fig.1.19. Derive the state-space equations.

1.6 Consider the network shown in Fig.1.20. Derive the network's transfer function $\dfrac{U_L(s)}{U_1(s)}, \dfrac{U_L(s)}{U_2(s)}$.

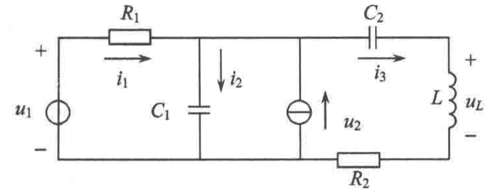


Fig.1.20

1.7 Consider the network shown in Fig.1.20.Derive the network's state-space equations.

1.8 Transition from state equations to transfer function.

(1) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 1 & -2 & 0\\ 0 & 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}6\\ 0\\ 0\end{bmatrix}u$ (2) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-5 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}2\\ 0\end{bmatrix}u$

$$y=\begin{bmatrix}0 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \qquad y=\begin{bmatrix}1 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

1.9 Transition from transfer function to state equations.

(1) $g(s)=\dfrac{s^3+s+1}{s^3+6s^2+11s+6}$

(2) $g(s)=\dfrac{s^2+2s+3}{s^3+2s^2+3s+1}$

1.10 Transition from differential equation to transfer function.

(1) $2\ddot{y}+3\dot{y}=\ddot{u}-u$

(2) $\dddot{y}+2\ddot{y}+3\dot{y}+5y=5\dddot{u}+7u$

1.11 Transition from transfer function to differential equation.

$$g(s)=\frac{Y(s)}{U(s)}=\frac{160s+720}{s^3+16s^2+194s+640}$$

Chapter 2
Linear Transformation of State Vector

2.1 Linear Algebra

This chapter reviews a number of concepts and results in linear algebra that are essential in the linear transformation of state vector.

As we saw in the preceding chapter, all parameters that arise in the real world are real numbers. Therefore we deal only with real numbers, except stated specially, throughout this chapter. Let A, B, C and D be, respectively, $n \times m, m \times r, l \times n$ and $r \times p$ dimensional real matrices. Let a_i be the i th column of A, and b_j the j th row of B. Then we have

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m \tag{2-1}$$

$$CA = C\begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} = \begin{bmatrix} Ca_1 & Ca_2 & \cdots & Ca_m \end{bmatrix} \tag{2-2}$$

$$BD = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} D = \begin{bmatrix} b_1 D \\ b_2 D \\ \vdots \\ b_m D \end{bmatrix} \tag{2-3}$$

These identities can easily be verified. Note that $a_i b_i$ is a $n \times r$ matrix; it is the product of a $n \times 1$ column vector and a $1 \times r$ row vector. The product $b_i a_i$ is not defined unless $n = r$; it becomes a scalar if $n = r$.

Consider an n-dimensional real linear space, denoted by R^n. Every vector in R^n has n real numbers such as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

To save space, we write it as $x=\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, where the superscript denotes the transpose.

The set of vectors $\{x_1, \ x_2, \ \cdots, \ x_m\}$ in R^n is said to be linearly dependent if there exists a set of real numbers $\alpha_1,\alpha_2,\cdots,\alpha_m$, which are not all zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = 0 \tag{2-4}$$

If the only set of α_i that makes (2-4) hold is $\alpha_1=0,\alpha_2=0,\cdots,\alpha_m=0$, then the set of vectors $\{x_1, \ x_2, \ \cdots, \ x_m\}$ is said to be linearly independent.

If the set of vectors in (2-4) is linearly dependent, then there exists at least one a_i, say, a_1, that is nonzero. Then (2-4) implies

$$x_1 = -\frac{1}{\alpha_1}\left[\alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_m x_m\right] = \beta_2 x_2 + \beta_3 x_3 + \cdots + \beta_m x_m$$

where $\beta_i = -\alpha_i / \alpha_1$. Such an expression is called a linear combination.

The dimension of a linear space can be defined as the maximum number of linearly independent vectors in the space. Thus in R^n, we can find n linearly independent vectors.

Basis and representation A set of linearly independent vectors in R^n is called a basis if every vector in R^n can be expressed as a unique linear combination of the set. In R^n, any set of n linearly independent vectors can be used as a basis. Let $\{q_1,q_2,\cdots,q_n\}$ be such a set. Then every vector x can be expressed uniquely as

$$x = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n \tag{2-5}$$

Define the $n \times n$ square matrix

$$Q=\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \tag{2-6}$$

Then (2-5) can be written as

$$x = Q\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = Q\overline{x} \tag{2-7}$$

We call $\overline{x}=\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}^T$ the representation of the vector x with respect to the basis $\{q_1,q_2,\cdots,q_n\}$.

We will associate every R^n with the following orthonormal basis:

$$i_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad i_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \cdots, \quad i_{n-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad i_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2-8}$$

With respect to this basis, we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 i_1 + x_2 i_2 + \cdots + x_n i_n = I_n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where I_n is the $n \times n$ unit matrix. In other words, the representation of any vector x with respect to the orthonormal basis in (2-8) equals itself.

Norms of vectors The concept of norm is a generalization of length or magnitude. Any real-valued function of x, denoted by $\|x\|$, can be defined as a norm if it has the following properties:

(1) $\|x\| \geqslant 0$ for every x and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\alpha x\| = |\alpha| \|x\|$, for any real α.

(3) $\|x_1 + x_2\| \leqslant \|x_1\| + \|x_2\|$ for every x_1 and x_2.

orthonormalization A vector x is said to be normalized if its Eucildeam norm is 1 or $x^T x = 1$. Note that $x^T x$ is scalar and xx^T is $n \times n$. Two vectors x_1 and x_2 are said to be orthogonal if $x_1^T x_2 = x_2^T x_1 = 0$. A set of vectors x_i, $i = 1, 2, \cdots, m$, is said to be orthonormal if

$$x_i^T x_j = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$$

Given a set of linearly independent vectors $e_1, e_2, \cdots, e_m$, we can obtain an orthonormal set using the procedure that follows:

$$\begin{aligned} u_1 &= e_1 & q_1 &= u_1 / \|u_1\| \\ u_2 &= e_2 - \left(q_1^T e_2\right) q_1 & q_2 &= u_2 / \|u_2\| \\ &\vdots & &\vdots \\ u_m &= e_m - \sum_{k=1}^{m-1} \left(q_k^T e_m\right) q_k & q_m &= u_m / \|u_m\| \end{aligned}$$

The first equation normalizes the vector e_1 to have norm 1. The vector $\left(q_1^T e_2\right) q_1$ is the projection of the vector e_2 along q_1. Its subtraction from e_2 yields the vertical part u_2. It is then normalized to 1. Using this procedure, we can obtain an orthonormal set. This is called the *Schmidt orthonormalization procedure.*

Let $A=\begin{bmatrix}a_1 & a_2 & \cdots & a_m\end{bmatrix}$ be an $n\times m$ matrix with $m\leqslant n$. If all columns of A or $\{a_i, i=1,2,\cdots,m\}$ are orthonormal, then

$$A^TA=\begin{bmatrix}a_1^T\\ a_2^T\\ \vdots\\ a_m^T\end{bmatrix}\begin{bmatrix}a_1 & a_2 & \cdots & a_m\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}=I_m$$

where I_m is the unit matrix of order m. Note that, in general, $AA^T\neq I_n$.

Similarity transformation Consider a $n\times n$ matrix A. It maps R^n into itself. If we associate with R^n the orthonormal basis $\{i_1,\ i_2,\ \cdots,\ i_n\}$ in (2-8), then the ith column of A is the representation of Ai_i with respect to the orthonormal basis. Now if we select a different set of basis $\{q_1,\ q_2,\ \cdots,\ q_n\}$, then the matrix A has a different representation $\bar{A}$. It turns out that the ith column of $\bar{A}$ is the representation of Aq_i with respect to the basis $\{q_1,\ q_2,\ \cdots,\ q_n\}$. This is illustrated by the following example.

Example 2.1 Consider the matrix

$$A=\begin{bmatrix}3 & 2 & -1\\ -2 & 1 & 0\\ 4 & 3 & 1\end{bmatrix} \tag{2-9}$$

Let $b=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. Then we have

$$Ab=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix},\quad A^2b=A(Ab)=\begin{bmatrix}-4\\ 2\\ -3\end{bmatrix},\quad A^3b=A\left(A^2b\right)=\begin{bmatrix}-5\\ 10\\ -13\end{bmatrix}$$

It can be verified that the following relation holds:

$$A^3b=17b-15Ab+5A^2b \tag{2-10}$$

Because the three vectors b, Ab, and A^2b are linearly independent, they can be used as a basis. We now compute the representation of A with respect to the basis. It is clear that

$$A(b)=\begin{bmatrix}b & Ab & A^2b\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$A(Ab)=\begin{bmatrix}b & Ab & A^2b\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

$$A\left(A^2 b\right)=\begin{bmatrix} b & Ab & A^2 b \end{bmatrix}\begin{bmatrix} 17 \\ -15 \\ 5 \end{bmatrix}$$

where the last equation is obtained from (2-10). Thus the representation of A with respect to the basis $\{b, \quad Ab, \quad A^2 b\}$ is

$$\overline{A}=\begin{bmatrix} 0 & 0 & 17 \\ 1 & 0 & -15 \\ 0 & 1 & 5 \end{bmatrix} \tag{2-11}$$

The preceding discussion can be extended to the general case. Let A be a $n\times n$ matrix. If there exists a $n\times 1$ vector b such that the n vectors $b, Ab, \cdots, A^{n-1}b$ are linearly independent and if

$$A^n=\beta_1 b+\beta_2 Ab+\cdots+\beta_n A^{n-1}b$$

the representation of A with respect to the basis $\{b, \quad Ab, \quad \cdots, \quad A^{n-1}b\}$ is

$$\overline{A}=\begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_1 \\ 1 & 0 & \cdots & 0 & \beta_2 \\ 0 & 1 & \cdots & 0 & \beta_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_{n-1} \\ 0 & 0 & \cdots & 1 & \beta_n \end{bmatrix} \tag{2-12}$$

This matrix is said to be in a companion form.

Consider the equation

$$Ax=y \tag{2-13}$$

The square matrix A maps x in R^n into y in R^n. With respect to the basis $\{q_1, \quad q_2, \quad \cdots, \quad q_n\}$, the equation becomes

$$\overline{A}\,\overline{x}=\overline{y} \tag{2-14}$$

where $\overline{x}$ and $\overline{y}$ are the representations of x and y with respect to the basis $\{q_1, \quad q_2, \quad \cdots \quad q_n\}$. As discussed in (2-7), they are related by

$$x=Q\,\overline{x} \qquad y=Q\,\overline{y}$$

with

$$Q=\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \tag{2-15}$$

to be an $n\times n$ nonsingular matrix. Substituting these into (2-13) yields

$$AQ\,\overline{x}=Q\,\overline{y} \quad \text{or} \quad Q^{-1}AQ\,\overline{x}=\overline{y} \tag{2-16}$$

Comparing this with (2-14) yields

$$\bar{A}=Q^{-1}AQ \quad \text{or} \quad A=Q\bar{A}Q^{-1} \tag{2-17}$$

This is called the similarity transformation and A and $\bar{A}$ are said to be similar. We write (2-17) as

$$AQ=Q\bar{A}$$

or

$$A\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}=\begin{bmatrix} Aq_1 & Aq_2 & \cdots & Aq_n \end{bmatrix}=\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}\bar{A}$$

This shows that the i th column of $\bar{A}$ is indeed the representation of Aq_i with respect to the basis $\{q_1, \ q_2, \ \cdots, \ q_n\}$.

2.2 Transform to Diagonal Form and Jordan Form

A square matrix A has different representations with respect to different sets of basis. In this section, we introduce a set of basis so that the representation will be diagonal or block diagonal.

A real or complex number λ is called an eigenvalue of the $n\times n$ real matrix A if there exists a nonzero vector x such that $Ax=\lambda x$ is called a (right) eigenvector of A associated with eigenvalue λ. In order to find the eigenvalue of A, we write $Ax=\lambda x=\lambda Ix$ as

$$(A-\lambda I)x=0 \tag{2-18}$$

where I is the unit matrix of order n. This is a homogeneous equation. If the matrix $(A-\lambda I)$ is nonsingular, then the only solution of (2-18) is $x=0$. Thus in order for (2-18) to have a nonzero solution x, the matrix $(A-\lambda I)$ must be singular or have determinant. We define

$$\Delta(\lambda)=\det(\lambda I-A)$$

It is a monic polynomial of degree n with real coefficients and is called the characteristic polynomical of A. A polynomical is called monic if its leading coefficient is 1. If λ is a root of the characteristic polynomical, then the determinant of $(A-\lambda I)$ is 0 and (2-18) has at least one nonzero solution. Thus every root of $\Delta(\lambda)$ is an eigenvalue of A. Because $\Delta(\lambda)$ has degree n, the $n\times n$ matrix A has n eigenvalues (not necessarily all distinct).

We mention that the matrices

$$\begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & 0 & 0 & -\alpha_3 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{bmatrix} \qquad \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and their transposes

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} \qquad \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix}$$

have the following characteristic polynomial:

$$\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4$$

These matrices can easily be formed from the coefficients of $\Delta(\lambda)$ and are called companion-form matrices. The companion-form matrices will arise repeatedly later. The matrix in (2-12) is such a form.

Eigenvalues of *A* are all distinct Let λ_i, $i = 1, 2, \cdots, n$, be the eigenvalues of A and be all distinct. Let q_i be an eigenvector of A associated with λ_i; that is, $Aq_i = \lambda_i q_i$. Then the set of eigenvectors $\{q_1, \ q_2, \ \cdots, \ q_n\}$ is linearly independent and can be used as a basis. Let $\hat{A}$ be the representation of A with respect to this basis. Then the first column of $\hat{A}$ is the representation of $Aq_1 = \lambda_1 q_1$ with respect to $\{q_1, \ q_2, \ \cdots, \ q_n\}$. From

$$Aq_1 = \lambda_1 q_1 = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

we conclude that the first column of $\hat{A}$ is $[\lambda_1 \ \ 0 \ \ \cdots \ \ 0]^T$. The second column of $\hat{A}$ is the representation of $Aq_2 = \lambda_2 q_2$ with respect to $\{q_1, \ q_2, \ \cdots, \ q_n\}$, that is, $[0 \ \ \lambda_2 \ \ 0 \ \ \cdots \ \ 0]^T$. Proceeding forward, we can establish

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_N \end{bmatrix} \tag{2-19}$$

This is a diagonal matrix. Thus we conclude that every matrix with distinct eigenvalues has a diagonal matrix representation by using its eigenvectors as a basis. Different orderings of eigenvectors will yield different diagonal matrices for the same A.

If we define

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \tag{2-20}$$

then the matrix $\hat{A}$ equals

$$\hat{A} = Q^{-1}AQ \tag{2-21}$$

as derived in (2-17). Computing (2-21) by hand is not simple because of the need to compute the inverse of Q. However, if we know $\hat{A}$, then we can verify (2-20) by checking $Q\hat{A} = AQ$.

Example 2.2 Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Its characteristic polynomial is

$$\Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda & -2 \\ 0 & -1 & \lambda - 1 \end{bmatrix}$$

$$= \lambda\left[\lambda(\lambda-1)-2\right] = (\lambda-2)(\lambda+1)\lambda$$

Thus the eigenvalues of A are 2, −1, and 0. The eigenvector associated with $\lambda = 2$ is any nonzero solution of

$$(A-2I)q_1 = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} q_1 = 0$$

Thus $q_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is an eigenvector associated with $\lambda = 2$. Note that the eigenvector is not unique, $\begin{bmatrix} 0 & \alpha & \alpha \end{bmatrix}^T$ for any nonzero real α can also be chosen as an eigenvector. The eigenvector associated with $\lambda = -1$ is any nonzero solution of

$$(A-(-1)I)q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} q_2 = 0$$

which yields $q_2 = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix}^T$. Similarly, the eigenvector associated with $\lambda = 0$ can be computed as $q_3 = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$. Thus the representation of A with respect to

$\{q_1, \quad q_2, \quad q_3\}$ is

$$\hat{A}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2-22}$$

It is a diagonal matrix with eigenvalues on the diagonal. This matrix can also be obtained by computing

$$\hat{A}=Q^{-1}AQ$$

with
$$Q=\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

However, it is simpler to verify $Q\hat{A}=AQ$ or

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Eigenvalues of *A* are not all distinct An eigenvalue with multiplicity 2 or higher is called a repeated eigenvalue. In contrast, an eigenvalue with multiplicity 1 is called a simple eigenvalue. If A has only simple eigenvalues, it always has a diagonal-form representation. If A has repeated eigenvalues, then it may not have a diagonal form representation. However, it has a block-diagonal and triangular-form representation as we will discuss next.

Consider an $n \times n$ matrix A with eigenvalue λ and multiplicity n. In other words, A has only one distinct eigenvalue. To simplify the discussion, we assume $n=4$. Suppose the matrix $(A-\lambda I)$ has rank $n-1=3$ or, equivalently, nullity 1; then the equation

$$(A-\lambda I)q=0$$

has only one independent solution. Thus A has only one eigenvector associated with λ. We need $n-1=3$ more linearly independent vectors to form a basis for R^4. The three vectors q_2, q_3, q_4 will be chosen to have the properties $(A-\lambda I)^2 q_2=0$, $(A-\lambda I)^3 q_3=0$, and $(A-\lambda I)^4 q_4=0$.

A vector v is called a *generalized eigenvector* of grade n if

$$(A-\lambda I)^n v=0$$

and

$$(A-\lambda I)^{n-1} v \neq 0$$

If $n=1$, they reduce to $(A-\lambda I)v=0$ and $v\neq 0$ and v is an ordinary eigenvector. For $n=4$, we define

$$
\begin{aligned}
v_4 &= v \\
v_3 &= (A-\lambda I)v_4 = (A-\lambda I)v \\
v_2 &= (A-\lambda I)v_3 = (A-\lambda I)^2 v \\
v_1 &= (A-\lambda I)v_2 = (A-\lambda I)^3 v
\end{aligned}
$$

They are called a chain of generalized eigenvectors of length $n=4$ and have the properties $(A-\lambda I)v_1=0, (A-\lambda I)^2 v_2=0, (A-\lambda I)^3 v_3=0$, and $(A-\lambda I)^4 v_4=0$. These vectors, as generated, are automatically linearly independent and can be used as a basis. From these equations, we can readily obtain

$$
\begin{aligned}
Av_1 &= \lambda v_1 \\
Av_2 &= v_1 + \lambda v_2 \\
Av_3 &= v_2 + \lambda v_3 \\
Av_4 &= v_3 + \lambda v_4
\end{aligned}
$$

Then the representation of A with respect to the basis $\{v_1,\ v_2,\ v_3,\ v_4\}$ is

$$
J=\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \tag{2-23}
$$

We verify this for the first and last columns. The first column of J is the representation of $Av_1=\lambda v_1$ with respect to $\{v_1,\ v_2,\ v_3,\ v_4\}$, which is $[\lambda\ \ 0\ \ 0\ \ 0]^T$. The last column of J is the representation of $Av_4=v_3+\lambda v_4$ with respect to $\{v_1,\ v_2,\ v_3,\ v_4\}$, which is $[0\ \ 0\ \ 1\ \ \lambda]^T$. This verifies the representation in (2-23). The matrix J has eigenvalues on the diagonal and 1 on the superdiagonal. If we reverse the order of the basis, then the 1 in (2-23) will appear on the subdiagonal. The matrix is called a Jordan block of order $n=4$.

If $(A-\lambda I)$ has rank $n-2$ or , equivalently, nillity 2, then the equation

$$(A-\lambda I)q=0$$

has two linearly independent solution. Thus A has two linearly independent eigenvectors and we need $(n-2)$ generalized eigenvectors. In this case, there exist two chains of generalized eigenvectors $\{v_1,\ v_2,\ \cdots,\ v_k\}$ and $\{u_1,\ u_2,\ \cdots,\ u_l\}$ with $k+l=n$. If v_1 and u_2 are linearly independent, then the set of n vectors $\{v_1,\ \cdots,\ v_k,\ u_1,\ \cdots,\ u_l\}$ is linearly independent and can be used as a basis. With respect to this basis, the representation of A is a block diagonal matrix of form

$$\hat{A} = diag\{J_1, \quad J_2\}$$

where J_1 and J_2 are, respectively, Jordan blocks of order k and l.

Now we discuss a specific example. Consider a 5×5 matrix A with repeated eigenvalue λ_1 with multiplicity 4 and simple eigenvalue λ_2. Then there exists a nonsingular matrix Q such that

$$\hat{A} = Q^{-1}AQ$$

assumes one of the following forms

$$\hat{A}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \qquad \hat{A}_2 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\hat{A}_3 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \qquad \hat{A}_4 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\hat{A}_5 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \tag{2-24}$$

The first matrix occurs when the nullity of $(A - \lambda_1 I)$ is 1. If the nullity is 2, then $\hat{A}$ has two Jordan blocks associated with λ_1; it may assume the form in $\hat{A}_2$ or $\hat{A}_3$. If $(A - \lambda_1 I)$ has nullity 3, then $\hat{A}$ has three Jordan blocks associated with λ_1 as shown in $\hat{A}_4$. Certainly, the positions of the Jordan blocks can be changed by changing the order of the basis. If the nullity is 4, then $\hat{A}$ is a diagonal matrix as shown in $\hat{A}_5$. All these matrices are triangular and block diagonal with Jordan blocks on the diagonal; they are said to be in Jordan form. A diagonal matrix is a degenerated Jordan form; its Jordan blocks all have order 1.

Jordan-form matrices are triangular and block diagonal and can be used to establish many general properties of matrices. For example, because $\det(CD) = \det C \det D$ and

$\det Q \det Q^{-1} = \det I = 1$, from $A = Q\hat{A}Q^{-1}$, we have

$$\det A = \det Q \det \hat{A} \det Q^{-1} = \det \hat{A}$$

The determinant of $\hat{A}$ is the product of all diagonal entries or, equivalently, all eigenvalues of A. Thus we have

$$\det A = product \quad of \quad all \quad \text{eigenvalues} \quad \text{of} \quad A$$

which implies that A is nonsingular if and only if it has non-zero eigenvalue.

We discuss a useful property of Jordan blocks to conclude this section. Consider the Jordan block in (2-23) with order 4. Then we have

$$(J-\lambda I)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (J-\lambda I)^2=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(J-\lambda I)^3=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2-25}$$

and $(J-\lambda I)^k = 0-$ for $k \geqslant 4$. This is called nilpotent.

Example 2.3 Consider the matrix

$$A=\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

Its characteristic polynomial is

$$\Delta(\lambda)=\det(\lambda I - A)=\det\begin{bmatrix} \lambda & -1 & 1 \\ 6 & \lambda+11 & -6 \\ 6 & 11 & \lambda-5 \end{bmatrix}=(\lambda+1)(\lambda+2)(\lambda+3)$$

Thus the eigenvalues of A are -1,-2, and -3.

$$q_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$$

$$q_2 = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$$

$$q_3 = \begin{bmatrix} 1 & 6 & 9 \end{bmatrix}^T$$

We can obtain the diagonal matrix by computing

$$\hat{A} = Q^{-1}AQ$$

With

$$Q = [q_1 \quad q_2 \quad q_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 1 & 4 & 9 \end{bmatrix}$$

Then we can get

$$Q^{-1} = \frac{1}{9}\begin{bmatrix} 2 & -5 & 2 \\ 6 & 3 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

We have the diagonal form

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\hat{B} = Q^{-1}B = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\hat{C} = CQ = [1 \quad 1 \quad 1]$$

The result in this example can easily be obtained using MATLAB. Typing

```
a=[0 1 -1;-6 -11 6;-6 -11 5];
[q,d]=eig(a)
yields
q =
    0.7071   -0.2182   -0.0921
    0.0000   -0.4364   -0.5523
    0.7071   -0.8729   -0.8285
d =
   -1.0000         0            0
     0       -2.0000            0
     0             0      -3.0000
```

where d is the diagonal matrix. The matrix is different from the Q, but their corresponding columns differ only by a constant. This is due to nonuniqueness of eigenvectors and every column of q is normalized to have norm 1 in MATLAB.

Example 2.4 Try to transform the state-space representation to the Jordan block:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \quad 0 \quad 0)x$$

Solution

First, we get the eigenvalues of A:

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & -3 & \lambda \end{vmatrix} = 0$$

That is,

$$\lambda^3 - 3\lambda - 2=0$$

We get:

$$\lambda_{1,2}=-1, \quad \lambda_3=2$$

Eigenvector Q_1 corresponding to $\lambda_1=-1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}\begin{pmatrix} q_{11} \\ q_{21} \\ q_{31} \end{pmatrix} = -\begin{pmatrix} q_{11} \\ q_{21} \\ q_{31} \end{pmatrix}$$

Then,

$$Q_1=\begin{pmatrix} q_{11} \\ q_{21} \\ q_{31} \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvector Q_2 corresponding to $\lambda_2=-1$

$$\lambda_1 Q_2 - AQ_2 = -Q_1$$

$$-\begin{pmatrix} q_{21} \\ q_{22} \\ q_{23} \end{pmatrix}-\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}\begin{pmatrix} q_{21} \\ q_{22} \\ q_{23} \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$Q_2=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Last, eigenvector Q_3 corresponding to $\lambda_3=2$

$$\lambda_3 Q_3 = AQ_3$$

$$Q_3=\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$T=\begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 4 \end{pmatrix}$$

It can be calculated,

$$T^{-1} = \frac{1}{9}\begin{pmatrix} 2 & -5 & 2 \\ 6 & 3 & -3 \\ 1 & 2 & 1 \end{pmatrix}$$

Then, we can calculate the matrix we want:

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$T^{-1}B = \begin{pmatrix} \frac{2}{9} \\ -\frac{1}{3} \\ \frac{1}{9} \end{pmatrix}$$

$$CT = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

Using MATLAB, typing

```
A=[0 1 0;0 0 1;2 3 0];
[v,J] = jordan(A);
yields
v =
    0.1111    0.6667    0.8889
    0.2222   -0.6667   -0.2222
    0.4444    0.6667   -0.4444
J =
     2     0     0
     0    -1     1
     0     0    -1
```

JORDAN(A) computes the Jordan Canonical/Normal Form of the matrix A. The columns of V are the generalized eigenvectors. J is the Jordan canonical form.

Exercise

2.1 Judge whether the following vectors are linearly dependent or not.

(1) $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ (2) $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

2.2 Find the characteristic polynomials of the following matrices.

(1) $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$ (2) $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$

(3) $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$ (4) $\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$

2.3 Find eigenvectors of the following matrices.

(1) $A=\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$ (2) $A=\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$

(3) $A=\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{pmatrix}$ (4) $A=\begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & -1 \\ 4 & 4 & 5 \end{pmatrix}$

2.4 Find Jordan-form representations of the following matrices:

$$A_1=\begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A_2=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$A_3=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad A_4=\begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

2.5 Find Jordan-form representations of the following state-space equations:

(1) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$

$$y=(1,\ \ 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(2) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 7 \\ 5 & 3 \end{pmatrix} u$

$$y=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Chapter 3
Solution of State Space Model

3.1 Introduction

It is shown in Chapter 2 that linear systems can be described by state-space equations. This chapter will discuss the solution of the state space equation for linear time invariant systems. Different methods to solve the state transition matrix for both continuous systems and discrete systems are discussed in detail.

3.2 Solution of LTI State Equations

Consider the linear time-invariant(LTI) state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3-1}$$

$$y(t) = Cx(t) + Du(t) \tag{3-2}$$

where A, B are, respectively $n \times n, n \times p$ dimensional constant matrices.

The problem is to find the solution excited by the initial state $x(t)\big|_{t=0} = x(0)$ and the input $u(t)$. The solution hinges on the exponential function of A. In particular, the following property

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A \tag{3-3}$$

is necessary to develop the solution.

Rewrite the equation as:

$$\dot{x}(t) - Ax(t) = Bu(t) \tag{3-4}$$

Pre-multiplying e^{-At} to both sides of (3-3) yields

$$e^{-At}[\dot{x}(t) - Ax(t)] = e^{-At}Bu(t) \tag{3-5}$$

which implies

$$\frac{d}{dt}(e^{-At}x(t)) = e^{-At}Bu(t)$$

Its integration from 0 to t yields

$$e^{-A\tau}x(\tau)\Big|_{\tau=0}^{t} = \int_{o}^{t} e^{-A\tau}Bu(\tau)d\tau$$

Thus we have

$$e^{-At}x(t) - e^{0}x(0) = \int_{o}^{t} e^{-A\tau}Bu(\tau)d\tau \tag{3-6}$$

Because the inverse of e^{-At} is e^{At} and $e^{0} = I$, equation (3-6) implies

$$x(t) = e^{At}x(0) + \int_{o}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{3-7}$$

or, equivalently

$$x(t) = \Phi(t)x(0) + \int_{o}^{t} \Phi(t-\tau)Bu(\tau)d\tau \tag{3-8}$$

where $\Phi(t) = e^{At}$ is called the state transition matrix. Equation (3-7), or (3-8), is the solution of (3-1).

To verify that (3-7) is the solution of (3-1), it is necessary to show that (3-7) satisfies (3-1) and the initial condition $x(t) = x(0)$ at $t = 0$. Indeed, at $t = 0$, (3-7) reduces to

$$x(0) = e^{A\cdot 0}x(0) = e^{0}x(0) = Ix(0) = x(0)$$

Thus (3-6) satisfies the initial condition. The equation

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau)d\tau = \int_{t_0}^{t} (\frac{\partial}{\partial t} f(t,\tau))d\tau + f(t,\tau)\big|_{\tau=t} \tag{3-9}$$

is needed to show that (3-7) satisfies (3-1). Differentiating (3-7) and using (3-9) obtain

$$\dot{x}(t) = \frac{d}{dt}\left[e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \right]$$

$$= Ae^{At}x(0) + \int_{0}^{t} Ae^{A(t-\tau)}Bu(\tau)d\tau + e^{A(t-\tau)}Bu(\tau)\big|_{\tau=t}$$

$$= A(e^{At}x(0) + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau) + e^{A\cdot 0}Bu(t)$$

substituting (3-7) into the above equation results in,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Thus (3-7) meets (3-1) and the initial condition $x(0)$ and is the solution of (3-1).

Substituting (3-7) into (3-2) yields the solution of (3-2) as

$$y(t) = Ce^{At}x(0) + C\int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{3-10}$$

This solution and (3-7) are computed directly in the time domain. It is also convenient to compute the solutions by using the Laplace transform. Applying the Laplace transform to (3-1) yields

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

Pre-multiplying $(sI - A)^{-1}$ to both sides of the above equation yields

$$x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \tag{3-11}$$

Notice that

$$(sI - A)^{-1} = \ell[\Phi(t)]$$
$$U(s) = \ell[u(t)]$$

From the property of Laplace transform function, the second term of the right side of (3-11) can be expressed as:

$$(sI - A)^{-1}BU(s) = \ell\left[\int_0^t \Phi(t-\tau)Bu(\tau)\right]d\tau \tag{3-12}$$

Substituting (3-12) into (3-11) and applying the inverse Laplace transform yields

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \tag{3-13}$$

Equation. (3-13) is just the time-domain solution of (3-1).

Example 3.1

Find the solution of the following system excited by the unit step function

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution

$$\Phi(t) = e^{At} = (sI - A)^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Substituting $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $u(t) = 1(t)$ into equation (3-8) yields

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} (2e^{-t} - e^{-2t})x_1(0) + (e^{-t} - e^{-2t})x_2(0) \\ (-2e^{-t} + 2e^{-2t})x_1(0) + (-e^{-t} + 2e^{-2t})x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + [2x_1(0) + x_2(0) - 1]e^{-t} - \left[x_1(0) + x_2(0) - \frac{1}{2}\right]e^{-2t} \\ -[2x_1(0) + x_2(0) - 1]e^{-t} - [2x_1(0) + 2x_2(0) - 1]e^{-2t} \end{bmatrix}$$

If the initial condition is zero, i.e., $x(0) = 0$, then the response of the system depends only on the excitation of the control action

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

In the following, the solution of the state space equation (3-8) is presented under three special control signals.

(1) The impulse response

When $u(t) = K\delta(t), x(0_-) = x_0$,

$$x(t) = e^{At}x_0 + e^{At}BK$$

where K is a constant vector having the same dimension with $u(t)$.

(2) The step response

When $u(t) = K \times 1(t), x(0_-) = x_0$,

$$x(t) = e^{At}x_0 + A^{-1}(e^{At} - 1)BK$$

(3) The scope response

When $u(t) = Kt \times 1(t), x(0_-) = x_0$,

$$x(t) = e^{At}x_0 + \left[A^{-2}(e^{At} - 1) - A^{-1t}\right]BK$$

3.3 State Transfer Matrix

3.3.1 Properties

Consider the linear time-invariant(LTI) state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3-14}$$

where, $x(t) \in R^n$, $u(t) \in R^r$, $A \in R^{n\times n}$, $B \in R^{n\times r}$.

The solution of the system is

$$x(t) = e^{At}x_0 \quad \text{or} \quad x(t) = e^{A(t-t_0)}x_0$$

which reflects a vector transition relation from the initial state vector x_0 to the state $x(t)$ of any $t > 0$ or $t > t_0$ and e^{At} is called transfer matrix. It is not a constant matrix and it is a $n \times n$ time-varying function matrix because the elements of the matrix are general functions of t. This means that it makes the state vector change constantly in the state space, so $\Phi(t) = e^{At}$ is also called state transition matrix. $\Phi(t) = e^{At}$ is the transition matrix from $x(0)$ to $x(t)$, while $\Phi(t - t_0) = e^{A(t-t_0)}$ is the transition matrix from $x(t_0)$ to $x(t)$. Therefore, the solution of $\dot{x}(t) = Ax(t) + Bu(t)$ can also be expressed as

$$x(t)=\Phi(t)x(0)$$

or
$$x(t)=\Phi(t-t_0)x(t_0)$$

Its geometric meaning, taking two-dimensional state vector for example, can be represented as Fig. 3.1.

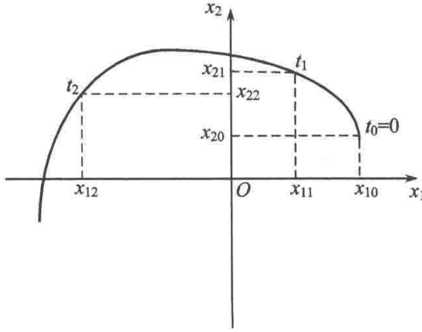


Fig. 3.1 State transition trajectory

From the Fig. 3.1, we can know $x(0)=\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ when $t=0$. If we consider this as initial condition, and $\Phi(t_1)$ is known, when $t=t_1$, the state will be

$$x(t_1)=\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}=\Phi(t_1)x(0) \tag{3-15}$$

If $\Phi(t_2)$ is known, when $t=t_2$, the state will be

$$x(t_2)=\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}=\Phi(t_2)x(0) \tag{3-16}$$

That is to say, the state $x(0)$ will transfer to the state $x(t_1)$ or $x(t_2)$ according to $\Phi(t_1)$ or $\Phi(t_2)$.

If we take $t=t_1$ as initial time, the state $x(t_1)$ is the initial state and the state transited from t_1 to t_2 will be

$$x(t_2)=\Phi(t_2-t_1)x(t_1) \tag{3-17}$$

Substituting $x(t_1)$ of equation (3-15) into the above equation can result in:

$$x(t_2)=\Phi(t_2-t_1)\Phi(t_1)x(0) \tag{3-18}$$

Equation (3-18) shows the transformation of the state x from $x(0)$ to $x(t_1)$, and then to $x(t_2)$.

Comparing equation (3-16) and (3-18), it is clear that:

$$\Phi(t_2-t_1)\Phi(t_1)=\Phi(t_2)$$

or $$e^{A(t_2-t_1)}e^{At_1}=e^{At_2} \tag{3-19}$$

Such relation is called the combination property.

From the above, for any given initial state vector, $x(t_0)$ can be transited to $x(t)$ at any t using state transition matrix. In other words, matrix differential equations can be solved in arbitrary time period. This is another advantage of state space representation on a dynamic system.

Property 1

$$\left.\begin{aligned} \Phi(t)\Phi(\tau)=\Phi(t+\tau) \\ or \quad e^{At}e^{A\tau}=e^{A(t+t)} \end{aligned}\right\} \tag{3-20}$$

This is combination property, which means a combination of transition from $-\tau$ to 0 and transition from 0 to t. That is to say,

$$\Phi(t-0)\Phi[0-(-\tau)]=\Phi[t-(-\tau)]=\Phi(t+\tau)$$

Property 2

$$\left.\begin{aligned} \Phi(t-t)=I \\ e^{A(t-t)}=I \end{aligned}\right\} \tag{3-21}$$

This property means that when the state vector transit from time instant t to t, the state vector is invariable.

Property 3

$$\left.\begin{aligned} [\Phi(t)]^{-1}=\Phi(-t) \\ or \quad \left[e^{At}\right]^{-1}=e^{-At} \end{aligned}\right\} \tag{3-22}$$

This property shows that the inverse of the transition matrix means the reversion of time. If $x(t)$ is known, we can have $x(t_0)$ at time t while $t_0<t$.

Property 4

For the transition matrix,

$$\left.\begin{aligned} \dot{\Phi}(t)=A\Phi(t)=\Phi(t)A \\ or \quad \frac{d}{dt}e^{At}=Ae^{At}=e^{At}\cdot A \end{aligned}\right\} \tag{3-23}$$

This property shows $\Phi(t)$ or e^{At} can change with matrix A.

Property 5

For the $n\times n$ square matrices A and B, if and only if $AB=BA$, there exists $e^{At}e^{Bt}=e^{(A+B)t}$; if $AB\neq BA$, then $e^{At}e^{Bt}\neq e^{(A+B)t}$.

This property shows unless A and B are matrix exchangeable, the product of their respective matrix exponential functions is not equivalent to matrix exponential function of the sum of A and B.

Here are some special matrix exponential functions.

(i) If A is diagonal matrix

$$A = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

then,

$$e^{At} = \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \tag{3-24}$$

(ii) If A can be diagonalized through nonsingular transformation,

$$T^{-1}AT = \Lambda$$

then,

$$e^{At} = \Phi(t) = T \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} T^{-1} \tag{3-25}$$

(iii) If A is Jordan matrix

$$A = J = \begin{bmatrix} \lambda & 1 & & & & 0 \\ & \lambda & 1 & & & \\ & & \lambda & 1 & & \\ & & & \lambda & 1 & \\ & & & & \lambda & 1 \\ 0 & & & & & \lambda \end{bmatrix} \tag{3-26}$$

then,

$$e^{Jt} = \Phi(t) = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(n-1)!}t^{n-1} \\ 0 & 1 & t & \cdots & \frac{1}{(n-2)!}t^{n-2} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3-27}$$

(iv) If $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

then, $$e^{At} = \Phi(t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \tag{3-28}$$

3.3.2 Calculating the state transition matrix

For the calculating the matrix e^{At}, many methods have been proposed. Here are the four most popular ones.

Method 1. Expansion of e^{At}

This method takes place entirely in the time domain and is based on the expansion of e^{At} in a power series. Namely, it is base on the definition of e^{At} or $\Phi(t)$:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \tag{3-29}$$

Example 3.2

Compute the matrix e^{At}, *where* $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Solution

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} t + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^2 \frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^3 \frac{t^3}{3!} + \cdots \\ &= \begin{bmatrix} 1 - t^2 + t^3 + \cdots & t - \frac{3}{2}t^2 - \frac{7}{6}t^3 + \cdots \\ -2t + 3t^2 - \frac{7}{3}t^3 + \cdots & 1 - 3t + \frac{7}{2}t^2 - \frac{5}{2}t^3 + \cdots \end{bmatrix} \end{aligned}$$

Method 2. Diagonal form

This method takes place entirely in the time domain and is based on the diagonalization of the matrix A. Indeed, if the eigenvalues of matrix A are distinct, then A can be transformed to a diagonal matrix Λ via the transformation matrix T as follows: $\Lambda = T^{-1}AT$. Matrix $\Phi(t)$, under the transformation T, becomes

$$\Phi(t) = e^{At} = Te^{\Lambda t}T^{-1}$$

Since

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

it follows that

$$\Phi(t)=e^{At}=T\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

Example 3.3

$A=\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ is known, and compute the matrix e^{At}.

Solution

$$|\lambda I-A|=\begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix}=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

so, $\lambda_1=-1,\quad \lambda_2=-2$

we can have the corresponding transfer matrix according to equation (3-25).

$$T=\begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad T^{-1}=\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{bmatrix}$$

Therefore,

$$e^{At}=\begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

If the matrix A has repeated eigenvalues, then A can be transformed to a jordan matrix Λ via the transformation matrix T.

$$J=T^{-1}AT$$

$$e^{At}=Te^{Jt}T^{-1}$$

Example 3.4

Compute the matrix e^{At}, where $A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

Solution

$$|\lambda I-A|=\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 5 & \lambda-4 \end{vmatrix}=(\lambda-1)^2(\lambda-2)=0$$

so, $\lambda_1=\lambda_2=1,\qquad \lambda_3=2$

According to (3-26)

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Since

$$T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}; \qquad T^{-1} = \begin{bmatrix} -2 & 5 & -2 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Then,

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 5 & -2 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & te^t - e^t & e^{2t} \\ e^t & te^t & 2e^{2t} \\ e^t & te^t + e^t & 4e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 5 & -2 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2te^t + e^{2t} & 3te^t + 2e^t - e^{2t} & -te^t - e^t + e^{2t} \\ 2\left(e^{2t} - te^t - e^t\right) & 3te^t + 5e^t - 4e^{2t} & -te^t - 2e^t + 2e^{2t} \\ -2te^t - 4e^t + 4e^{2t} & 3te^t + 8e^t - 8e^{2t} & -te^t - 3e^t + 4e^{2t} \end{bmatrix} \end{aligned}$$

Method 3. The inverse Laplace transformation

The inverse Laplace transformation method can be expressed as:

$$e^{At} = \Phi(t) = \ell^{-1}\left\{(sI - A)^{-1}\right\} \tag{3-30}$$

Proof Consider the differential equation

$$\dot{x}(t) = Ax(t)$$

with the initial state $x(0) = x_0$.

Apply the Laplace transform on both sides of the equation,

$$sX(s) - x(0) = AX(s)$$

in other words,

$$(sI - A)X(s) = x(0) = x_0$$

therefore,

$$X(s) = (sI - A)^{-1} x_0$$

Apply the inverse Laplace transform on both sides to get the solution of the differential equation:

$$x(t) = \ell^{-1}\left\{(sI - A)^{-1}\right\} x_0$$

Comparing the above equation with equation (3-15) can result in

$$e^{At} = \Phi(t) = \ell^{-1}\left\{(sI - A)^{-1}\right\}$$

Example 3.5

Compute the matrix e^{At}, where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Solution

$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{|sI - A|} adj(sI - A) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

Therefore,

$$e^{At} = \ell^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Method 4. Cayley - Hamilton theorem

(1) A square matrix A satisfies the characteristic equation of itself according to Cayley- Hamilton theorem ,

$$f(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

thus,

$$A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_1 A - a_0 I$$

which is the linear combination of $A^{n-1}, A^{n-2}, \cdots, A, I$.

In the same way,

$$A^{n+1} = A \cdot A^n = -a_{n-1}A^n - (a_{n-2}A^{n-1} + a_{n-3}A^{n-2} + \cdots + a_1 A^2 + a_0 A)$$

$$=-a_{n-1}\left(-a_{n-1}A^{n-1}-a_{n-2}A^{n-2}-\cdots-a_1A-a_0I\right)$$
$$-\left(a_{n-2}A^{n-1}+a_{n-3}A^{n-2}+\cdots+a_1A^2+a_0A\right)$$
$$=\left(a_{n-1}^2-a_{n-2}\right)A^{n-1}+\left(a_{n-1}a_{n-2}-a_{n-3}\right)A^{n-2}+\cdots+\left(a_{n-1}a_1-a_0\right)A+a_{n-1}a_0I$$

That is to say, $A^n, A^{n+1}\cdots$ can all be expressed with $A^{n-1}, A^{n-2},\cdots, A, I$.

(2) In the definition equation (3-29) of e^{At}, we can eliminate the terms of A with power equal to and above n applying the method in (1). In other words,

$$e^{At}=I+At+\frac{1}{2!}A^2t^2+\cdots+\frac{1}{(n-1)!}A^{n-1}t^{n-1}+\frac{1}{n!}A^nt^n+\frac{1}{(n+1)!}A^{n+1}t^{n+1}+\cdots$$
$$=a_{n-1}(t)A^{n-1}+a_{n-2}(t)A^{n-2}+\cdots+a_1(t)A+a_0(t)I \tag{3-31}$$

Example 3.6

Compute $a_i(t)$ in the expression of e^{At}, where $A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$.

Solution

The characteristic equation of A:

$$|\lambda I-A|=\begin{vmatrix}\lambda & -1\\ 2 & \lambda+3\end{vmatrix}=\lambda^2+3\lambda+2=0$$

According to Cayley - Hamilton theorem,

$$A^2+3A+2I=0$$

thus, $A^2=-3A-2I$.

While

$$A^3=A\cdot A^2=A(-3A-2I)=-3A^2-2A$$
$$=-3(-3A-2I)-2A=7A-6I$$
$$A^4=A\cdot A^2=7A^2+6A$$
$$=7(-3A-2I)+6A=-15A-14I$$

……

Substituting equations above into the following equation, we can eliminate the terms of A with power equal to and above 2.

$$e^{At}=I+At+\frac{1}{2!}A^2t^2+\frac{1}{3!}A^3t^3+\frac{1}{4!}A^4t^4+\cdots$$
$$=\left(t-\frac{3}{2!}t^2+\frac{7}{3!}t^3-\frac{15}{4!}t^4+\cdots\right)A+\left(1-t^2+t^3-\frac{14}{4!}t^4+\cdots\right)I$$
$$=a_1(t)A+a_0(t)I$$

therefore,

$$a_1(t)=t-\frac{3}{2!}t^2+\frac{7}{3!}t^3-\frac{15}{4!}t^4+\cdots$$

$$a_0(t)=1-t^2+t^3-\frac{14}{4!}t^4+\cdots$$

(3) When the eigenvalues of A are all distinct, we have

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix}=\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ & & \cdots & \cdots & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \tag{3-32}$$

Proof Matrix A satisfies the characteristic equation of itself and therefore eigenvalues λ and A are exchangeable. Thus, λ satisfies the equation (3-31),

$$\left.\begin{aligned} a_0(t)+a_1(t)\lambda_1+\cdots+a_{n-1}(t)\lambda_1^{n-1}&=e^{\lambda_1 t} \\ a_0(t)+a_1(t)\lambda_2+\cdots+a_{n-1}(t)\lambda_2^{n-1}&=e^{\lambda_2 t} \\ &\vdots \\ a_0(t)+a_1(t)\lambda_n+\cdots+a_{n-1}(t)\lambda_n^{n-1}&=e^{\lambda_n t} \end{aligned}\right\}$$

Solve the above equation for $\begin{bmatrix} a_0(t) & a_1(t) & \cdots & a_{n-1}(t) \end{bmatrix}^T$, and we can obtain (3-32).

When eigenvalues of A are all λ_1, we have

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-2}(t) \\ a_{n-1}(t) \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & (n-1)\lambda_1 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 1 & \cdots & & \frac{(n-1)(n-2)}{2!}\lambda_1^{n-3} \\ 0 & 1 & 2\lambda & \cdots & (n-1)\lambda_1^{n-2} & (n-1)\lambda_1^{n-2} \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} & \lambda_1^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{(n-1)!}t^{n-1}e^{\lambda_1 t} \\ \frac{1}{(n-2)!}t^{n-2}e^{\lambda_1 t} \\ \vdots \\ \frac{1}{2!}t^2 e^{\lambda_1 t} \\ te^{\lambda_1 t} \\ e^{\lambda_1 t} \end{bmatrix} \tag{3-33}$$

Proof

Since

$$a_0(t)+a_1(t)\lambda_1+a_2(t)\lambda_1^2\cdots+a_{n-1}(t)\lambda_1^{n-1}=e^{\lambda_1 t}$$

Differentiating both sides of the equation, we have

$$a_1(t)+2a_2(t)\lambda_1\cdots+(n-1)a_{n-1}(t)\lambda_1^{n-2}=te^{\lambda_1 t}$$

Differentiating both sides of the equation again, we obtain

$$2a_2(t)+6a_3(t)+\cdots+(n-1)(n-2)a_{n-1}(t)\lambda_1^{n-3}=t^2e^{\lambda_1 t}$$

Repeat the former step and finally we have

$$(n-1)!a_{n-1}(t)=t^{n-1}e^{\lambda_1 t}$$

The equation (3-33) can be reached from the above n equations, by solving $a_i(t)$.

Example 3.7

Compute e^{At}, where $A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$.

Solution

We know the eigenvalues of the matrix A is $\lambda_1=-1, \lambda_2=-2$. According to equation (3-33), we have

$$\begin{bmatrix}a_0\\ a_1\end{bmatrix}=\begin{bmatrix}1 & \lambda_1\\ 1 & \lambda_2\end{bmatrix}^{-1}\begin{bmatrix}e^{\lambda_1 t}\\ e^{\lambda_2 t}\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & -2\end{bmatrix}^{-1}\begin{bmatrix}e^{-t}\\ e^{-2t}\end{bmatrix}$$

$$=\begin{bmatrix}2 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-t}\\ e^{-2t}\end{bmatrix}=\begin{bmatrix}2e^{-t}-e^{-2t}\\ e^{-t}-e^{-2t}\end{bmatrix}$$

Therefore,

$$e^{At}=a_0(t)I+a_1(t)A$$

$$=\left(2e^{-t}-e^{-2t}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\left(e^{-t}-e^{-2t}\right)\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

Example 3.8

Compute the matrix e^{At}, where $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & -5 & 4\end{bmatrix}$.

Solution

The eigenvalues of the matrix A are $\lambda_1=\lambda_2=1, \lambda_3=2$. The part of $\lambda_1=\lambda_2=1$ can be calculated based on equation (3-33), while the part of $\lambda_3=2$ can be calculated based on equation (3-32).

$$\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 2\lambda_1\\ 1 & \lambda_1 & \lambda_1^2\\ 1 & \lambda_3 & \lambda_3^2\end{bmatrix}^{-1}\begin{bmatrix}te^{\lambda_1 t}\\ e^{\lambda_1 t}\\ e^{\lambda_3 t}\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\\ 1 & 2 & 4\end{bmatrix}^{-1}\begin{bmatrix}te^{t}\\ e^{t}\\ e^{2t}\end{bmatrix}=\begin{bmatrix}-2 & 0 & 1\\ 3 & 2 & -2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}te^{t}\\ e^{t}\\ e^{2t}\end{bmatrix}$$

Therefore, $$e^{At}=\left(-2te^{t}+e^{2t}\right)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\left(3te^{t}+2e^{t}-2e^{2t}\right)\begin{bmatrix}0&1&0\\0&0&1\\2&-5&4\end{bmatrix}$$

$$+\left(-te^{t}-e^{t}+e^{2t}\right)\begin{bmatrix}0&0&1\\2&-5&4\\8&-18&11\end{bmatrix}$$

$$=\begin{bmatrix}-2te^{t}+e^{2t} & 3te^{t}+2e^{t}-2e^{2t} & -te^{t}-e^{t}+e^{2t}\\ 2\left(e^{2t}-te^{t}-e^{t}\right) & 3te^{t}+5e^{t}-4e^{2t} & -te^{t}-2e^{t}+2e^{2t}\\ -2te^{t}-4e^{t}+4e^{2t} & 3te^{t}+8e^{t}-8e^{2t} & -te^{t}-3e^{t}+4e^{2t}\end{bmatrix}$$

Example 3.9 Consider the state equation

$$\dot{x}=\begin{bmatrix}-2&0&0\\1&0&1\\0&-2&-2\end{bmatrix}x+\begin{bmatrix}1\\0\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}1&-1&0\end{bmatrix}x$$

Suppose the input is a step function of various magnitudes. First we use MATLAB to find its unit-step response. We type

```
a=[-2 0 0;1 0 1;0 -2 -2];
b=[1;0;1];
c=[0 -2 -2];
d=0;
[y,x,t]=step(a,b,c,d);
plot(t,y,t,x)
```

The system response is shown in Fig.3.2.

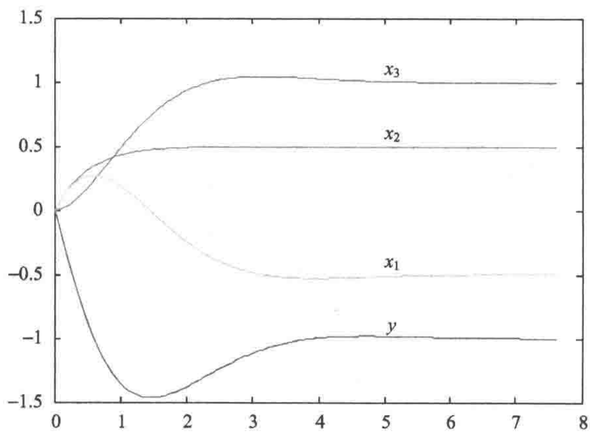


Fig.3.2 State variables of system step response

3.4 Discretization

Consider the continuous-time state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3-34}$$

If the set of equations is to be computed on a digital computer, it must be discretized. Because

$$\dot{x} = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T}$$

When the sample period T is quite small, usually, about 1/10 of the minimum time constant of the system, (3-34) can be approximated as

$$x[(k+1)T] = (TA+1)x(kT) + TBu(kT) \tag{3-35}$$

Proof

According to the definition of derivative, we have

$$\dot{x}(t_0) = \lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} \tag{3-36}$$

If we compute $x(t)$ and $y(t)$ from $t_0 = kT$ to $t = (k+1)T$ for $k = 0, 1, \cdots$, then (3-36) becomes

$$\dot{x}(kT) = \lim_{T \to 0} \frac{x[(k+1)T] - x(kT)}{T} \approx \frac{x[(k+1)T] - x(kT)}{T} \tag{3-37}$$

Substituting (3-37) into (3-34) yields

$$\frac{x[(k+1)T] - x(kT)}{T} = Ax(kT) + Bu(kT)$$

which can be rearranged as (3-35). Proof done.

This is a discrete-time state-space equation and can easily be computed on a digital computer. This discretization is the easiest to carry out but yields the least accurate results. The following is a different discretization way.

If an input $u(t)$ is generated by a digital computer followed by a digital-to-analog converter, then $u(t)$ will be piecewise constant. This situation often arises in computer control of control systems. Let

$$u(t) = u(kT) = constant \quad \text{for} \quad kT \leqslant t < (k+1)T$$

for $k = 0, 1, 2, \cdots$. This input changes value only at discrete-time instants. For this input, the solution of the state equation in (3-34) still equals (3-7). Computing (3-7) at $t = kT$ and $t = (k+1)T$ yields

$$x[(k+1)T] = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}Bd\tau u(kT) \tag{3-38}$$

Let $t=(k+1)T-\tau$, then $d\tau=-dt$. So the lower integral $\tau=kT$ becomes $t=T$; and the higher integral $\tau=(k+)T$ becomes $t=0$. Thus equation (3-38) can be simplified as

$$x[(k+1)T] = e^{AT}x(kT) + \int_0^T e^{At}dtBu(kT)$$

which equals

$$x(k+1) = e^{AT}x(k) + \int_0^T e^{At}dtBu(k) \tag{3-39}$$

This is a discrete-time state-space equation. Note that there is no approximation involved in this derivation. It is the exact solution of (3-34) at $t=kT$ if the input is piecewise constant.

Rewrite (3-39) as

$$x(k+1) = G(T)x(k) + H(T)u(k) \tag{3-40}$$

where

$$G(T) = e^{AT}, \quad H(T) = \int_0^T e^{At}dt \cdot B \tag{3-41}$$

The MATLAB function [ad,bd]=c2d(a,b,T) transforms the continuous state equation in (3-34) into the discrete-time state equation in (3-41).

Example 3.10 Try to discretize the following state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Solution

(1) equation (3-41) yields the exact result

$$e^{At} = \ell^{-1}\left[(sI-A)^{-1}\right] = \ell^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1}\right\} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix}$$

thus we have

$$G(T) = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix}$$

$$H = \int_0^T e^{At}dt \cdot B = \int_0^T \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} dt \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} T & \frac{1}{2}(T+\frac{1}{2}e^{-2T}-\frac{1}{2}) \\ 0 & -\frac{1}{2}e^{-2T}+\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(T+\frac{e^{-2T}-1}{2}) \\ \frac{1}{2}(1-e^{-2T}) \end{bmatrix}$$

According to equation.(3-40), the discretized state equation is

$$x(k+1)=\begin{bmatrix}1 & \frac{1}{2}(1-e^{-2T})\\ 0 & e^{-2T}\end{bmatrix}x(k)+\begin{bmatrix}\frac{1}{2}(T+\frac{e^{-2T}-1}{2})\\ \frac{1}{2}(1-e^{-2T})\end{bmatrix}u(k)$$

(2) Equation (3-35) yields the approximate result

$$TA+I=\begin{bmatrix}0 & T\\ 0 & -2T\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & T\\ 0 & 1-2T\end{bmatrix}$$

$$H=TB=\begin{bmatrix}0\\ T\end{bmatrix}$$

According to equation(3-45), the discretized state equation is

$$x\left[(k+1)T\right]=\begin{bmatrix}1 & T\\ 0 & 1-2T\end{bmatrix}x(kT)+\begin{bmatrix}0\\ T\end{bmatrix}u(kT)$$

which can be rewritten as

$$x(k)=\begin{bmatrix}1 & T\\ 0 & 1-2T\end{bmatrix}x(k)+\begin{bmatrix}0\\ T\end{bmatrix}u(k)$$

Then we use MATLAB to finish this task. We type

```
a=[0 1;0 -2];
b=[0;1];
T=0.1;
[ad,bd]=c2d(a,b,T)
Yield
ad =
    1.0000    0.0906
    0         0.8187
bd =
    0.0047
    0.0906
```

We can get

$$x(k)=\begin{bmatrix}1 & 0.09\\ 0 & 0.8\end{bmatrix}x(k)+\begin{bmatrix}0.0047\\ 0.09\end{bmatrix}u(k)$$

3.5 Solution of Discrete-Time Equation

Method 1. Recursive Method

Consider the discrete-time state-space equation

$$x(k+1) = G(T)x(k) + H(T)u(k)$$
$$x(k)\big|_{k=0} = x(0) \tag{3-42}$$

The solution of the first-order matrix differential equation is

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j) \tag{3-43a}$$

or

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^j Hu(k-j-1) \tag{3-43b}$$

which equals

$$x(k) = G^k x(0) + G^{k-1}Hu(0) + G^{k-2}Hu(1) + \cdots + GHu(k-2) + Hu(k-1) \tag{3-43c}$$

Proof

Solve the matrix differential equation (3-42) by iterative method

For $i=0$, $x(1) = Gx(0) + Hu(0)$

For $i=1$, $x(2) = Gx(1) + Hu(1) = G^2x(0) + GHu(1) + Hu(1)$

For $i=2$, $x(3) = Gx(2) + Hu(2) = G^3x(0) + G^2Hu(0) + GHu(1) + Hu(2)$

$\vdots$

For $i=k-1$, $x(k) = Gx(k-1) + Hu(k-1) = G^k x(0) + G^{k-1}Hu(0) + \cdots + GHu(k-2) + Hu(k-1)$

The last general formula is just equation (3-43c).

Equation (3-43c) can be expressed in matrix form as

$$\begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(k) \end{bmatrix} = \begin{bmatrix} G \\ G^2 \\ G^3 \\ \vdots \\ G^k \end{bmatrix} x(0) \begin{bmatrix} H & 0 & 0 & \cdots & 0 \\ GH & H & 0 & \cdots & 0 \\ G^2H & GH & H & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ G^{k-1}H & G^{k-2}H & G^{k-3}H & \cdots & H \end{bmatrix} \tag{3-43d}$$

Solution (3-43) is derived from the initial time instant $k=0$, if we start from the time $k=h$, the corresponding initial state is $x(h)$, then the solution becomes

$$x(k) = G^{k-h} x(0) + \sum_{j=h}^{k-1} G^{k-j-1} Hu(j) \tag{3-44a}$$

or

$$x(k) = G^{k-h} x(0) + \sum_{j=h}^{k-1} G^j Hu(k-j-1) \tag{3-44b}$$

Obviously, the solution of discrete-time state-space equation is similar to that of the continuous-time state-space equation. It consists of two parts of responses, the response excited by the initial state and the response excited by the input signal. Differently, the solution of the discrete-time state-space equation is a discrete track in state space. Besides, in the response excited by the input, the state $x(k)$ is only related with the sample values of input before time instant k.

Similarly, we define

$$\Phi(k) = G^k \quad \text{or} \quad \Phi(k-h) = G^{k-h} \tag{3-45}$$

as the state transition matrix of discrete-time system. Obviously,

$$\Phi(k+1) = G\Phi(k); \quad \Phi(0) = I \tag{3-46}$$

and the following properties:

$$\Phi(k-h) = \Phi(k-h_1)\Phi(h_1-h) \quad \text{for} \quad k > h_1 \geqslant h \tag{3-47}$$

$$\Phi^{-1}(k) = \Phi(-k) \tag{3-48}$$

Using the state transition matrix $\Phi(k)$, the solution (3-43) can be expressed as

$$x(k) = \Phi(k)x(0) + \sum_{j=0}^{k-1} \Phi(k-j-1)Hu(j) \tag{3-49a}$$

or

$$x(k) = \Phi(k)x(0) + \sum_{j=0}^{k-1} \Phi(k-j-1)Hu(j) \tag{3-49b}$$

thus equation (3-44) can be written as

$$x(k) = \Phi(k-h)x(h) + \sum_{j=h}^{k-1} \Phi(k-j-1)Hu(j) \tag{3-50a}$$

$$x(k) = \Phi(k-h)x(0) + \sum_{j=h}^{k-1} \Phi(j)Hu(k-j-1) \tag{3-50b}$$

Example 3.11 The state equation of a discrete-time system is

$$x(k+1) = Gx(k) + Hu(k)$$

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with initial state $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and control action $u(k) = 1$. Try to solve $\Phi(k), x(k)$.

Solution

As defined

$$\Phi(k) = G^k = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}^k$$

For simplicity, we transform the original equation into Jordan Canonical Form, i.e., transform G into diagonal form.

Let $x(k) = T\bar{x}(k)$, thus the original equation becomes

$$\tilde{x}(k+1) = T^{-1}GT\tilde{x}(k) + T^{-1}Hu(k)$$

Again let $$T^{-1}GT = \Lambda;\ \tilde{\Phi}(k) = (T^{-1}GT)^k = \Lambda^k$$

so

$$\tilde{x}(k) = \tilde{\Phi}(k)\tilde{x}(0) + \sum_{j=0}^{k-1} \tilde{\Phi}(j)T^{-1}Hu(k-j-1) \tag{3-51}$$

$$|\lambda I - G| = \begin{vmatrix} \lambda & -1 \\ 0.16 & \lambda+1 \end{vmatrix} = (\lambda+0.2)(\lambda+0.8) = 0$$

$$\lambda_1 = -0.2;\quad \lambda_2 = -0.8$$

therefore, $\Lambda = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix};\quad \tilde{\Phi}(k) = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix}^k = \begin{bmatrix} (-0.2)^k & 0 \\ 0 & (-0.8)^k \end{bmatrix}$

thus

$$T = \begin{bmatrix} 1 & 1 \\ -0.2 & -0.8 \end{bmatrix},\quad T^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{5}{3} \\ -\frac{1}{3} & -\frac{5}{3} \end{bmatrix}$$

then it's easy to derive

$$\Phi(k) = T\tilde{\Phi}(k)T^{-1} = \begin{bmatrix} 1 & 1 \\ -0.2 & -0.8 \end{bmatrix} \begin{bmatrix} (-0.2)^k & 0 \\ 0 & (-0.8)^k \end{bmatrix} \begin{bmatrix} \frac{4}{3} & \frac{5}{3} \\ -\frac{1}{3} & -\frac{5}{3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4(-0.2)^k - (-0.8)^k & 5\left[(-0.2)^k - (-0.8)^k\right] \\ -0.8\left[(-0.2)^k - (-0.8)^k\right] & -(-0.2)^k + 4(-0.8)^k \end{bmatrix}$$

Now compute $\tilde{x}(k)$ according to equation (3-51), the first term of the right side is

$$\tilde{\Phi}(k)\tilde{x}(0) = \tilde{\Phi}(k)T^{-1}x(0) = \begin{bmatrix} (-0.2)^k & 0 \\ 0 & (-0.8)^k \end{bmatrix} \begin{bmatrix} \frac{4}{3} & \frac{5}{3} \\ -\frac{1}{3} & -\frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -(-0.2)^k \\ 4(-0.8)^k \end{bmatrix}$$

the second term of the right side is

$$\sum_{j=0}^{k-1} \tilde{\Phi}(j)T^{-1}Hu(k-j-1) = \sum_{j=0}^{k-1} \tilde{\Phi}(j) \begin{bmatrix} \frac{4}{3} & \frac{5}{3} \\ -\frac{1}{3} & -\frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1]$$

$$= \sum_{j=0}^{k-1} \begin{bmatrix} (-0.2)^j & 0 \\ 0 & (-0.8)^j \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \sum_{j=0}^{k-1} \begin{bmatrix} 3(-0.2)^j \\ -2(-0.8)^j \end{bmatrix}$$

$$=\begin{bmatrix} 3\left[1+(-0.2)+(-0.2)^2+\cdots+(-0.2)^{k-1}\right] \\ -2\left[1+(-0.8)+(-0.8)^2+\cdots+(-0.8)^{k-1}\right] \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{3\left[1-(-0.2)^k\right]}{1.2} \\ \dfrac{-2\left[1-(-0.8)^k\right]}{1.8} \end{bmatrix}$$

thus

$$\tilde{x}(k)=\frac{1}{3}\begin{bmatrix} -(-0.2)^k \\ 4(-0.8)^k \end{bmatrix}+\begin{bmatrix} \dfrac{1}{0.4}\left[1-(-0.2)^k\right] \\ -\dfrac{1}{0.9}\left[1-(-0.8)^k\right] \end{bmatrix}$$

$$=\begin{bmatrix} -\dfrac{17}{6}(-0.2)^k+\dfrac{5}{2} \\ \dfrac{22}{9}(-0.8)^k-\dfrac{10}{9} \end{bmatrix}$$

therefore

$$x(k)=T\tilde{x}(k)=\begin{bmatrix} 1 & 1 \\ -0.2 & -0.8 \end{bmatrix}\begin{bmatrix} -\dfrac{17}{6}(-0.2)^k+\dfrac{5}{2} \\ \dfrac{22}{9}(-0.8)^k-\dfrac{10}{9} \end{bmatrix}$$

$$=\begin{bmatrix} -\dfrac{17}{6}(-0.2)^k+\dfrac{22}{9}(-0.8)^k+\dfrac{25}{18} \\ \dfrac{3.4}{6}(-0.2)^k-\dfrac{17.6}{9}(-0.8)^k+\dfrac{7}{18} \end{bmatrix}$$

Method 2. z Transform Method

For the linear time-invariant discrete system state equation, we can find its solution by z Transform Method.

Consider the discrete-time state-space equation

$$x(k+1)=G(T)x(k)+H(T)u(k)$$

Applying z transform to the above equation yields

$$zx(z)-zx(0)=Gx(z)+Hu(z)$$

or

$$(zI-G)x(z)=zx(0)+Hu(z)$$

thus

$$x(z)=(zI-G)^{-1}zx(0)+(zI-G)^{-1}Hu(z)$$

Take inverse z transform,

$$x(k) = \ell^{-1}\left[(zI - G)^{-1} zx(0)\right] + \ell^{-1}\left[(zI - G)^{-1} Hu(z)\right] \tag{3-52}$$

Comparing (3-43) with (3-52) yields

$$G^k x(0) = \ell^{-1}\left[(zI - G)^{-1} zx(0)\right] \tag{3-53}$$

$$\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) = \ell^{-1}\left[(zI - G)^{-1} Hu(z)\right] \tag{3-54}$$

Using the solution of the continuous state equation

$$x(t) = \Phi(t - kT)x(kT) + \int_{kT}^{t} \Phi(t - \tau)Bu(kT)d\tau$$

let $t = (k + \Delta)T$ $(0 \leqslant \Delta \leqslant 1)$, then the above equation becomes

$$x\left[(k + \Delta)T\right] = \Phi(\Delta T)x(kT) + \int_{0}^{\Delta T} \Phi(\Delta T - \tau)d\tau Bu(kT) \tag{3-55}$$

Comparing (3-43) with (3-52) yields

$$G^k = \Phi(k) = \ell^{-1}\left[(zI - G)^{-1} z\right] \tag{3-56}$$

$$\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) = \ell^{-1}\left[(zI - G)^{-1} Hu(z)\right] \tag{3-57}$$

Proof

First we compute the z transform of G^k

$$\ell\left[G^k\right] = \sum_{k=0}^{\infty} G^k z^{-k} = I + Gz^{-1} + G^2 z^{-2} + \cdots \tag{3-58}$$

Pre-multifying Gz^{-1} to both sides of (3-58),

$$Gz^{-1}\ell\left[G^k\right] = Gz^{-1} + G^2 z^{-12} + G^3 z^{-3} + \cdots \tag{3-59}$$

Subtract (3-58) from (3-59),

$$(I - Gz^{-1})\ell\left[G^k\right] = I$$

because

$$\ell\left[G^k\right] = \left(I - Gz^{-1}\right)^{-1} = (zI - G)^{-1} z \tag{3-60}$$

take z inverse transform of equation (3-60), we can get equation (3-56).

Next we use convolution formula to prove equation (3-57).

$$\ell\left[\sum_{j=0}^{k-1} G^{k-j-1} Hu(j)\right] = \ell\left[G^{k-1}\right] H \ell\left[u(k)\right]$$

$$= \ell\left[G^k\right] z^{-1} H \ell\left[u(k)\right] = (zI - G)^{-1} Hu(z)$$

take z inverse transform of the above equation, then equation (3-57) is derived.

$$\sum_{j=0}^{k-1} G^{k-j-1}Hu(j) = \ell^{-1}\left[(zI-G)^{-1}Hu(z)\right]$$

Example 3.12 Consider the state equation in example 3.11, try to find $\Phi(k)$ and $x(k)$ using z Transform Method.

Solution

As $u(k)=1$, we have

$$u(z)=\frac{z}{z-1}$$

According to equation(3-56),

$$\begin{aligned}
\varPhi(k) &= \ell^{-1}\left[(zI-G)^{-1}z\right] \\
&= \ell^{-1}\left\{\begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}^{-1} z\right\} = \ell^{-1}\left\{\frac{z}{(z+0.2)(z+0.8)}\begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}\right\} \\
&= \ell^{-1}\left\{\frac{z}{3}\begin{bmatrix} \dfrac{4}{z+0.2}+\dfrac{-1}{z+0.8} & \dfrac{5}{z+0.2}+\dfrac{-5}{z+0.8} \\ \dfrac{-0.8}{z+0.2}+\dfrac{0.8}{z+0.8} & \dfrac{-1}{z+0.2}+\dfrac{4}{z+0.8} \end{bmatrix}\right\} \\
&= \frac{1}{3}\begin{bmatrix} 4(-0.2)^k-(-0.8)^k & 5(-0.2)^k-5(-0.8)^k \\ -0.8(-0.2)^k+(-0.8)^k & -(-0.2)^k+4(-0.8)^k \end{bmatrix}
\end{aligned}$$

then compute

$$zx(0)+Hu(z)=\begin{bmatrix} z \\ -z \end{bmatrix}+\begin{bmatrix} \dfrac{z}{z-1} \\ \dfrac{z}{z-1} \end{bmatrix}=\begin{bmatrix} \dfrac{z^2}{z-1} \\ \dfrac{-z^2+2z}{z-1} \end{bmatrix}$$

thus

$$\begin{aligned}
x(z) &= (zI-G)^{-1}\left[zx(0)+Hu(z)\right] \\
&= \begin{bmatrix} \dfrac{(z^2+2)z}{(z+2)(z+0.8)(z-1)} \\ \dfrac{(-z^2+1.84z)z}{(z+2)(z+0.8)(z-1)} \end{bmatrix} = \begin{bmatrix} \dfrac{-(17/6)z}{z+2}+\dfrac{(22/9)z}{z+0.8}+\dfrac{(25/18)z}{z-1} \\ \dfrac{(3.4/6)z}{z+2}+\dfrac{(-17.6/9)z}{z+0.8}+\dfrac{(7/18)z}{z-1} \end{bmatrix}
\end{aligned}$$

hence

$$x(k)=\ell^{-1}\left[x(z)\right]=\begin{bmatrix} -\dfrac{17}{6}(-0.2)^k+\dfrac{22}{9}(-0.8)^k+\dfrac{25}{18} \\ \dfrac{3.4}{6}(-0.2)^k-\dfrac{17.6}{9}(-0.8)^k+\dfrac{7}{18} \end{bmatrix}$$

3.6 Summary

The solution to both the continuous-time and discrete-time state space equation is studied in this chapter. The state transfer matrix is a very important parameter matrix which plays a big role in the solution of a state space equation. Different algorithms are discussed to obtain the state transfer matrix.

Exercise

3.1 Consider the matrix A

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Use the Laplace transform to find e^{-At}.

3.2 Use three different methods to find e^{-At}

(1) $A=\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$ (2) $A=\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

3.3 Examine the following matrix whether they meet the conditions of state transition matrix. If it does, try to find out the corresponding matrix A.

(1) $\Phi(t)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin t & \cos t \\ 0 & -\cos t & \sin t \end{pmatrix}$ (2) $\Phi(t)=\begin{pmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{pmatrix}$

(3) $\Phi(t)=\begin{pmatrix} 2e^{-t}-e^{-2t} & 2e^{-t}-2e^{-2t} \\ e^{-t}-e^{-2t} & 2e^{-t}-e^{-2t} \end{pmatrix}$ (4) $\Phi(t)=\begin{pmatrix} \frac{1}{2}(e^{-t}-e^{-3t}) & -\frac{1}{4}(e^{-t}+2e^{3t}) \\ (-e^{-t}+e^{3t}) & \frac{1}{2}(e^{-t}+e^{3t}) \end{pmatrix}$

3.4 Solve the state space model

$$\dot{x}=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}x+\begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$

$$y=\begin{pmatrix} 1 & 0 \end{pmatrix}x$$

Initial state $x(0)=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the input $u(t)$ is a unit-step response.

3.5 Calculate $\Phi(t,0)$ and $\Phi^{-1}(t,0)$

(1) $A=\begin{pmatrix} 2t & 0 \\ 0 & 1 \end{pmatrix}$ (2) $A=\begin{pmatrix} 0 & e^{-t} \\ -e^{-t} & 0 \end{pmatrix}$

3.6 The discrete time system is listed as follows, try to calculate $x(k)$.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{3} \end{pmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$x_1(0) = -1, \quad x_2(0) = 4$$

$u_1(k)$ is sampled from a ramp function t, $u_2(k)$ is sampled from e^{-t}.

Chapter 4
Stable Analysis

4.1 Introduction

Stability is an important property for a system because only when a system is stable can it finish the target task. In this chapter, a group of conceptions of stability in the sense of Lyapunov are given at the beginning, which are somewhat different from the definitions of stability given in classical control theory. Then the theorems to decide whether a system is stable or not are introduced.

4.2 Definition

The response of linear systems can always be decomposed as the zero-state response and the zero-input response. The stabilities of these two responses are commonly studied separately. The BIBO (bounded-input bounded-output) stability is for the zero-state response, while marginal and asymptotic stabilities are for the zero-input response.

Definition 4.1 [external stability]

An input $u(t)$ is said to be bounded if $u(t)$ does not grow to positive infinity or negative infinity. Equivalently, there exist constants β_1 and β_2 and

$$u(t) \leqslant \beta_1 < \infty \quad \text{holds for all } t \geqslant 0 \tag{4-1}$$

A system is said to be BIBO stable if every bounded input excites a bounded output, i.e.

$$y(t) \leqslant \beta_2 < \infty \quad \text{holds for all } t \geqslant 0 \tag{4-2}$$

This stability is defined for the zero-state response.

Conclusion 4.1 [BIBO stability of linear time-variant system]

Consider a continuous linear time-variant system with p inputs, m outputs and zero initial condition, if we define $[t_0, \infty]$ as the time domain, the system is BIBO stable at time t_0 if and only if there exists a limited positive number β, which satisfies the following relationship

$$\int_{t_0}^{t} \left| h_{ij}(t,\tau) \right| d\tau \leqslant \beta < \infty \tag{4-3}$$

where

$$h_{ij}(t,\tau), \quad i = 1,2,\cdots,m, \quad j = 1,2,\cdots,p \tag{4-4}$$

are elements of the impulse response matrix $H(t,\tau)$ at any t ($t \in [t_0,\infty]$).

Proof The proof is divided into two parts.

(i) The system is SISO system , that is, $p = m = 1$.

First, if $h_{ij}(t,\tau)$ is absolutely integrable, then every bounded input excites a bounded output.

Let $u(t)$ be an arbitrary input with $u(t) \leqslant \beta_1 < \infty$ for all $t \geqslant 0$, then

$$\begin{aligned} |y(t)| &= \left| \int_{t_0}^{t} h(t,\tau) u(\tau) d\tau \right| \leqslant \int_{t_0}^{t} |h(t,\tau)| |u(\tau)| d\tau \\ &\leqslant \beta_1 \int_{t_0}^{t} |h(t,\tau)| d\tau \leqslant \beta_1 \beta = \beta_2 < \infty \end{aligned} \tag{4-5}$$

the output is bounded.

Second, it can be seen that if $h_{ij}(t,\tau)$ is not absolutely integrable, the system is not BIBO stable.

If $h_{ij}(t,\tau)$ is not absolutely integrable, for any absolutely large N, there exists a $t_1 \in [t_0,\infty]$ such that

$$\int_{t_0}^{t_1} |h(t_1,\tau)| d\tau \geqslant N$$

Let us choose

$$u(t) = \operatorname{sgn} h(t_1,t) = \begin{cases} +1, & h(t_1,t) > 0 \\ 0, & h(t_1,t) = 0 \\ -1, & h(t_1,t) < 0 \end{cases}$$

It is very clear that u is bounded. The output excited by the input is

$$y(t_1) = \int_{t_0}^{t_1} h(t,\tau) u(\tau) d\tau = \int_{t_0}^{t_1} |h(t_1,\tau)| d\tau = \infty$$

Because $y(t_1)$ can be absolutely large, we conclude that a similar bounded input can excite an unbounded output, which is contrary to the definition *external stability*. Therefore, the assumption does not hold, and we have

$$\int_{t_0}^{t} |h_{ij}(t,\tau)| d\tau \leqslant \beta < \infty, \ \forall t \in [t_0,\infty]$$

(ii) The system is MIMO system. Note that any element $y_i(t)$ of output $y(t)$ is

$$
\begin{aligned}
|y_i(t)| &= \left|\int_{t_0}^{t}\left[h_{i1}(t,\tau)u_1(\tau)+\cdots+h_{ip}(t,\tau)u_p(\tau)\right]d\tau\right| \\
&\leqslant \left|\int_{t_0}^{t} h_{i1}(t,\tau)u_1(\tau)d\tau\right|+\cdots+\left|\int_{t_0}^{t} h_{ip}(t,\tau)u_p(\tau)d\tau\right|, \quad i=1,2,\cdots,m
\end{aligned}
$$

The sum of a finite number of bounded functions remains bounded. Therefore, we can have the conclusion based on the condition of SISO. Proof done.

Conclusion 4.2 [BIBO stability of linear time-invariant system]

Considering a continuous linear time-invariant system with p inputs, m outputs and zero initial condition, if we define the initial time as $t_0 = 0$, the system is BIBO stable if and only if there exists a limited positive number β, which satisfies the following relationship

$$
\int_0^{\infty} |h_{ij}(t)| dt \leqslant \beta < \infty
$$

where $h_{ij}(t)$, $i=1,2,\cdots,m$, $j=1,2,\cdots p$ are elements of the impulse response matrix $H(t)$.

Conclusion 4.3 [BIBO stability of linear time-invariant system]

Considering a continuous-time linear time-invariant system with p inputs , m outputs and zero initial condition, if we define the initial time as $t_0 = 0$, the system with proper rational transfer function matrix $G(s)$ is BIBO stable if and only if every pole of $G(s)$ has a negative real part or every pole of $G(s)$ lies in the left-half s-plane.

Proof The characteristic polynomial of $G(s)$ is $\alpha_G(s)$. The pole of $G(s)$ is s_l ($l=1,2,\cdots,m$), which is the roots of $\alpha_G(s)=0$. Therefore, any rational fraction of $G(s)$ is $g_{ij}(s)$ ($i=1,2,\cdots,q$, $j=1,2,\cdots p$). Its expansion contains the partial fractions

$$
\frac{\beta_l}{(s-s_l)^{\alpha_{l_r}}}, \ l=1,2,\cdots,m, \ \alpha_{l_r}=1,2,\cdots,\sigma_l
$$

where β_l is zero or nonzero constant, the pole s_l with multiplicity σ_l.

Thus the inverse Laplace transform of $g_{ij}(s)$ is

$$
\rho_{l_r} t^{\alpha_{l_r}-1} e^{s_l t}, \qquad l=1,2,\cdots,m
$$

If $\dfrac{\beta_l}{(s-s_l)^{\alpha_{lr}}} = \beta_l$, the corresponding inverse Laplace transform is the impulse function δ.

Therefore, the element $h_{ij}(t)$ of the impulse response matrix $H(t)$ derived from the inverse Laplace transform of element transfer function $g_{ij}(s)$ is the sum of the finite terms as $\rho_{l_r} t^{\alpha_{l_r}-1} e^{s_l t}$. It may contain the function δ. It is straightforward to verify that every $\rho_{l_r} t^{\alpha_{l_r}-1} e^{s_l t}$ ($\forall i=1,2,\cdots,q$, $\forall j=1,2,\cdots,p$) is absolutely integrable if and only if the pole s_l ($l=1,2,\cdots,m$) has a negative real part, i.e. $h_{ij}(t)$ ($\forall i=1,2,\cdots,q$, $\forall j=1,2,\cdots,p$) is absolutely integrable.

Therefore, according to Conclusion 4.2, the system is BIBO stable.

The BIBO stability is defined for the zero-state response.

Now we study the stability of the zero-input response or the response of

$$\dot{x}(t)=Ax(t)$$

which is excited by nonzero initial state x_0. Clearly, the solution is

$$x(t)=e^{At}x_0$$

Definition 4.2 [internal stability]

The zero-input response of equation $\dot{x}(t)=Ax(t)$ is marginally stable or stable in the sense of Lyapunov if every finite internal state x_0 excites a bounded response. It is asymptotically stable if every finite initial state excites a bounded response, which, in addition, approaches 0 as $t\to\infty$.

Conclusion 4.4 [internal stability of linear time-variant system]

The zero-input response of equation $\dot{x}(t)=Ax(t)$ is internal stable or marginally stable if every finite internal state x_0 excites a bounded state transition matrix $\phi(t,t_0)$, which, in addition, approaches 0 as $t\to\infty$.

Proof If $x(t_0)=x_0$ at time t_0, the zero-input response is

$$x_{0u}=\phi(t,t_0)x_0,\quad \forall t\in[t_0,\infty]$$

It is straightforward to verify that x_{0u} is bounded if and only if $\phi(t,t_0)$ is bounded and $\lim\limits_{t\to\infty}x_{0u}(t)=0$ if and only if $\lim\limits_{t\to\infty}\phi(t,t_0)=0$.

Conclusion 4.5 [internal stability of linear time-invariant system]

The zero-input response of equation $\dot{x}(t)=Ax(t)+Bu$ with the initial state $x(0)=x_0$ is internal stable or marginal stable if and only if $\lim\limits_{t\to\infty}e^{At}=0$.

Proof For a linear time-invariant system, the state transfer matrix $\phi(t)=e^{At}$ and e^{At} is bounded for any $t>0$, then we can obtain Conclusion 4.5 from Conclusion 4.4.

Conclusion 4.6 [internal stability of linear time-invariant system]

The zero-input response of equation $\dot{x}(t)=Ax(t)+Bu(t)$ with the initial state $x(0)=x_0$ is internal stable or marginal stable if and only if every eigenvalue $\lambda_i(A)$ $(i=1,2,\cdots,n)$ has a negative real part, in other words,

$$\mathrm{Re}\{\lambda_i(A)\}<0,\quad i=1,2,\cdots,n$$

Conclusion 4.7 [The relationship between internal stability and external stability]

Consider the continuous linear time-invariant system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \geqslant 0$$
$$y = Cx + Du$$

where x is n-dimensional state vector, u is p-dimensional input vector, and y is m-dimensional output vector. If the above system is internal stable or marginal stable, it must be BIBO stable or external stable.

Proof For the above linear time-invariant system, from the analysis of the system dynamics we know that the impulse response matrix $H(t)$

$$H(t) = Ce^{At}B + D\delta(t)$$

From Conclusion 4.5 we know that if the system is internal stable, e^{At} is bounded and $\lim_{x\to\infty} e^{At} = 0$. With the above contents we can get all elements of the impulse response matrix $H(t)$, where $h_{ij}(t)$, ($i=1,2,\cdots, m,\quad j=1,2,\cdots,p$) satisfies the following relationship:

$$\int_0^{\infty} \left|h_{ij}(t)\right| dt \leqslant \beta < \infty$$

The system is BIBO stable according to Conclusion 4.2.

Conclusion 4.8 [The relationship between external stability and internal stability]
Consider the continuous linear time-invariant system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \geqslant 0$$
$$y = Cx + Du$$

BIBO stability or external stability can not guarantee internal stability or marginal stability.

Proof When some poles and zeros are same, the order of transfer function for a system is lower than that of state space description, i.e., the number of poles is less than the number of eigenvalues. The system is BIBO stable, in other words, every pole of $G(s)$ has a negative real part, cannot guarantee that the eigenvalues of the system have negative real parts. Therefore, BIBO stability can not guarantee the internal stability of the system.

Conclusion 4.9 [The equivalence between external stability and internal stability]

Consider the continuous linear time-invariant system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \geqslant 0$$
$$y = Cx + Du$$

Without the zero-pole cancellation, the system is internal stable if and only if the system is external stable.

Proof From Conclusion 4.7, we can know internal stability means external stability of the system. If the system has no zero-pole cancellation, external stability means internal stability of the system according to proof of Conclusion 4.8. Therefore, external stability is equivalent to internal stability of a system if the system has no zero-pole cancellation.

Definition 4.3 [autonomous system]

A dynamic system without external or input excitation is defined as an autonomous system.

Generally, the state equation of a continuous nonlinear time-variant autonomous system can be described as follows:

$$\dot{x} = f(x,t),\quad x(t_0) = x_0,\quad t \in [t_0, \infty] \tag{4-6}$$

where x is n-dimensional state vector, $f(x,t)$ is n-dimensional vector function. For a continuous nonlinear time-invariant system, state equation can be written as $\dot{x} = f(x)$.

For a continuous linear time-variant system, the vector function $f(x,t)$ of equation (4-6) can be further described as a linear vector function of state x. The state equation of autonomous system can be rewritten as

$$\dot{x} = A(t)x,\quad x(t_0) = x_0,\quad t \in [t_0, \infty] \tag{4-7}$$

And the state equation of a continuous linear time-invariant autonomous can be written as $\dot{x} = Ax$.

Definition 4.4 [Equilibrium state]

For a continuous nonlinear time-variant system, the equilibrium state of the autonomous system (4-7) is x_e, which satisfies the following equation:

$$\dot{x}_e = f(x,t) = 0,\quad \forall t \in [t_0, \infty] \tag{4-8}$$

Some notes about the equilibrium state:

(i) Intuitive meaning of the equilibrium state

Equilibrium state x_e is a class of state which always satisfies $\dot{x}_e = 0$.

(ii) The form of the equilibrium state

The equilibrium state x_e can be solved from equation (4-8). For a 2-dimensional autonomous system, the form of x_e can be points or a line in the state space.

(iii) Non-uniqueness

The equilibrium state x_e of an autonomous system is not always unique. For a continuous linear time-invariant system, the equilibrium state x_e is the solution of Ax_e=0. If the matrix A is nonsingular, we have a unique solution x_e=0. If the matrix A is singular, the solution is not unique.

(iv) Zero equilibrium state

For the autonomous systems (4-6) or (4-7), x_e=0 must be an equilibrium state for the system.

(v) Isolated equilibrium states

The isolated equilibrium states are equilibrium states in the form of isolated equilibrium point in the state space. An important feature of the isolated equilibrium states is that they can be transferred to state-space origin by moving coordinates.

(vi) Agreement on the equilibrium states

In the direct method of Lyapunov, the stability analysis is mainly aimed at the equilibrium states. Therefore, we always set the state-space origin as the equilibrium states, i.e., $x_e=0$ in the following sections of the stability analysis.

Definition 4.5 [disturbed dynamics]

The disturbed dynamics of a dynamic system is a class of state dynamics caused by the initial state x_0.

In nature, the disturbed dynamics is the state response of zero-input. We call it disturbed dynamics because a non-zero initial state x_0 will be regarded as a state disturbance relative to the zero equilibrium state $x_e=0$ in stability analysis.

Usually, for more clearly description of the relationship of time and causality in the disturbed dynamics, we further represent the disturbance dynamics as the following form

$$x_{0u}(t)=\phi(t;x_0,t_0), \qquad t\in[t_0,\infty]$$

where ϕ is a vector function. When $t=t_0$, the vector function of the disturbed dynamics satisfies $\phi(t_0;x_0,t_0)=x_0$.

In the sense of geometry, the disturbed dynamics $\phi(t;x_0,t_0)$ presents a trajectory from the initial state x_0 in the state space. We can constitute a trajectory cluster of disturbed dynamics $\phi(t;x_0,t_0)$ according to different initial states.

Definition 4.6 [the stability in the sense of Lyapunov]

The isolated equilibrium state $x_e=0$ of the autonomous system is considered to be stable in the sense of Lyapunov at the time instant t if for any real number $\varepsilon>0$, there exists a corresponding real number $\delta(\varepsilon,t_0)>0$, when

$$\|x_0-x_e\|\leqslant\delta(\varepsilon,t_0) \tag{4-9}$$

the disturbed dynamics $\phi(t;x_0,t_0)$ from the initial x_0 satisfies the following inequality:

$$\|\phi(t;x_0,t_0)-x_e\|\leqslant\varepsilon, \qquad \forall t\geqslant t_0 \tag{4-10}$$

Notes of the stability in the sense of Lyapunov:

(1) Geometric significance of stability

There is direct geometric significance about the stability in the sense of Lyapunov. Hence, inequality (4-10) can be considered as a super-sphere in the state space whose core is x_e and the radius is ε. Its field can be represented with $S(\varepsilon)$. Inequality (4-9) can be considered as a super-sphere whose core is x_e and the radius is $\delta(\varepsilon,t_0)$ in the state space. Its field can be represented with $S(\delta)$, which is function of ε and t_0. The geometric explaination of stability in the sense of Lyapunov is that the dynamics trajectories $\phi(t;x_0,t_0)$ starting from any initial state within the field $S(\delta)$ will never exceed the boundary $H(\varepsilon)$ of the filed $S(\varepsilon)$, as shown in Fig. 4.1.

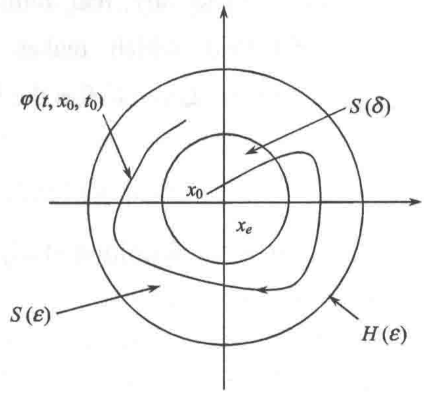


Fig.4.1 The stability in the sense of Lyapunov

(2) Uniform stability in the sense of Lyapunov

In the definition of stability in the sense of Lyapunov, if there exists a real number $\delta(\varepsilon)>0$ which is not related to the initial time t_0, i.e., when $\|x_0 - x_e\| \leqslant \delta(\varepsilon)$ holds, $\|\phi(t;x_0,t_0)-x_e\| \leqslant \varepsilon \ \forall t \geqslant t_0$ always holds, then we call the equilibrium state x_e is uniform stable in the sense of Lyapunov. In general, for the time-variant systems, uniform stability is of more practical significance than stability. Uniform stability means that, if the system is stable in the sense of Lyapunov at an initial time instant t_0, the system is stable in the sense of Lyapunov at all initial time t_0 in the definition interval of time.

(3) The stability properties of the time-invariant system

For the time-invariant system, no matter it is linear system or nonlinear system, continuous system or time-discrete system, stability in the sense of Lyapunov must be equivalent to uniform stability. In other words, if the equilibrium state x_e for a time-invariant system is stable in the sense of Lyapunov, x_e must be uniform stable in the sense of Lyapunov.

(4) The nature of stability in the sense of Lyapunov

The definition shows that the stability in the sense of Lyapunov can only guarantee the boundedness of the system's disturbed dynamics instead of the asymptotic characteristic relative to the equilibrium state. Therefore, compared to the project understanding, the essence of stability in the sense of Lyapunov is critical instability in the meaning of industrial process.

Definition 4.7 [the asymptotic stability]

The isolated equilibrium state x_e=0 of the autonomous system is considered to be asymptotic stable if the following conditions hold.

(i) x_e=0 is stable in the sense of Lyapunov at time t_0.

(ii) For a real number $\delta(\varepsilon,t_0)>0$ and any real number $\mu>0$, there exists corresponding real number $T(\mu,\delta,t_0)>0$ which makes the disturbed dynamics $\phi(t;x_0,t_0)$ starting from any initial state x_0 that satisfies the inequality (4-9) satisfy the following inequality:

$$\left\|\phi(t;x_0,t_0)-x_e\right\|\leqslant\mu,\qquad\forall t\geqslant t_0+T(\mu,\delta,t_0)\tag{4-11}$$

We give the following points based on the definition of asymptotic stability.

(1) Geometric significance of asymptotic stability

Take a two-dimensional system for example. The geometric meaning of asymptotic stability is shown in Fig.4.2 and Fig.4.3. The response from a initial state x_0 in the sphere $S(\delta)$ will not exceed the sphere $S(\varepsilon)$ (as shown in Fig.4.2) and converges to the sphere μ as time goes (as shown in Fig.4.3).

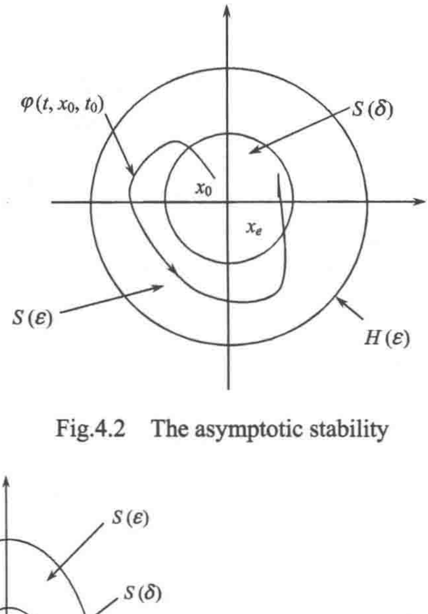


Fig.4.2 The asymptotic stability

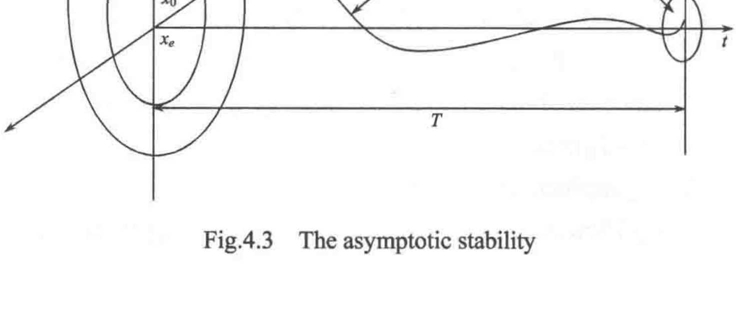


Fig.4.3 The asymptotic stability

(2) The equivalent definition of asymptotic stability

In the definition of asymptotic stability, if we select $\mu \to 0$, then $T(\mu, \delta, t_0) \to \infty$. Therefore, the equivalent definition of the asymptotic stability can be introduced which reflects the asymptotic characteristics of the stable process in a more intuitive form. The equivalent definition can be expressed that the isolated equilibrium state x_e=0 of an autonomous system (4-7) is asymptotic stable at time t_0 if the following two conditions hold:

(i) The disturbed dynamics $\phi(t; x_0, t_0)$ starting from any initial state $x_0 \in S(\delta)$ is bounded to any $t \in [t_0, \infty)$ relative to equilibrium state x_e=0.

(ii) The disturbed dynamics relative to equilibrium state x_e=0 meets the asymptotic characteristic, that is,

$$\lim_{t \to \infty} \phi(t; x_0, t_0) = 0, \qquad \forall x_0 \in S(\delta)$$

(3) Uniform asymptotic stability

In the definition of asymptotic stability, if $\delta(\varepsilon)$ has nothing to do with t_0, and other conditions holds, the equilibrium state x_e is uniform asymptotic stable. Similarly, for time-variant systems, the uniform asymptotic stability is more meaningful than the asymptotic stability.

(4) The properties of asymptotic stability for time-invariant systems

For time-invariant systems, no matter the system is linear or nonlinear, time-continuous or time-discrete, the asymptotic stability is equivalent to the uniform asymptotic stability of the equilibrium state x_e. In other words,

the asymptotic stability of the equilibrium state x_e

$\Leftrightarrow$ the uniform asymptotic stability of the equilibrium state x_e (4-12)

(5) Large-scale and small-scale asymptotic stability

Small-scale asymptotic stability is also known as local asymptotic stability. The definition of local asymptotic stability is:

There exists a super-sphere $S(\delta)$ around x_e=0, $\forall 0 \neq x_0 \in S(\delta)$,

x_e is asymptotically stable. (4-13)

Where $S(\delta)$ is the attraction domain, representing the property that all the states within $S(\delta)$ can be attracted to the equilibrium state x_e.

Large-scale asymptotic stability is also known as global asymptotic stability. The definition of global asymptotic stability is

$$\forall 0 \neq x_0 \in \mathrm{R}^n, \ x_e = 0 \text{ is asymptotic stable.} \tag{4-14}$$

(6) The necessary condition of large-scale asymptotic stability

From the definition of large-scale asymptotic stability (4-14), the necessary condition for the equilibrium state x_e=0 to be large-scale asymptotic stable is that there are no other asymptotic stable equilibrium states in the state space R^n.

(7) The properties of asymptotic stability for linear systems

For linear systems, no matter the system is time-invariant or time-variant, time-continuous or time-discrete, if the equilibrium state x_e=0 is asymptotic stable, it is large-scale asymptotic stable.

(8)The asymptotic stability in the sense of Lyapunov

$\Leftrightarrow$ The stability in the meaning of industrial process

Definition 4.8 [the instability]

The isolated equilibrium state x_e=0 of the autonomous system is considered to be unstable if for $\varepsilon > 0$ and even ε is big enough, there does not exist a corresponding real number $\delta(\varepsilon, t_0) > 0$ which makes the disturbed dynamics $\phi(t; x_0, t_0)$ starting from any initial state x_0 that satisfies the inequality $\|x_0 - x_e\| \leqslant \delta(\varepsilon, t_0)$ satisfy the following inequality:

$$\|\phi(t; x_0, t_0) - x_e\| \leqslant \varepsilon, \qquad \forall t \geqslant t_0$$

Take a two-dimensional system for example. The geometric meaning of instability is shown in Fig.4.4. If the equilibrium state x_e=0 is unstable, no matter how large or small $S(\delta)$ is, there exists non-zero point $x_0^* \in S(\delta)$ which makes the disturbed dynamics trajectory starting from which exceed the field $S(\delta)$. In essence, instability in the sense of Lyapunov is equivalent to divergent instability in the meaning of industrial process.

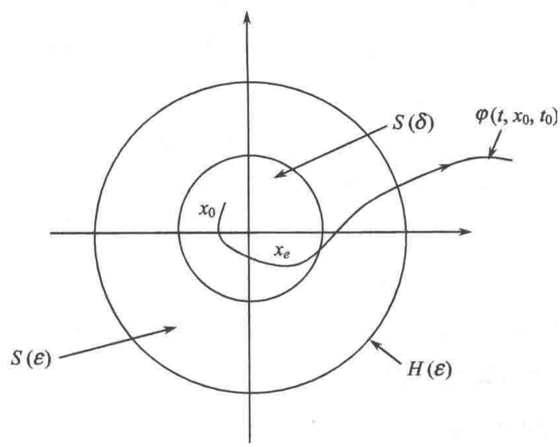


Fig.4.4 Instability

4.3 Stability Criteria

4.3.1 Lyapunov's Second Method

Lyapunov's second method proposes such a visual revelation in physics as: the dynamic process of a system is accompanied by changes of energy. If the change rate of the system energy always remains negative, i.e., the energy decreases monotonously, the disturbed dynamics of the system will eventually return to the equilibrium state. Based on this fact, we present the following stability criteria.

- **Large-Scale Asymptotically Stability Theorem**

Consider a nonlinear time-variant autonomous system described by

$$\dot{x} = f(x,t)\,,\quad t \in \left[t_0,\infty\right) \tag{4-15}$$

where $x \in R^{n\times 1}$, and $f(0,t)=0$ for all $t \in [t_0,\infty)$, which means the origin of the state space is an isolated equilibrium state.

Theorem 4.1 The origin of (4-15) is large-scale uniformly and asymptotically stable if there exists a scalar function $V(x,t)$ that satisfies $V(0,t)=0$ and has continuous first-order partial derivatives for x and t, and qualifies the following terms for all the non-zero sates in state space R^n:

(i) $V(x,t)$ is positive definite and bounded, i.e., there exist such two continuous non-decreasing scalar function $\alpha\left(\|x\|\right)$ and $\beta\left(\|x\|\right)$ ($\alpha(0)=0$, $\beta(0)=0$) that:

$$\beta\left(\|x\|\right) \geqslant V(x,t) \geqslant \alpha\left(\|x\|\right) > 0\,, \text{ for all } x \neq 0 \text{ and } t \in [t_0,\infty) \tag{4-16}$$

(ii) The derivative of $V(x,t)$ on t: $\dot{V}(x,t)$ is negative definite and bounded, i.e., there exists such a continuous non-decreasing scalar function $\gamma\left(\|x\|\right)$ ($\gamma(0)=0$) that:

$$\dot{V}(x,t) \leqslant -\gamma\left(\|x\|\right) < 0\,, \text{ for all } x \neq 0 \text{ and } t \in [t_0,\infty) \tag{4-17}$$

(iii) $\alpha\left(\|x\|\right) \to \infty$ when $\|x\| \to \infty$; equivalently, $V(x,t) \to \infty$.

Proof

1) Prove that the origin equilibrium x_e=0 is uniformly stable.

From the above term i, we know that $\beta\left(\|x\|\right)$ is continuous non-decreasing and $\beta(0)=0$. Thus for any real number $\varepsilon > 0$, there must exist such a real number $\delta(\varepsilon) > 0$ that $\beta(\delta) \leqslant \alpha(\varepsilon)$. Besides, as $\dot{V}(x,t)$ is negative definite, we have

$$V(\phi(t;x_0,t_0),t) - V(x_0,t_0) = \int_{t_0}^{t} \dot{V}(\phi(\tau;x_0,t_0),\tau)d\tau \leqslant 0 \tag{4-18}$$

Then for any initial time t_0 and any nonzero initial state x_0 with $\|x_0\| \leqslant \delta(\varepsilon)$, we have

$$\alpha(\varepsilon) \geqslant \beta(\delta) \geqslant V(x_0,t_0) \geqslant V(\phi(t;x_0,t_0),t) \geqslant \alpha\left(\|\phi(t;x_0,t_0)\|\right), \text{ for any } t \in [t_0,\infty) \quad (4\text{-}19)$$

As $\alpha(\|x\|)$ is continuous non-decreasing and $\alpha(0)=0$, from the above equation, we can deduce that for any initial time t_0 and any nonzero initial state x_0 which has $\|x_0\| \leqslant \delta(\varepsilon)$, we have

$$\|\phi(t;x_0,t_0)\| \leqslant \varepsilon, \quad \forall t \geqslant t_0 \quad (4\text{-}20)$$

Therefore, for any real number $\varepsilon > 0$, we can find a $\delta(\varepsilon) > 0$ ($\delta(\varepsilon)$ is independent from the initial time t_0) which makes the response $\phi(t;x_0,t_0)$ excited by any initial time t_0 and any non-zero initial state x_0 with $\|x_0\| \leqslant \delta(\varepsilon)$ qualify equation (4-20). According to the definition, the origin equilibrium x_e=0 is uniformly stable. Proof done.

2) Prove that for any initial time t_0, the dynamics $\phi(t;x_0,t_0)$ excited by any nonzero state x_0 which meets $\|x_0\| \leqslant \delta(\varepsilon)$ converges to the origin equilibrium state x_e=0.

First, for any real number $\mu > 0$ and the deduced real number $\delta(\varepsilon) > 0$, we can construct a real number $T(\mu,\delta) > 0$. Suppose the initial time t_0 is random and the non-zero x_0 satisfies $\|x_0\| \leqslant \delta(\varepsilon)$. Without loss of generality, we assume $0 < \mu \leqslant \|x_0\|$. Then, as $V(x,t)$ is bounded, for the given $\mu > 0$, we can find a corresponding real number $v(\mu) > 0$ which makes $\beta(v) \leqslant \alpha(\mu)$. Besides, $\gamma(\|x\|)$ is continuous and non-decreasing, suppose $\rho(\mu,\delta)$ is the minimum value of $\gamma(\|x\|)$ in the interval $v(\mu) \leqslant \|x\| \leqslant \varepsilon$, we assume

$$T(\mu,\delta) = \frac{\beta(\delta)}{\rho(\mu,\delta)} \quad (4\text{-}21)$$

In accordance with this principle, for any given real number $\mu > 0$, we can construct a corresponding $T(\mu,\delta)$ which is independent of the initial time t_0.

Furthermore, for some time t_2 ($t_0 \leqslant t_2 \leqslant t_0 + T(\mu,\delta)$), we prove that $\phi(t_2;x_0,t_0) = v(\mu)$. Let $t_1 = t_0 + T(\mu,\delta)$, and suppose $\phi(t_2;x_0,t_0) > v(\mu)$ for any t in the interval $t_0 \leqslant t \leqslant t_1$. Then using equation (4-21) and the negative definite property of $\dot{V}(x,t)$, we can deduce that:

$$\begin{aligned} 0 < \alpha(v) &\leqslant V(\phi(t_1;x_0,t_0),t_1) \leqslant V(x_0,t_1) \\ &\leqslant V(x_0,t_0) - (t_1 - t_0)\rho(\mu,\delta) \\ &\leqslant \beta(\delta) - T(\mu,\delta)\rho(\mu,\delta) \\ &= \beta(\delta) - \beta(\delta) = 0 \end{aligned} \quad (4\text{-}22)$$

Obviously, the above equation is a contradictory result. So the hypothesis does not hold, which means, there must exist a time t_2 in the time interval $t_0 \leqslant t \leqslant t_1$ which makes

$\phi(t_2;x_0,t_0)=v(\mu)$.

Finally, we deduce that for all $t \geqslant t_0+T(\mu,\delta)$ we have $\|\phi(t;x_0,t_0)\| \leqslant \mu$. In this respect, considering $\phi(t_2;x_0,t_0)=v(\mu)$ and using the bound of $V(x,t)$ and the negative definite property of $\dot{V}(x,t)$, for all the $t \geqslant t_2$, we have

$$\alpha\left(\|\phi(t;x_0,t_0)\|\right) \leqslant V(\phi(t;x_0,t_0),t) \leqslant V(\phi(t_2;x_0,t_0),t_2) \leqslant \beta(v) \leqslant \alpha(\mu) \tag{4-23}$$

Thus, based on the fact that $\alpha\left(\|x\|\right)$ is a continuous non-decreasing function, we can deduce from equation (4-23) that for all $t \geqslant t_2$,we have

$$\|\phi(t;x_0,t_0)\| \leqslant \mu \tag{4-24}$$

Besides, from $t_0+T(\mu,\delta) \geqslant t_2$ we can know that (4-24) holds for all t when $t \geqslant t_0+T(\mu,\delta)$, and $T \to \infty$ when $\mu \to 0$.

As proven above, for any initial time instant t_0, the dynamics excited by any non-zero initial state x_0 with $\|x_0\| \leqslant \delta(\varepsilon)$ converges to the origin equilibrium state x_e=0 when $t \to \infty$.

3) Prove that for any non-zero initial state x_0 in the state space R^n, its forced dynamics $\phi(t;x_0,t_0)$ is uniformly bounded. As $\alpha(\|x\|) \to \infty$ when $\|x\| \to \infty$, there must exist a finite real number $\varepsilon(\delta)>0$ which makes $\beta(\delta)<\alpha(\varepsilon)$ for any arbitrarily large real number $\delta>0$. Using the bound of $V(x,t)$ and the negative definite property of $\dot{V}(x,t)$, we can know that for all $t \in [t_0,\infty)$ and any non-zero $x_0 \in R^n$, we have

$$\alpha(\varepsilon)>\beta(\delta) \geqslant V(x_0,t_0) \geqslant V(\phi(t;x_0,t_0),t) \geqslant \alpha(\|\phi(t;x_0,t_0)\|) \tag{4-25}$$

Therefore, considering that $\alpha\left(\|x\|\right)$ is a continuous non-decreasing function, we have

$$\|\phi(t;x_0,t_0)\| \leqslant \varepsilon(\delta), \quad \forall t \geqslant t_0, \quad \forall x_0 \in R^n \tag{4-26}$$

$\varepsilon(\delta)$ is independent of the initial time t_0. This indicates that for any non-zero initial state $x_0 \in R^n$, $\phi(t;x_0,t_0)$ is uniformly bounded. Thus, the whole proof is done.

Notes of Theorem 4.1:

(1) Physical implication

For Theorem 4.1, in physical sense, the positive definite bounded scalar function $V(x,t)$ is regarded as some kind of "generalized energy" and $\dot{V}(x,t)$ as the change rate of the generalized energy. This idea reflects an institutive fact that, if the energy of the system is limited and the change rate of the energy is negative definite, the system energy is bounded and eventually decreases to zero. Correspondingly, the dynamics of the system is bounded and eventually converges to the origin equilibrium.

(2) Lyapunov function

In Theorem 4.1, $V(x,t)$ is not equivalent to the energy, meanwhile the meaning and form of $V(x,t)$ varies with the physical property. Thus in the theory of system stability, $V(x,t)$ which qualifies the theorem is called Lyapunov function. To judge the asymptotical stability of the system, we construct a Lyapunov function $V(x,t)$ for the system.

(3) The selection of Lyapunov function

For a comparatively simple system, we usually select a quadratic function of state x as the Lyapunov function. If the function does not satisfy the theorem, we can try to select a more complicated one. For a complex system, the construction of Lyapunov function is difficult and we sometime select the Lyapunov function with experience.

(4) The sufficiency of the criterion

Theorem 4.1 is a sufficient but not necessary condition to judge the large-scale uniformly and asymptotically stability of the system (4-15). The limitation is that, if we can not find the Lyapunov function $V(x,t)$ which meets the theorem, we can not determine whether the system is stable or not.

(5) The principles in using the criterion

Considering the sufficient property of Theorem 4.1 in determining the stability, we first judge whether the system is large-scale asymptotically stability or not. If the answer is no, then we judge the small-scale asymptotically stability of the system. If the result is no either, we will judge whether the system is Lyapunov stability till the stability is determined. The above principle is helpful but not always works.

Now, we discuss the continuous nonlinear time-invariant system, the state equation is

$$\dot{x} = f(x), \quad t \geqslant 0 \tag{4-27}$$

where $x \in R^{n\times n}$ and $f(0)=0$ for all $t \in [0,\infty)$, i.e., the state space origin x=0 is an isolated equilibrium state of the system.

We obtain the corresponding conclusion for the time-invariant case directly from Theorem 4.1, since time-invariant system is a special case. Besides, we can see that the requirement is largely simplified in its form for time-invariant cases.

Theorem 4.2 For a continuous nonlinear time-invariant autonomous system (4-27), if there exists a scalar function $V(x)$, which has continuous first-order partial derivatives for x, and qualifies the following terms for all the non-zero sates in state space R^n and $V(0)=0$:

(i) $V(x)$ is positive definite;

(ii) $\dot{V}(x) = dV(x)/dt$ is negative definite;

(iii) $V(x) \to \infty$ when $\|x\| \to \infty$.

the origin equilibrium state of (4-27) is large-scale asymptotically stable.

Example 4.1 Consider a continuous nonlinear time-invariant autonomous system

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

Discuss the stability of the system.

Solution

Obviously, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T$ is the equilibrium state.

First, we select a quadratic function of state x as Lyapunov function $V(x)$

$$V(x) = x_1^2 + x_2^2$$

It is obviously that $V(x)$ is positive definite.

Second, by calculating $\dot{V}(x)$,

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V(x)}{\partial x_2}\frac{dx_2}{dt} = \begin{bmatrix} \frac{\partial V(x)}{\partial x_1} & \frac{\partial V(x)}{\partial x_2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}\begin{bmatrix} x_2 - x_1(x_1^2 + x_2^2) \\ -x_1 - x_2(x_1^2 + x_2^2) \end{bmatrix} = -2(x_1^2 + x_2^2)^2$$

It's easy to see that $\dot{V}(x)$ is negative definite.

Last, when $\|x\| = \sqrt{x_1^2 + x_2^2} \to \infty$, we have

$$V(x) = \|x\|^2 = (x_1^2 + x_2^2) \to \infty$$

According to Theorem 4.2, the system origin equilibrium state x=0 is large-scale asymptotically stable.

The research shows that the item "$\dot{V}(x)$ is negative definite" is the main difficulty in the construction of Lyapunov function. Simultaneously, "$\dot{V}(x)$ is negative definite" is a conservative condition of the theorem. Next, we will give a relaxed stable criterion for continuous nonlinear time-invariant system.

Theorem 4.3 For a continuous nonlinear time-invariant autonomous system (4-27), if there exists a scalar function $V(x)$, which has continuous first-order partial derivatives for x, and qualifies the following terms for all the nonzero states in state space R^n and $V(0)$=0:

(i) $V(x)$ is positive definite;

(ii) $\dot{V}(x) = dV(x)/dt$ is semi-negative definite;

(iii) $\dot{V}\left(\varphi(t, x_0, 0)\right)$ is not identically equal to zero for any nonzero $x_0 \in R^n$;

(iv) $V(x) \to \infty$ when $\|x\| \to \infty$.

the origin equilibrium state of (4-27) is large-scale asymptotically stable.

Example 4.2 Consider a continuous nonlinear time-invariant autonomous system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - (1+x_2)^2 x_2$$

Discuss the stability of the system.

Solution

Obviously, $[x_1, x_2]^{\mathrm{T}}=[0,0]^{\mathrm{T}}$ is the only equilibrium state.

First, we select a quadratic function of state x as the Lyapunov function $V(x)$

$$V(x) = x_1^2 + x_2^2$$

$V(x)$ is positive definite.

Second, by computation,

$$\dot{V}(x) = \begin{bmatrix} \dfrac{\partial V(x)}{\partial x_1} & \dfrac{\partial V(x)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 - (1+x_2)^2 x_2 \end{bmatrix} = -2x_2^2(1+x_2)^2$$

We can see that there are two cases which make $\dot{V}(x)=0$:

case 1, x_1 is arbitrary and $x_2=0$;

case 2, x_1 is arbitrary and $x_2=-1$.

Except for these two cases, we have $\dot{V}(x)<0$ when $x \neq 0$. Thus $\dot{V}(x)$ is semi-negative definite.

Now we check whether $\dot{V}\left(\varphi(t, x_0, 0)\right)$ identically equal to zero or not, and the problem comes down to judge if the above two cases are the disturbed response of system.

For case 1,

$$\bar{\phi}(t; x_0, 0) = \left[x_1(t), 0\right]^T$$

from $x_2(t) \equiv 0$ we can deduce $\dot{x}_2(t)=0$. Substituting this into the system equation yields

$$\dot{x}_1(t) = x_2(t) = 0$$
$$0 = \dot{x}_2(t) = -(1+x_2(t))^2 x_2(t) - x_1(t) = -x_1(t)$$

therefore, $\bar{\phi}(t; x_0, 0) = \left[x_1(t), 0\right]^T$ is not the solution of the system disturbed dynamics except for the origin ($x_1 = 0, x_2 = 0$).

For case 2,

$$\bar{\phi}(t;x_0,0)=\left[x_1(t),-1\right]^T$$

from $x_2(t)=-1$ we can deduce $\dot{x}_2(t)=0$. Substituting this into the system equation yields

$$\dot{x}_1(t)=x_2(t)=-1$$
$$0=\dot{x}_2(t)=-(1+x_2(t))^2x_2(t)-x_1(t)=-x_1(t)$$

Obviously, this is a contradictory result. Hence, $\bar{\phi}(t;x_0,0)=\left[x_1(t),-1\right]^T$ is not the solution of the system. Therefore, the item iii in Theorem 4.3 is satisfied.

Last, when $\|x\|=\sqrt{(x_1^2+x_2^2)}\to\infty$, we have

$$V(x)=\|x\|^2=(x_1^2+x_2^2)\to\infty$$

According to Theorem 4.3, the origin equilibrium state of the system x=0 is large-scale asymptotically stable. Besides, we can see that the Lyapunov function we choose for the system does not qualifies Theorem 4.2 but meets Theorem 4.3.

- **Small-Scale Asymptotically Stability Theorem**

In the application of the Lyapunov Second method, when a system is not large-scale asymptotically stable, we turn to judge the small-scale asymptotically stability. This section presents some basic theorems about small-scale asymptotically stability in Lyapunov Second method.

For continuous nonlinear time-variant systems, we have the following conclusion.

Theorem 4.4 For a continuous nonlinear time-variant autonomous system (4-15), if there exists a scalar function $V(x,t)$ ($V(0,t)=0$), which has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin, qualifies the following terms for all the non-zero sates $x\in\Omega$ and all $t\in\left[t_0,\infty\right)$:

(i) $V(x,t)$ is positive definite and bounded;

(ii) $\dot{V}(x,t)=dV(x,t)/dt$ is negative definite and bounded.

then the origin equilibrium state of the system x=0 is uniformly and asymptotically stable in the area Ω.

For continuous nonlinear time-invariant systems, we have the following two conclusions.

Theorem 4.5 For a continuous nonlinear time-invariant autonomous system (4-27), if there exists a scalar function $V(x)(V(0)=0)$, which has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin, qualifies the following terms for all non-zero states $x\in\Omega$ and all $t\in\left[t_0,\infty\right)$:

(i) $V(x)$ is positive definite;

(ii) $\dot{V}(x,t) = dV(x)/dt$ is negative definite.

then the origin equilibrium state of the system x=0 is asymptotically stable in the area Ω.

Theorem 4.6 For a continuous nonlinear time-invariant autonomous system (4-27), if there exists a scalar function $V(x)$ ($V(0)$=0), which has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin, qualifies the following terms for all non-zero states $x \in \Omega$ and all $t \in [t_0, \infty)$:

(i) $V(x)$ is positive definite;

(ii) $\dot{V}(x,t) = dV(x)/dt$ is semi-negative definite.

(iii) $\dot{V}(\varphi(t, x_0, 0))$ is not identically equal to zero for any non-zero state $x \in \Omega$

then the origin equilibrium state $x = 0$ is asymptotically stable in the area Ω.

- **Theorem for Stability in the Sense of Lyapunov**

In the same way, when a system is not small-scale asymptotically stable, we turn to judge the stability in the sense of Lyapunov. In this section, we will provide some rules to determine the stability in the sense of Lyapunov.

Theorem 4.7 For a continuous nonlinear time-variant autonomous system (4-15), if there exists a scalar function $V(x,t)$ ($V(0,t)$=0), which has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin, qualifies the following terms for all the non-zero sates $x \in \Omega$ and all $t \in [t_0, \infty)$:

(i) $V(x,t)$ is positive definite and bounded;

(ii) $\dot{V}(x,t) = dV(x,t)/dt$ is semi-negative definite and bounded.

then the origin equilibrium state of the system x=0 is stable in the sense of Lyapunov in the area Ω.

For continuous nonlinear time-variant systems, we have the following conclusion.

Theorem 4.8 For a continuous nonlinear time-invariant autonomous system (4-27), if there exists a scalar function $V(x)$, $V(0)$=0, $V(x)$ has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin which qualifies the following terms for all non-zero states $x \in \Omega$ and all $t \in [t_0, \infty)$:

(i) $V(x)$ is positive definite;

(ii) $\dot{V}(x) = dV(x)/dt$ is semi-negative definite.

then the origin equilibrium state of the system x=0 is stable in the sense of Lyapunov in the area Ω.

- **Theorem for Instability**

For a continuous nonlinear time-variant system, the criterion for instability is presented as follows.

Theorem 4.9 For a continuous nonlinear time-variant autonomous system (4-15), if there exists a scalar function $V(x,t)$ ($V(0,t)=0$), which has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin, qualifies the following terms for all the nonzero sates $x \in \Omega$ and all $t \in [t_0, \infty)$:

(i) $V(x,t)$ is positive definite and bounded;

(ii) $\dot{V}(x,t) = dV(x,t)/dt$ is positive definite and bounded.

then the origin equilibrium state of the system x=0 is unstable.

For continuous nonlinear time-variant systems, we have the following criteria.

Theorem 4.10 For a continuous nonlinear time-invariant autonomous system (4-27), if there exists a scalar function $V(x)$ ($V(0)$=0), which has continuous first-order partial derivatives for x and t and an attractive area Ω around the state space origin, qualifies the following terms for all non-zero states $x \in \Omega$ and all $t \in [t_0, \infty)$:

(i) $V(x)$ is positive definite;

(ii) $\dot{V}(x) = dV(x)/dt$ is positive definite.

then the origin equilibrium state of the system x=0 is unstable.

Note: From the above two conclusions, we can see that the system is unstable when $V(x,t)$ or $V(x)$ have the same sign with $\dot{V}(x,t)$ or $\dot{V}(x)$, theoretically, the disturbed dynamics trajectories of the system will diverge to infinity.

4.3.2 State Dynamics Stability Criteria for Continuous Linear Systems

This section discusses the stability for continuous linear systems. Based on the concepts and results of Lyapunov Second method, similar to the linear time-invariant system and linear time-variant system, we will discuss the stability of the disturbed dynamics first. Then some stable criteria will be presented.

- **Stability Criteria for Linear Time-Invariant Systems**

Consider a continuous linear time-invariant system, the autonomous state equation is

$$\dot{x} = Ax, \quad x(0) = x_0, \quad t \geqslant 0 \tag{4-28}$$

where $x \in R^n$, the origin of the state space x=0 is an equilibrium state of the system.

Next, we present the stability criteria for linear time-invariant systems based on eigenvalues.

Theorem 4.11 For a continuous linear time-invariant system (4-28), the origin equilibrium state x=0 is stable in the sense of Lyapunov if and only if all the eigenvalus of matrix A have non-positive real parts, i.e., zero or negative real parts, and the eigenvalue whose real part is zero is distinct.

Proof The proof is divided into two steps.

1) Prove that the system is stable if and only if $\left\|e^{At}\right\| \leqslant \beta < \infty$. From the autonomous dynamics equation of the linear time-invariant system, we can obtain the disturbed dynamics of states

$$\phi(t;x_0,0) = x_{0u}(t) = e^{At}x_0 \tag{4-29}$$

The equilibrium state is $x_e = 0$, we notice that $x_e = e^{At}x_e$, thus we further deduce that the disturbed dynamics relative to equilibrium state $x_e = 0$ is

$$\phi(t;x_0,0) - x_e = e^{At}(x_e - x_0), \quad \forall t \geqslant 0 \tag{4-30}$$

This indicates that if and only if $\left\|e^{At}\right\| \leqslant \beta < \infty$, for any real number ε there exists a real number $\delta(\varepsilon) = \varepsilon/\beta$ which is independent of the initial time and makes the disturbed dynamics from any non-zero initial state $\left\|x_0 - x_e\right\| \leqslant \delta(\varepsilon)$ $(x_0 \in R^n)$ qualify the following inequality:

$$\left\|\phi(t;x_0,0) - x_e\right\| \leqslant \left\|e^{At}\right\| \cdot \left\|x_0 - x_e\right\| \leqslant \beta \cdot \frac{\varepsilon}{\beta} = \varepsilon, \quad \forall t \geqslant 0 \tag{4-31}$$

As defined, the system is stable in the sense of Lyapunov. Proof done.

2) Prove the conclusion. Introducing the linear non-singular transformation $\hat{x} = Q^{-1}x$ to ensure $\hat{A} = Q^{-1}AQ$ as the Jordan Canonical:

$$\left\|e^{\hat{A}t}\right\| \leqslant \left\|Q^{-1}\right\|\left\|e^{At}\right\|\left\|Q\right\|, \quad \left\|e^{At}\right\| \leqslant \left\|Q\right\|\left\|e^{\hat{A}t}\right\|\left\|Q^{-1}\right\| \tag{4-32}$$

This indicates that, the bound of $\left\|e^{At}\right\|$ is equivalent to the bound of $\left\|e^{\hat{A}t}\right\|$. From the Jordan Canonical, we can know that the element of $e^{\hat{A}t}$ is the combination of the following items:

$$t^{\beta_i - 1}e^{\alpha_i t + j\omega_i t}, \quad \lambda_i(\hat{A}) = \lambda_i(A) = \alpha_i + j\omega_i, \quad i = 1,2,\cdots,\mu, \beta_i = 1,2,\cdots,\sigma_i \tag{4-33}$$

where $\lambda(\cdot)$ is the eigenvalue of the corresponding matrix, σ_i means that λ_i is a σ_i duplicate eigenvalue. When $\alpha_i < 0$, the corresponding items are bounded in the interval $[0,\infty)$ for any limited positive integral β_i; when $\alpha_i = 0$, the corresponding items are bounded in the interval $[0,\infty)$ only for $\beta_i = 1$. Further, the bound of the elements of $e^{\hat{A}t}$ means the bound of $\left\|e^{\hat{A}t}\right\|$. This indicates that, $\left\|e^{\hat{A}t}\right\|$, i.e., $\left\|e^{At}\right\|$ is bounded if and only if all the eigenvalues of matrix A have zero or negative real parts and the eigenvalues whose parts are zero are distinct. Using the proposition given in the first part, we can prove that the above condition is the necessary and sufficient one for the stability in the sense of Lyapunov. Proof done.

Theorem 4.12 For a continuous linear time-invariant system (4-28), the origin equilibrium state x=0 is asymptotically stable if and only if all the eigenvalues of matrix A have negative real parts.

Proof From Theorem 4.11, the equilibrium state x=0 is stable in the sense of Lyapunov if and only if all the eigenvalues of matrix A have zero or negative real parts and the eigenvalues whose real parts are zero are distinct. Further, from equations (4-29), (4-32) and (4-33), we can know that

$$\lim_{t\to\infty}\phi(t;x_0,0)=\lim_{t\to\infty}e^{At}x_0=0$$

$$\Leftrightarrow \lim_{t\to\infty}\left\|e^{At}\right\|=0$$

$$\Leftrightarrow \lim_{t\to\infty}t^{\beta_i-1}e^{\alpha_i t+j\omega_i t}=0,\qquad i=1,2,\cdots,\mu,\qquad \beta_i=1,2,\cdots,\sigma_i$$

$\Leftrightarrow$ The eigenvalues of A all have negative real parts.

As defined, the system is asymptotically stable. Proof done.

Note: We can see that the asymptotically stability equals to the internal stability illustrated beforehand.

Further, based on the Lyapunov Second method, we can provide the Lyapunov stability criteria for linear time-invariant systems.

Theorem 4.13 For an n-dimensional continuous linear time-invariant system (4-28), the origin equilibrium state x_e=0 is asymptotically stable if and only if for any given $n\times n$ dimensional positive definite symmetry matrix Q, the Lyaponov equation

$$A^T P+PA=-Q$$

has a unique $n\times n$ dimensional positive definite symmetry matrix solution P.

Proof

First we prove the sufficiency. Given $n\times n$ positive definite matrix P, we want to prove the asymptotically stability of x_e=0. For this, we select the Lyapunov function $V(x)=x^T Px$. As $P=P^T>0$, $V(x)$ is positive definite. Further, we have

$$\begin{aligned}V(x)&=\dot{x}^T Px+x^T P\dot{x}=(Ax)^T Px+x^T P(Ax)\\&=x^T(A^T P+PA)x=-x^T Qx\end{aligned}\tag{4-34}$$

And from $Q=Q^T>0$ we know that $\dot{V}(x)$ is negative. According to large-scale asymptotically stability theorems, x_e=0 is asymptotically stable. Sufficiency proven.

Then we prove the necessity. Given the asymptotically stability of x_e=0, we want to prove that matrix P is positive definite. For this, considering the matrix equation:

$$\dot{X} = A^T X + XA, \quad X(0) = Q, \quad t \geqslant 0 \tag{4-35}$$

the matrix X is

$$X(t) = e^{A^T t} Q e^{At}, \quad t \geqslant 0 \tag{4-36}$$

The integration of (4-35) from $t=0$ to $t \to \infty$ is

$$X(\infty) - X(0) = A^T (\int_0^\infty X(t)dt) + (\int_0^\infty X(t)dt)A \tag{4-37}$$

As the system is asymptotically stable, e.g., $e^{At} \to 0$ when $t \to \infty$, thus from (4-36) we have $X(\infty) = 0$. Considering $X(0) = Q$, let $P = \int_0^\infty X(t)dt$, so (4-37) can be further expressed as

$$A^T P + PA = -Q \tag{4-38}$$

Hence, $P = \int_0^\infty X(t)dt$ is the solution of the Lyapunov equation. From the facts that $X(t)$ is unique and $X(\infty) = 0$, $P = \int_0^\infty X(t)dt$ is unique. While from

$$P^T = \int_0^\infty \left[e^{A^T t} Q e^{At} \right]^T dt = \int_0^\infty e^{A^T t} Q e^{At} dt = P \tag{4-39}$$

So $P = \int_0^\infty X(t)dt$ is symmetry. Again for any nonzero $x_0 \in R^n$, we have

$$x_0^T P x_0 = \int_0^\infty (e^{At} x_0)^T Q (e^{At} x_0) dt \tag{4-40}$$

where the positive definite matrix $Q = N^T N$ is nonsingular. From equation (4-40) we can further deduce:

$$x_0^T P x_0 = \int_0^\infty (e^{At} x_0)^T N^T N (e^{At} x_0) dt$$

$$= \int_0^\infty \left\| N e^{At} x_0 \right\|^2 dt > 0 \tag{4-41}$$

Therefore, P is unique and positive definite. Necessity proven. Proof done.

Example 4.3 Consider the stability of the following continuous linear time-invariant system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} x$$

Solution

For simplicity, we select $Q = I_2$. Further, from Lyapunov equation

$$A^T P + PA = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -Q$$

we can deduce:

$$\begin{aligned} -2p_1 + 0p_2 + 4p_3 &= -1 \\ 0p_1 - 6p_2 + 2p_3 &= -1 \\ p_1 + 2p_2 - 4p_3 &= 0 \end{aligned}$$

By algebraic equations solving method, we have

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 4 \\ 0 & -6 & 2 \\ 1 & 2 & -4 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{8} & -\frac{1}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{3}{8} \\ \frac{5}{8} \end{bmatrix}$$

the solution of the Lyapunov equation is

$$P = \begin{bmatrix} \frac{7}{4} & \frac{5}{8} \\ \frac{5}{8} & \frac{3}{8} \end{bmatrix} > 0$$

P is positive definite, so the system is asymptotically stable.

MATLAB can be adopted for the solution of the above question.

```
>> A=[-1 1;2 -3];
>> Q=[1 0;0 1];
>> P=lyap(A',Q)


P =
    1.7500    0.6250
    0.6250    0.3750
```

The function"posdef"can be used to judge whether a matrix is positive definite or not.

The format of the MATLAB function is

```
[key,sdet]=posdef(P)
```

The codes are shown as follow.

```
function [key,sdet]=posdef(P)
[nr,nc]=size(P);
sdet=[];
for i=1:nr
      sdet=[sdet,det(P(1:i,1:i))];
```

```
end
key=1;
if any(sdet<=0)
      key=0;
end
```

The running result of posdef (P) is

```
key =
     1
sdet =
     1.7500    0.2656
```

If key=1, the result represents that P is positive definite , else P is negative definite. Sdet is the determinant of every matrix in the upper left corner.

Theorem 4.14 For a n-dimensional continuous linear time-invariant system (4-28) and any given real number $\sigma \geqslant 0$, suppose the eigenvalues of matrix A are $\lambda_i(A), i=1,2,\cdots,n$, then all the eigenvalues locate in the left-half plane of the straight line $-\sigma + j\omega$ on s plane, i.e.,

$$\operatorname{Re}\lambda_i(A) < -\sigma, \quad i=1,2,\cdots,n$$

if and only if for any given $n \times n$ dimensional positive definite symmetry matrix Q, the expanded Lyapunov function

$$2\sigma P + A^T P + PA = -Q \tag{4-42}$$

has unique positive definite solution matrix P.

Proof Let $\tilde{A} = A + \sigma I$, then from

$$\begin{aligned}\det(\tilde{s}I - \tilde{A}) &= \det(\tilde{s}I - A - \sigma I) = \det\left[(\tilde{s} - \sigma)I - A\right] \\ &= \det(sI - A), \quad \tilde{S} = s + \sigma\end{aligned} \tag{4-43}$$

We can know that

$$\lambda_i(\tilde{A}) = \lambda_i(A) + \sigma, \quad i=1,2,\cdots,n \tag{4-44}$$

From Theorem 4.13, we know that all the eigenvalues of matrix $\tilde{A}$ have negative real parts if and only if for any positive definite symmetry matrix Q, the following Lyapunov function has unique positive definite solution matrix P:

$$\tilde{A}^T P + P\tilde{A} = -Q \tag{4-45}$$

Hence, substituting $\tilde{A} = A + \sigma I$ into (4-45), we can deduce (4-42). While from (4-44), we have the following equivalent relationship:

$$\operatorname{Re}\lambda_i(\tilde{A}) < 0 \Leftrightarrow \operatorname{Re}\lambda_i(A) < -\sigma, \quad i=1,2,\cdots,n$$

Therefore, if and only if (4-42) has unique positive definite solution matrix P, $\operatorname{Re}\lambda_i(A) < -\sigma$, $i=1,2,\cdots,n$. Proof done.

- **Stability Criteria for Linear Time-Variant Systems**

Now we turn to discuss the continuous linear time-variant systems, the autonomous state equation is

$$\dot{x} = A(t)x\,,\ \ x(t_0) = x_0\,,\ \ t \in \left[t_0,\infty\right],\ \ t_0 \in \left[t_0,\infty\right) \tag{4-46}$$

where $x \in R^{n\times n}$, $A(t)$ qualifies the condition which guarantees the existence and uniqueness of the solution, x_e=0 is a equilibrium state of the system. Usually, there is a non-zero equilibrium state x_e besides x_e=0.

For linear time-variant systems, we can adopt two ways to judge the stability of equilibrium states, i.e., the method based on state transfer matrix and the method based on Lyapunov criteria. Next, we will introduce these two methods.

Theorem 4.15 For a continuous linear time-variant system (4-46), $\phi(t,t_0)$ is the state transfer matrix of the system, then the origin equilibrium state $x_e = 0$ is stable in the sense of Lyapunov at time t_0 if and only if there exists a real number $\beta(t_0) > 0$ which makes the following equation valid:

$$\left\|\phi(t,t_0)\right\| \leqslant \beta(t_0) < \infty, \quad \forall t \geqslant t_0 \tag{4-47}$$

Further, if and only if there exists independent real numbers $\beta > 0$ for all t_0, the origin equilibrium state $x_e = 0$ is stable in the sense of Lyapunov.

Theorem 4.16 For a continuous linear time-variant system (4-46), $\phi(t,t_0)$ is the state transfer matrix of the system, then the origin equilibrium state $x_e = 0$ is asymptotically stable at time t_0, if and only if there exists a real number $\beta(t_0) > 0$ which qualifies the following two items:

$$\begin{aligned} &\left\|\phi(t,t_0)\right\| \leqslant \beta(t_0) < \infty, \quad \forall t \geqslant t_0 \\ &\lim_{t\to\infty}\left\|\phi(t,t_0)\right\| = 0 \end{aligned} \tag{4-48}$$

Furthermore, the origin equilibrium state $x_e = 0$ is uniformly and asymptotically stable if and only if there exist independent real numbers $\beta_1 > 0$ and $\beta_2 > 0$ for all $t_0 \in \left[0,\infty\right]$ that qualifies the following equation

$$\left\|\phi(t,t_0)\right\| \leqslant \beta_1 e^{-\beta_2(t-t_0)} \tag{4-49}$$

Proof

First we prove the sufficiency. Given equation (4-48), we need to prove that $x_e = 0$ is uniformly and asymptotically stable. From (4-49) and using the disturbed dynamics equation we have

$$\|\phi(t;x_0,t_0)\| = \|\phi(t,t_0)x_0\| \leqslant \|\phi(t,t_0)\|\|x_0\| \leqslant \beta_1\|x_0\|e^{-\beta_2(t-t_0)} \tag{4-50}$$

This indicates that the disturbed dynamics $\phi(t;x_0,t_0)$ is bounded for all $t \geqslant t_0$, and for all $t_0 \in [0,\infty)$ we have $\|\phi(t;x_0,t_0)\| \to 0$ when $t \to \infty$. Thus $x_e = 0$ is uniformly and asymptotically stable. Sufficiency proven.

Then we prove the necessity. Given that $x_e = 0$ is uniformly and asymptotically stable, we need to prove equation (4-48). As $x_e = 0$ is uniformly and asymptotically stable, $x_e = 0$ is stable in the sense of Lyapunov, i.e., there exists a real number $\beta_3 > 0$ which satisfies

$$\|\phi(t,t_0)\| \leqslant \beta_3,\ \forall t_0 \in [0,\infty],\ \forall t \geqslant t_0 \tag{4-51}$$

Further, for a fixed real number $\delta > 0$ and any given real $\mu > 0$, there exists a real number $T > 0$ that satisfies the following equation for all initial states x_0 and all $t_0 \in [0,\infty)$

$$\|\phi(t_0+T;x_0,t_0)\| = \|\phi(t_0+T;t_0,x_0)\| \leqslant \mu \tag{4-52}$$

Select randomly a x_0 to satisfy

$$\|x_0\| = \delta \text{ and } \|\phi(t_0+T,t_0)x_0\| = \|\phi(t_0+T,t_0)\| \cdot \|x_0\| \tag{4-53}$$

Then, by selecting $\mu = \delta/2$ from equations (4-52) and (4-53), we can further deduce that:

$$\|\phi(t_0+T,t_0)\| \leqslant \frac{1}{2},\ \forall t_0 \in [0,\infty) \tag{4-54}$$

Hence, using equations (4-51) and (4-54) we can get

$$\|\phi(t,t_0)\| \leqslant \beta_3,\quad \forall t \in [t_0, t_0+T)$$

$$\begin{aligned}\|\phi(t,t_0)\| &= \|\phi(t,t_0+T)\phi(t_0+T,t_0)\| \\ &\leqslant \|\phi(t,t_0+T)\|\|\phi(t_0+T,t_0)\| \leqslant \frac{\beta_3}{2},\quad \forall t \in [t_0+T, t_0+2T)\end{aligned}$$

$$\|\phi(t,t_0)\| \leqslant \|\phi(t,t_0+2T)\|\|\phi(t_0+2T,t_0+T)\|\|\phi(t_0+T),t_0\| \leqslant \frac{\beta_3}{2^2},$$

$$\forall t \in [t_0+2T, t_0+3T)$$

$$\cdots\cdots$$

$$\|\phi(t,t_0)\| \leqslant \frac{\beta_3}{2^m},\quad \forall t \in [t_0+mT, t_0+(m+1)T)$$

Again we construct an exponential function $\beta_1 e^{-\beta_2(t-t_0)}$ which makes the following equation valid:

$$\left[\beta_1 e^{-\beta_2(t-t_0)}\right]_{t=t_0+mT} = \frac{\beta_3}{2^{m-1}},\qquad m = 1, 2, \cdots \tag{4-55}$$

Further, we can get

$$\beta_1(e^{-\beta_2 T})^m = 2\beta_3\left(\frac{1}{2}\right)^m \tag{4-56}$$

We can see that, by selecting $\beta_1 = 2\beta_3$ and an adequate β_2, we can make $e^{-\beta_2 T} = 1/2$ hold. Thus we proved that there exist real numbers $\beta_1 > 0$ and $\beta_2 > 0$ to validate equation (4-49). Necessity proven. Proof done.

Theorem 4.17 For a continuous linear time-variant system (4-46), suppose x_e=0 is the unique equilibrium state of the system. The elements of $n \times n$ dimensional matrix $A(t)$ are segmented continuous uniform and bounded real function, then the origin equilibrium state x_e=0 is uniformly and asymptotically stable if and only if there exist two real numbers $\beta_1 > 0$ and $\beta_2 > 0$, when $0 < \beta_1 I \leqslant Q(t) \leqslant \beta_2 I$ holds, the $n \times n$ solution matrix $P(t)$ of the Lyapunov equation:

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t), \quad \forall t \geqslant t_0 \tag{4-57}$$

is real symmetry, uniformly bounded and uniformly positive definite. Equivalently, there exist two real numbers $\alpha_1 > 0$ and $\alpha_2 > 0$ making $0 < \alpha_1 I \leqslant P(t) \leqslant \alpha_2 I$, $\forall t \geqslant t_0$.

4.3.3 State Dynamics Stability Criteria for Discrete Systems

- **Lyapunov Stability Theorem for Discrete Nonlinear Time-Invariant Systems**

Consider a discrete nonlinear time-invariant system, the autonomous equation is

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k = 0,1,2,\cdots \tag{4-58}$$

where $x \in R^{n \times n}$, $f(0) = 0$, i.e., the origin of the state space x=0 is an equilibrium state.

Next, we will present some Lyapunov stability theorems for discrete nonlinear time-invariant systems.

Theorem 4.18 For a discrete nonlinear time-invariant system (4-58), if there exists a scalar function $V(x(k))$ for discrete state $x(k)$ which meets the following items for any $x(k) \in R^n$:

(i) $V(x(k))$ is positive definite;

(ii) let $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$, $\Delta V(x(k))$ is negative;

(iii) $V(x(k)) \to \infty$ when $x(k) \to \infty$.

then the origin equilibrium state x=0 is large-scale asymptotically stable.

Note: The conservative property of ii may result in the failure of judgment for many systems. Thus, we can release this condition as follows.

Theorem 4.19 For a discrete nonlinear time-invariant system (4-58), if there exists a scalar function $V(x(k))$ for discrete state $x(k)$ which meets the following items for any $x(k) \in R^n$:

(i) $V(x(k))$ is positive definite;

(ii) let $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$, $\Delta V(x(k))$ is semi-negative;

(iii) $\Delta V(x(k))$ is not identically zero for any free dynamics started from any non-zero initial state $x(0) \in R^n$, i.e., equation (4-58);

(iv) $V(x(k)) \to \infty$ when $x(k) \to \infty$.

then the origin equilibrium state $x = 0$ is large-scale asymptotically stable.

Based on the above stability theorems, we can easily deduce a more intuitive and convenient stability criterion for discrete systems.

Theorem 4.20 For a discrete nonlinear time-invariant system (4-58), suppose $f(0)=0$, $x=0$ is an equilibrium state of the system, if $f(x(k))$ is convergent, i.e., for $x(k) \neq 0$ we have

$$\|f(x(k))\| < \|x(k)\| \tag{4-59}$$

then the origin equilibrium state $x=0$ is large-scale asymptotically stable.

Proof For a given discrete system, we select the Lyapunov function

$$V(x(k)) = \|x(k)\|$$

obviously, $V(x(k))$ is positive definite. Further, we can deduce that:

$$\begin{aligned}\Delta V(x(k)) &= V(x(k+1)) - V(x(k)) = \|x(k+1)\| - \|x(k)\| \\ &= \|f(x(k))\| - \|x(k)\|\end{aligned}$$

From equation (4-59), we can see that $\Delta V(x(k))$ is negative. And $V(x(k)) \to \infty$ when $x(k) \to \infty$. According to Theorem 4.18, the origin equilibrium state $x = 0$ is large-scale asymptotically stable. Proof done.

- **Stability Criteria for Discrete Linear Time-Invariant Systems**

Consider a discrete nonlinear time-invariant system, the autonomous equation is

$$x(k+1) = Gx(k), \quad x(0) = x_0, \quad k = 0,1,2,\cdots \tag{4-60}$$

where $x \in R^{n \times n}$, the solution state x_e of $Gx_e = 0$ is an equilibrium state. If the matrix G is singular, there are nonzero equilibrium states besides $x_e=0$. While if the matrix G is nonsingular, there is only one equilibrium state $x_e=0$.

Next, we will give the corresponding equilibrium states stability criteria for linear time-invariant systems.

Theorem 4.21 For a discrete linear time-invariant autonomous system (4-60), the origin equilibriums state $x_e=0$ is stable in the sense of Lyapunov if and only if all the amplitudes of the eigenvalues of $G: \lambda_i(G)(i = 1,2,\cdots,n)$ are equal to or less than 1, and the eigenvalue whose amplitude is 1 is the single root of the polynomial of G.

Theorem 4.22 For a discrete linear time-invariant autonomous system (4-60), the origin equilibriums state x_e=0 is asymptotically stable if and only if all the amplitudes of the eigenvalues of G: $\lambda_i(G)(i=1,2,\cdots,n)$ are less than 1.

Theorem 4.23 For an n-dimensional discrete linear time-invariant autonomous system (4-60), the origin equilibriums state x_e=0 is asymptotically stable, i.e., all the amplitudes of the eigenvalues of G: $\lambda_i(G)(i=1,2,\cdots,n)$ are less than 1, if and only if for any given $n \times n$ dimensional positive definite symmetry matrix Q the discrete Lyapunov function

$$G^T PG - P = -Q \tag{4-61}$$

has unique $n \times n$ dimensional positive definite symmetry solution matrix P.

Theorem 4.24 For an n-dimensional discrete linear time-invariant autonomous system (4-60), the origin equilibriums state x_e=0 is exponentially stable with the index of $\sigma > 0$, i.e., the eigenvalues of G satisfy

$$|\lambda_i(G)| < \sigma,\ 0 \leqslant \sigma \leqslant 1,\ i=1,2,\cdots,n \tag{4-62}$$

if and only if for any given $n \times n$ dimensional positive definite symmetry matrix Q the expanded discrete Lyapunov function

$$(1/\sigma)^2 G^T PG - P = -Q \tag{4-63}$$

has unique $n \times n$ dimensional positive definite symmetry solution matrix P.

Example 4.4 the state equation of a discrete linear system is

$$x(k+1) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x(k)$$

Try to determine the condition for the asymptotic stability of the equilibrium state.

Solution

According to $G^T PG - P = -I$, we have

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then

$$p_{11}(1-\lambda_1^2)=1;\quad p_{12}(1-\lambda_1\lambda_2)=0;\quad p_{22}(1-\lambda_2^2)=1$$

$$\text{So}\qquad P = \begin{bmatrix} \dfrac{1}{1-\lambda_1^2} & 0 \\ 0 & \dfrac{1}{1-\lambda_2^2} \end{bmatrix}$$

the condition for asymptotic stability of the equilibrium state are

$$|\lambda_1| < 1 \text{ and } |\lambda_2| < 1$$

MATLAB can be adopted for the solution of the above question.

```
P=dlyap(G',Q);
```

If P is positive definite, the system is asymptotically stable.

4.4 Summary

Stability is very important for a system. In this chapter, the definitions of stability in the sense of Lyapunov are given for equilibrium state. Different stable criteria are listed and proven for different kinds of system and examples are selected to show how to use the criteria.

Exercise

4.1 Determine whether the following functions are positive definite or not.

(1) $V(x)=2x_1^2+3x_2^2+x_3^2-2x_1x_2+2x_1x_3$

(2) $V(x)=\frac{1}{2}\left[(x_1+x_2)^2+2x_1^2+x_2^2\right]$

(3) $V(x)=x_1^2+x_3^2-2x_1x_2+x_2x_3$

(4) $V(x)=x_1^2+3x_2^2+11x_3^2-2x_1x_2+4x_2x_3+2x_1x_3$

4.2 Given a continuous-time nonlinear time-invariant system, try to analyze the stability of its equilibrium state:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^2x_2-x_1\end{aligned}$$

4.3 Consider a continuous-time nonlinear time-invariant system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1-x_2(1+x_2)^2\end{aligned}$$

Try to determine the stability of the origin equilibrium state $x_e=0$.

4.4 Consider a continuous-time linear time-invariant system

$$\dot{x}=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}x$$

Try to determine the stability of its equilibrium state.

4.5 Consider a continuous-time nonlinear time-invariant system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(1-|x_1|)x_2-x_1\end{aligned}$$

Try to analyze the stability of its equilibrium state.

4.6 Given the state-space equation

$$\dot{x}=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}x$$

Try to determine the stability of the origin equilibrium state $x_e=0$.

4.7 Given the state-space equation

$$\dot{x}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}x$$

Try to determine the stability of the equilibrium point.

4.8 Consider a continuous-time linear time-varying system

$$\dot{x}=\begin{bmatrix}0 & 1\\ -\dfrac{1}{t+1} & -10\end{bmatrix}x, \qquad t\geqslant 0$$

Try to determine whether the origin equilibrium state $x_e=0$ is large-scale asymptotically stable. (hint: let $V(x,t)=\frac{1}{2}\left[x_1^2+(1+t)x_2^2\right]$)

4.9 Try to analyze the BIBO stability and the asymptotical stability of the system equilibrium state $x_e=0$ of the following two systems:

(1) $\dot{x}=\begin{bmatrix}0 & 6\\ 1 & -1\end{bmatrix}x+\begin{bmatrix}-2\\ 1\end{bmatrix}u$

$y=\begin{bmatrix}0 & 1\end{bmatrix}x$

(2) $\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 250 & 0 & -5\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 10\end{bmatrix}u$

$y=\begin{bmatrix}-25 & 5 & 0\end{bmatrix}x$

4.10 Consider a linear discrete-time system

$$x(k+1)=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}x(k)$$

Try to determine the asymptotically stable condition for the equilibrium state.

4.11 Given a discrete-time linear time-invariant system

$$x(k+1)=\begin{bmatrix}1 & 4 & 0\\ -3 & -2 & -3\\ 2 & 0 & 0\end{bmatrix}x(k)$$

Use two methods to determine whether the system is asymptotically stable.

Chapter 5
Controllability and Observability

5.1 Introduction

This chapter introduces the concepts of controllability and observability. Controllability deals with whether or not the state of a state-space equation can be controlled from the input, and observability deals with whether or not the initial state can be observed from the output. These concepts can be illustrated using the network shown in Fig.5.1. In Fig. 5.1(a), the network has two state variables. Let x_i be the voltage across the capacitor with capacitance C_i, for $i=1,2$. The input u is a voltage source. From the network, it can be seen that, when, $C_1=C_2$, $R_1=R_2$, we always have $x_1=x_2$,the input u cannot change x_1 and x_2 to any value, i.e. the system is uncontrollable. In Fig. 5.1(b), when the initial value of x_1 and x_2 have $x_1(t)=x_2(t)$, the outputy can't reflect the value of $x_1(t)$ and $x_2(t)$. So the system is unobservable.

These concepts are essential in describing the internal structure of linear systems. They are also needed in studying control and filtering problems. In this chapter, we will discuss continuous-time linear time-invariant (LTI) state equations.

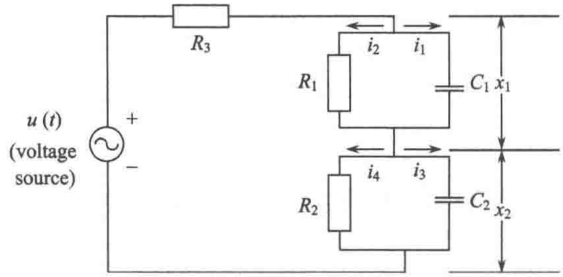


Fig.5.1 (a) Network

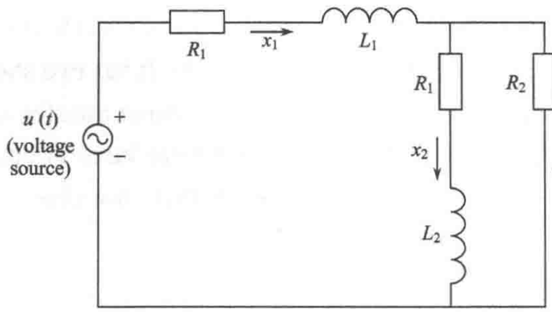


Fig.5.1 (b) Network

5.2 Definition

5.2.1 Controllability

Consider the n-dimensional p-input state equation

$$\dot{x} = Ax + Bu \tag{5-1}$$

where A and B are, respectively, $n \times n$ and $n \times p$ real constant matrices. Because the output does not play any role in controllability, we will disregard the output equation in this study.

Definition 5.1 The state equation (5-1) or the pair (A, B) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1, there exists an input that transfers x_0 to x_1 in a finite time. Otherwise (5-1) or (A, B) is said to be uncontrollable.

This definition requires only that the input be capable of moving any state in the state space to any other state in finite time; what trajectory the state should take is not specified. Furthermore, there is no constraint imposed on the input; its magnitude can be as large as desired.

Consider the n-dimensional p-input q-output state equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{5-2}$$

where A,B,C and D are, respectively, $n \times n$, $n \times p$, $q \times n$ and $q \times p$ constant matrices.

Example 5.1

Consider the network shown in Fig.5.2(a). Its state variable x is the voltage across the capacitor. If $x(0) = 0$,then $x(t) = 0$ for all $t \geqslant 0$ no matter what input is applied. This is due to the symmetry of the network, and the input has no effect on the voltage across the capacitor. Thus the system or, more precisely, the state equation that describes the system is not controllable.

Next we consider the network shown in Fig.5.2(b). It has two state variables x_1 and x_2. The input can transfer x_1 or x_2 to any values; but it cannot transfer x_1 and x_2 to any values. For example, if $x_1(0) = x_2(0) = 0$, then no matter what input is applied, $x_1(t)$ always equals $x_2(t)$ for all $t \geqslant 0$. Thus the equation that describes the network is not controllable.

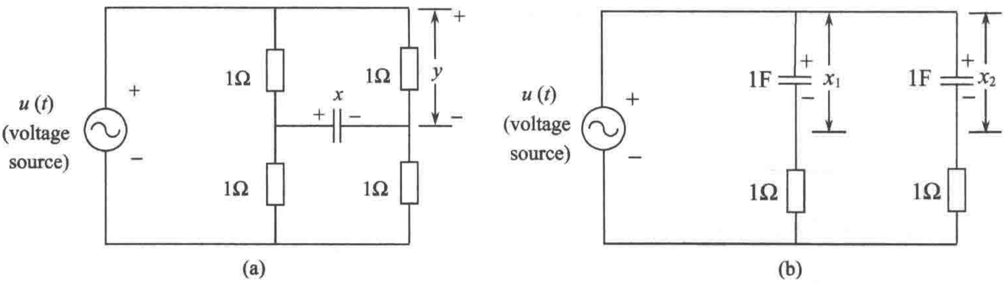


Fig.5.2 Uncontrollable networks

5.2.2 Observability

Definition 5.2 The state equation (5-2) is said to be observable if for any unknown initial state $x(0)$, there exists a finite $t_1 > 0$ such that the knowledge of the input u and the output y over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable.

Example 5.2 Consider the network shown in Fig.5.3. If the input is zero, no matter what the initial voltage across the capacitor is, the output is identically zero because of the symmetry of the four resistors. We know the input and output (both are identically zero), but we cannot determine uniquely the initial state. Thus the network or, more precisely, the state equation that describes the network is not observable.

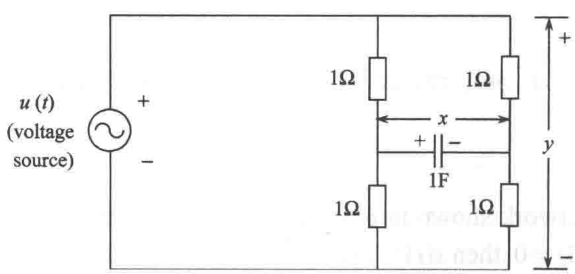


Fig.5.3 Unobservable network

The response of (5-2) excited by the initial state $x(0)$ and the input $u(t)$ is

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{5-3}$$

In the study of observability, the output y and the input u are assumed to be known; the initial state $x(0)$ is the only unknown one. Thus we can write (5-3) as

$$Ce^{At}x(0) = \overline{y}(t) \tag{5-4}$$

where

$$\overline{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

is a known function. Thus the observability problem reduces to solving $x(0)$ from (5-4). If $u \equiv 0$, then $\overline{y}(t)$ reduces to the zero-input response $Ce^{At}x(0)$. There fore definition 2 can be modified as follows: Equation (5-2) is observable if and only if the initial state $x(0)$ can be determined uniquely from its zero-input response over a finite time interval.

5.3 Criteria

5.3.1 Controllable Criteria

Consider the continuous linear time-invariant system, the state equation is expressed as:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \geqslant 0 \tag{5-5}$$

where $x \in R^n$; $u \in R^r$; $A_{n\times n}$, $B_{n\times r}$.

Theorem 5.1 [Controllability Gram-Matrix Criteria]

The system (5-5) is controllable if and only if the $n \times n$ matrix

$$W_c[0,t_1] \triangleq \int_0^{t_1} e^{-At}BB^Te^{-A^Tt}dt \tag{5-6}$$

is nonsingular for any $t_1 > 0$.

Proof First we show that if $W_c[0,t_1]$ is nonsingular, then (5-5) is controllable. For any non-zero state x_0, the response of (5-5) at time t_1 is derived as

$$\begin{aligned} x(t_1) &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-t)}Bu(t)dt \\ &= e^{At_1}x_0 - [e^{At_1}\int_0^{t_1} e^{-At}BB^Te^{A^Tt}dt]W_c^{-1}[0,t_1]x_0 \\ &= e^{At_1}x_0 - e^{At_1}W_c[0,t_1]W_c^{-1}[0,t_1]x_0 \\ &= e^{At_1}x_0 - e^{At_1}x_0 = 0, \quad \forall x_0 \in R^n \end{aligned} \tag{5-7}$$

This shows that all nonzero states in R^n are controllable. As defined, the system is completely controllable. We show the converse by contradiction. Suppose $W_c[0,t_1]$ is singular, then there exists a nonzero state $\overline{x}_0$ such that $\overline{x}_0{}^T W_c[0,t_1]x_0 = 0$, so we have

$$0 = \overline{x}_0{}^T W_c[0,t_1]x_0 = \int_0^{t_1} \overline{x}_0{}^T e^{-At} BB^T e^{-A^T t} \overline{x}_0 dt$$

$$= \int_0^{t_1} \left[B^T e^{-A^T t} \overline{x}_0 \right]^T \left[B^T e^{-A^T t} \overline{x}_0 \right] dt$$

$$= \int_0^{t_1} \left\| B^T e^{-A^T t} \overline{x}_0 \right\|^2 dt \tag{5-8}$$

which implies

$$B^T e^{-A^T t} \overline{x}_0 = 0, \quad \forall t \in [0, t_1]. \tag{5-9}$$

If (5-5) is controllable, there exists an input that ensures

$$0 = x(t_1) = e^{At_1} \overline{x}_0 + \int_0^{t_1} e^{At_1} e^{-At} Bu(t) dt \tag{5-10}$$

thus

$$\overline{x}_0 = -\int_0^{t_1} e^{-At} Bu(t) dt$$

$$\left\| \overline{x}_0 \right\|^2 = \overline{x}_0{}^T \overline{x}_0 = [-\int_0^{t_1} e^{-At} Bu(t) dt]^T \overline{x}_0 = -\int_0^{t_1} u^T(t) [B^T e^{-At} \overline{x}_0] dt \tag{5-11}$$

from equation (5-9), (5-11) can be derived as

$$\left\| \overline{x}_0 \right\|^2 = 0 \Rightarrow \overline{x}_0 = 0 \tag{5-12}$$

which contradicts $\overline{x}_0 \neq 0$. Thus $W_c[0,t_1]$ is nonsingular. Proof done.

Theorem 5.2 [Controllability Rank Criteria]

The system (5-5) is controllable if and only if the $n \times np$ controllability matrix

$$Q_c = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \tag{5-13}$$

has rank n (full row rank).

Proof First we show that if $rank Q_c = n$, then (5-5) is controllable.

Suppose the system is not completely controllable, then we can get from Gram-Matrix Criteria that the Gram matrix

$$W_c[0,t_1] \triangleq \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt, \quad \forall t_1 > 0 \tag{5-14}$$

is singular, which means that there exists a nonzero state α such that

$$0 = \alpha^T W_c[0,t_1]\alpha = \int_0^{t_1} \alpha^T e^{-At} BB^T e^{-A^T t} dt$$

$$= \int_0^{t_1} \left[\alpha^T e^{-At} B \right] \left[\alpha^T e^{-At} B \right]^T dt \tag{5-15}$$

therefore, we have

$$\alpha^T e^{-At} B = 0, \quad \forall t \in [0, \ t_1] \tag{5-16}$$

compute the $n-1$ order derivative of the above equation and let $t=0$, we have

$$\alpha^T B = 0, \alpha^T AB = 0, \alpha^T A^2 B = 0, \cdots, \alpha^T A^{n-1} B = 0$$

which equals

$$\alpha^T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = \alpha^T Q_c = 0 \tag{5-17}$$

Because $\alpha \neq 0$, we can know that all the rows of Q_c are linearly independent, equivalently, $rankQ_c < n$, which contradicts the hypothesis that $rankQ_c = n$. So the system is controllable. We also prove the converse by contradiction.

Suppose $rankQ_c < n$, then there exists a nonzero state α such that

$$\alpha^T Q_c = \alpha^T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = 0$$

which implies

$$\alpha^T A^i B = 0, \quad i = 0, 1, \cdots, n-1 \tag{5-18}$$

thus, for any $t_1 > 0$, we have

$$\pm \frac{A^i t^i}{i!} B = 0, \quad \forall t \in [0, t_1], \quad i = 0, 1, 2, \cdots$$

or

$$0 = \alpha^T \left[I - At + \frac{1}{2!} A^2 t^2 - \frac{1}{3!} A^3 t^3 + \cdots \right] B = \alpha^T e^{-At} B, \quad \forall t \in [0, t_1] \tag{5-19}$$

hence we can get

$$0 = \alpha^T \int_0^{t_1} e^{-At} BB^T e^{-A^T t} \alpha dt = \alpha^T W_c [0, t_1] \alpha \tag{5-20}$$

which indicates that the Gram matrix $W_c[0, t_1]$ is singular, the system is not totally controllable. This contradicts the hypothesis that the system is controllable. Proof done.

Theorem 5.3 [Controllability PBH Criteria]

The system (5-5) is controllable if and only if

$$rank[sI - A, B] = n, \quad \forall s \in \ell \tag{5-21}$$

or

$$rank[\lambda_i I - A, B] = n, \quad i = 1, 2, \cdots, n \tag{5-22}$$

where ℓ is pluralism field, $\lambda_i (i = 1, 2, \cdots, n)$ is the eigenvalue.

Proof First we prove that if (5-5) is controllable, then equation (5-21) and (5-22) are correct.

Suppose for a certain eigenvalue λ_i, there exists $rank[\lambda_i I - A, B] < n$, which implies that all the rows of Q_c are linearly independent. Hence, there must exist a non-zero n-dimensional constant vector α such that

$$\alpha^T\left[\lambda_i I - A, B\right] = 0 \tag{5-23}$$

that is, $\alpha^T A = \lambda_i \alpha^T$, $\alpha^T B = 0$.

Further, $\alpha^T B = 0$, $\alpha^T AB = \lambda_i \alpha^T B = 0, \cdots, \alpha^T A^{n-1}B = 0$, which equals

$$\alpha^T\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = a^T Q_c = 0 \tag{5-24}$$

Because $\alpha \neq 0$, we have $rankQ_c < n$.

From Rank Criteria, we know that the system is completely controllable. So the hypothesis does not establish. Besides, for all s in pluralism field ℓ except the eigenvalues λ_i, we have $rank\left[sI - A, B\right] = n$, so (5-22) equals (5-21).

Conversely, we suppose the system is not completely controllable, there must exist a linear nonsingular transformation which transforms (A, B) into the following form:

$$\begin{aligned} \overline{A} &= PAP^{-1} = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{\overline{c}} \end{bmatrix} \\ \overline{B} &= PB = \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix} \end{aligned} \tag{5-25}$$

where $(\overline{A}_c \in R^{n\times n}, \overline{B}_c \in R^{n\times p})$ and $(\overline{A}_{\overline{c}} \in R^{(n-h)\times(n-h)}, \overline{B}_c \in R^{(n-h)\times p})$ respectively denote the controllable part and uncontrollable part after decomposed.

$$\lambda_i = \text{an eigenvalue of } \overline{A}_{\overline{c}} = \text{an eigenvalue of } A$$

$$\overline{q_{\overline{c}}} \in \ell^{1\times(n-h)} i = \text{one left characteristic vector of } \lambda_i$$

On basis of this, we can construct a nonzero n-dimensional row vector

$$q^T = \left[0, \overline{q_{\overline{c}}}^T\right]P \cdot P^{-1}\begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix} = 0$$

$$\begin{aligned} q^T A &= \left[0, \overline{q_{\overline{c}}}^T\right]P \cdot P^{-1}\begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{\overline{c}} \end{bmatrix}P \\ &= \left[0, \overline{q_{\overline{c}}}^T \overline{A}_{\overline{c}}\right]P = \left[0, \lambda_i \overline{q_{\overline{c}}}^T\right]P = \lambda_i\left[0, \overline{q_{\overline{c}}}^T\right]P = \lambda_i q^T \end{aligned}$$

This shows that there exists a n-dimensional row vector $q^T = 0$ such that

$$q^T\left[\lambda_i I - A, B\right] = 0 \tag{5-26}$$

equivalently, there exists a $\lambda_i \in \ell$ such that

$$rank\left[\lambda_i I - A, B\right] < n \tag{5-27}$$

Obviously, this contradicts "$rank\left[sI - A, B\right] = n$, $\forall s \in \ell$", thus the hypothesis does not establish, and (5-5) is controllable. Proof done.

Theorem 5.4 [Controllability Jordan-Canonical Form Criteria I]

Consider system (5-5), suppose the n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are pair-wise differently, then the system is controllable if and only if the Jordan canonical of (5-5):

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \bar{x} + \bar{B}u \tag{5-28}$$

$\bar{B}$ does not contain zero row vector, which means each row vector of $\bar{B}$ satisfies

$$\bar{b}_i \neq 0, \qquad i = 1, 2, \cdots, n \tag{5-29}$$

Proof For the Jordan canonical (5-28), we construct the PBH Criteria matrix:

$$\left[sI - \bar{A}, \bar{B}\right] = \begin{bmatrix} s - \lambda_i & & & & \bar{b}_1 \\ & s - \lambda_2 & & & \bar{b}_2 \\ & & \ddots & & \vdots \\ & & & s - \lambda_n & \bar{b}_n \end{bmatrix} \tag{5-30}$$

From the unit structure which makes up the matrix, we have $s = \lambda_i, i \in [1, 2, \cdots, n]$, $rank[sI - A, B] = n$, if and only if "$b_i \neq 0, \forall i \in [1, 2, \cdots, n]$". Proof done.

Theorem 5.5 [Controllability Jordan-Canonical Form Criteria Π]

Consider system (5-5), suppose the n eigenvalues $\lambda_i(\sigma_1$ layers, α_1 layers$)$, $\lambda_i(\sigma_2$ layers, α_2 layers$)$, $\cdots, \lambda_l(\sigma_l$ layers, α_l layers$)$, and $\sigma_1 + \sigma_2 + \cdots + \sigma_l = n$, $\lambda_i \neq \lambda_j$, $\forall i \neq j$, then the system is controllable if and only if the Jordan canonical derived from linear nonsingular transformation of state equation (5-5):

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \tag{5-31}$$

where

$$\underset{n \times n}{\hat{A}} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{bmatrix}, \quad \underset{n \times p}{\hat{B}} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_l \end{bmatrix} \tag{5-32}$$

$$\underset{\sigma_i \times \sigma_i}{J_i} = i \begin{bmatrix} J_{i1} & & & \\ & J_{i2} & & \\ & & \ddots & \\ & & & J_{i\alpha_i} \end{bmatrix}, \quad \underset{\sigma_i \times p}{\hat{B}_i} = \begin{bmatrix} \hat{B}_{i1} \\ \hat{B}_{i2} \\ \vdots \\ \hat{B}_{i\alpha_i} \end{bmatrix} \tag{5-33}$$

$$
\underset{n_{ik}\times n_{ik}}{J_{ik}} = i\begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}, \quad \underset{n_k\times p}{\hat{B}_{ik}} = \begin{bmatrix} \hat{b}_{1ik} \\ \hat{b}_{2ik} \\ \vdots \\ \hat{b}_{rik} \end{bmatrix} \tag{5-34}
$$

satisfies such condition as: for $i=1,2,\cdots,l$, the last row vector of $\hat{B}_{i1},\hat{B}_{i2},\cdots,\hat{B}_{i\alpha_i}$ are all pair-wise linear independent, which means

$$
rank\begin{bmatrix} \hat{b}_{ri1} \\ \hat{b}_{ri2} \\ \vdots \\ \hat{b}_{ri\alpha_i} \end{bmatrix} = \alpha_i, \quad \forall i=1,2,\cdots,l \tag{5-35}
$$

Proof For simplicity, let

$$
\hat{A} = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_1 & 1 & & \\ & & & & \lambda_1 & & \\ & & & & & \lambda_2 & 1 \\ & & & & & & \lambda_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{b}_{111} \\ \hat{b}_{211} \\ \hat{b}_{r11} \\ \hat{b}_{112} \\ \hat{b}_{r12} \\ \hat{b}_{121} \\ \hat{b}_{r21} \end{bmatrix} \tag{5-36}
$$

where $\lambda_1 \neq \lambda_2$. For the above Jordan canonical, we construct the PBH Criteria matrix:

$$
\left[sI-\hat{A},\hat{B}\right] = \begin{bmatrix} s-\lambda_1 & -1 & & & & & & \hat{b}_{111} \\ & s-\lambda_1 & -1 & & & & & \hat{b}_{211} \\ & & s-\lambda_1 & & & & & \hat{b}_{r11} \\ & & & s-\lambda_1 & -1 & & & \hat{b}_{112} \\ & & & & s-\lambda_1 & & & \hat{b}_{r12} \\ & & & & & s-\lambda_2 & -1 & \hat{b}_{121} \\ & & & & & & s-\lambda_2 & \hat{b}_{r21} \end{bmatrix} \tag{5-37}
$$

We implies the Rank Criteria when $s=\lambda_1$, then get

$$\left[\lambda_1 I-\hat{A},\hat{B}\right]=\begin{bmatrix} 0 & -1 & & & & & & \hat{b}_{111} \\ & 0 & -1 & & & & & \hat{b}_{211} \\ & & 0 & & & & & \hat{b}_{r11} \\ & & & 0 & -1 & & & \hat{b}_{112} \\ & & & & 0 & & & \hat{b}_{r12} \\ & & & & & \lambda_1-\lambda_2 & -1 & \hat{b}_{121} \\ & & & & & & \lambda_1-\lambda_2 & \hat{b}_{r21} \end{bmatrix} \tag{5-38}$$

where $\lambda_1-\lambda_2 \neq 0$. Obviously, $\left[\lambda_1 I-\hat{A},\hat{B}\right]$ has full rank for rows, namely, $rank\left[\lambda_1 I-\hat{A},\hat{B}\right]=n=7$ if and only if

$$rank\begin{bmatrix}\hat{b}_{r11}\\ \hat{b}_{r12}\end{bmatrix}=\alpha_1=2 \tag{5-39}$$

Similarly, for $s=\lambda_2$, $\left[\lambda_1 I-\hat{A},\hat{B}\right]$ has full rank for rows: $rank\left[\lambda_1 I-\hat{A},\hat{B}\right]=n=7$ if and only if

$$rank\hat{b}_{r21}=\alpha_2=1 \tag{5-40}$$

Thereby equation (5-35) is proved. Proof done.

5.3.2 Controllable Examples

Example 5.3 Consider the controllability of the following continuous-time linear time-invariant system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 2\end{bmatrix}u,\ \ n=2$$

Solution

The controllability matrix is:

$$Q_c=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}1 & 4\\ 2 & -10\end{bmatrix}$$

Obviously, $rankQ_c=2=n$. According to Rank Criteria, the system is controllable.

Example 5.4 Consider the controllability of the following continuous-time linear time-invariant system

$$\dot{x}=\begin{bmatrix}-1 & -4 & -2\\ 0 & 6 & -1\\ 1 & 7 & -1\end{bmatrix}x+\begin{bmatrix}2 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}u,\ \ n=3$$

Solution

The controllability matrix is:

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 & * & * & * \\ 0 & 1 & -1 & * & * & * \\ 1 & 1 & 1 & * & * & * \end{bmatrix}$$

from the first three columns of Q_c,

$$\det \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \neq 0$$

$$rankQ_c = 3 = n$$

Thus, there is no need to compute the last three columns of Q_c. According to Rank Criteria, the system is controllable.

Example 5.5 Consider the controllability of the following continuous-time linear time-invariant system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -2 & 0 \end{bmatrix} u, \quad n = 4$$

Solution

First, compute the matrix

$$\begin{bmatrix} sI - A, B \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & 0 & 0 & 1 \\ 0 & s & 1 & 0 & 1 & 0 \\ 0 & 0 & s & -1 & 0 & 1 \\ 0 & 0 & -5 & s & -2 & 0 \end{bmatrix}$$

The eigenvalus of A are calculated as:

$$\lambda_1 = \lambda_2 = 0, \ \lambda_3 = \sqrt{5}, \ \lambda_4 = -\sqrt{5}$$

Next, we check the rank of $\begin{bmatrix} sI - A, B \end{bmatrix}$ for each eigenvalue. For $s = \lambda_1 = \lambda_2 = 0$,

$$rank\begin{bmatrix} sI - A, B \end{bmatrix} = rank \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & -2 & 0 \end{bmatrix}$$

$$= rank\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -5 & 0 & -2 \end{bmatrix} = 4 = n$$

for $s = \lambda_3 = \sqrt{5}$,

$$rank\left[sI - A, B\right] = rank\begin{bmatrix} \sqrt{5} & -1 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{5} & 1 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} & -1 & 0 & 1 \\ 0 & 0 & -5 & \sqrt{5} & -2 & 0 \end{bmatrix}$$

$$= rank\begin{bmatrix} \sqrt{5} & -1 & 0 & 1 \\ 0 & \sqrt{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 4 = n$$

for $s = \lambda_4 = -\sqrt{5}$,

$$rank\left[sI - A, B\right] = rank\begin{bmatrix} -\sqrt{5} & -1 & 0 & 0 & 0 & 1 \\ 0 & -\sqrt{5} & 1 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{5} & -1 & 0 & 1 \\ 0 & 0 & -5 & -\sqrt{5} & -2 & 0 \end{bmatrix}$$

$$= rank\begin{bmatrix} -\sqrt{5} & -1 & 0 & 1 \\ 0 & -\sqrt{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 4 = n$$

This shows that the given system qualifies the PBH Criteria, and it is controllable.

Example 5.6 Considering a continuous-time linear time-invariant system with pair-wise different eigenvalues, suppose the Jordan canonical sate equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Solution

We can see directly that the matrix $\overline{B}$ does not contain zero row vectors. From Jordan-Canonical Form Criteria I, the system is controllable.

Example 5.7 Considering a continuous-time linear time-invariant system with duplicate eigenvalues, suppose the Jordan canonical sate equation is

$$\dot{\hat{x}} = \begin{bmatrix} -2 & 1 & & & & & \\ 0 & -2 & & & & & \\ & & -2 & & & & \\ & & & -2 & & & \\ & & & & 3 & 1 & \\ & & & & 0 & 3 & \\ & & & & & & 3 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} u$$

Considering the last rows of Jordan blocks for $\lambda_1 = -2$ and $\lambda_2 = 3$, we find the corresponding rows in $\hat{B}$ and construct the following two matrices:

$$\begin{bmatrix} \hat{b}_{r11} \\ \hat{b}_{r12} \\ \hat{b}_{r13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} \hat{b}_{r21} \\ \hat{b}_{r22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Hence, we can see that both of them have full ranks. From Jordan-Canonical Form: Criteria Π, the system is controllable.

5.3.3 Observable Criteria

Consider the continue-time LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{5-41}$$

where A,B,C and D are, respectively, $n \times n$, $n \times p$, $q \times n$ and $q \times p$ constant matrices.

Theorem 5.6 [Gram Matrix Criteria]

The state equation (5-41) is observable if and only if the $n \times n$ matrix

$$W_o\left[0, t_1\right] \triangleq \int_0^{t_1} e^{A^T t} C^T C e^{At} dt \tag{5-42}$$

is nonsingular for any $t_1 > 0$.

Proof We pre-multiply (5-4) by $e^{A^T t} C^T$ and then integrate it over $\left[0, t_1\right]$ to yield

$$\left(\int_0^{t_1} e^{A^T t} C^T C e^{At} dt \right) x(0) = \int_0^{t_1} e^{A^T t} C^T \overline{y}(t) dt \tag{5-43}$$

If $W_o\left[0, t_1\right]$ is nonsingular, then

$$x(0) = W_o^{-1}\left[0, t_1\right] \int_0^{t_1} e^{A^T t} C^T \overline{y}(t) dt \tag{5-44}$$

This yields a unique $x(0)$. This shows that if $W_o[0,t_1]$, for any $t_1>0$, is nonsingular, then (5-41) is observable. Next we shown that if $W_o[0,t_1]$ is singular or, equivalently, positive semi-definite for all t_1, then (5-41) is not observable. If $W_o[0,t_1]$ is positive semi-definite, there exists an $n\times 1$ non-zero constant vector v such that

$$v^T W_o[0,t_1]v=\int_0^{t_1} v^T e^{A^T t}C^T Ce^{At}vdt=\int_0^{t_1}\left\|Ce^{At}v\right\|^2 dt=0$$

which implies

$$Ce^{At}v\equiv 0 \tag{5-45}$$

for all t in $[0,t_1]$. If $u\equiv 0$, then $x_1(0)=v\neq 0$ and $x_2(0)=0$ both yield the same

$$y(t)=Ce^{At}x_i(0)\equiv 0$$

Two different initial states yield the same zero-input response; therefore we can not uniquely determine $x(0)$. Thus (5-41) is not observable. This completes the proof of Theorem 1.

Theorem 5.7[Theorem of Duality]

The pair (A,B) is controllable if and only if the pair (A^T,B^T) is observable.

Poof The pair (A,B) is controllable if and only if

$$W_c[0,t_1]=\int_0^{t_1} e^{At}BB^T e^{A^T t}dt$$

is nonsingular for any t_1. The pair (A^T,B^T) is observable if and only if, by replacing A with A^T and C with B^T in (5-42), $W_o[0,t_1]=\int_0^{t_1} e^{At}BB^T e^{A^T t}dt$ is nonsingular for any t.

Theorem 5.8 [Rank Criteria]

The state equation (5-41) is observable if and only if the $nq\times n$ observability matrix

$$Q_o=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} \qquad \text{or} \qquad Q_o^T=\begin{bmatrix} C^T & \vdots A^T C^T & \vdots\cdots & \vdots\left(A^T\right)^{n-1}C^T\end{bmatrix}$$

has rank n.

Theorem 5.9 [Observable PBH Rank Criteria]

The state equation (5-41) is observable if and only if

$$rank\begin{bmatrix}\lambda_i I-A \\ C\end{bmatrix}=n, \qquad i=1,2,\cdots,n$$

at the eigenvalue, λ, of A.

Theorem 5.10 [Observable PBH Characteristic Vector Criteria]

The state equation (5-41) is observable if and only if there not exist orthogonal non-zero right characteristic vector for all row of matrix C in matrix A, equivalently, the only right characteristic vector at every eigenvalue λ, of A,that can satisfy the following equations

$$A\bar{\alpha} = \lambda_i\bar{\alpha}, \quad C\bar{\alpha} = 0$$

is $\bar{\alpha} = 0$.

Theorem 5.11 [Observable Jordan-canonical Form Criteria]

Assuming $\lambda_i \neq \lambda_j, \forall i \neq j$, the eigenvalues of system (5-41) are λ_i (σ_i layers, α_i layers) with i varying from 1 to l, and $(\sigma_1 + \sigma_2 + \cdots \sigma_l) = n$. The Jordan-canonical form of system is obtained by linearly nonsingular transformation.

$$\dot{\hat{x}} = \hat{A}\hat{x}$$
$$y = \hat{C}\hat{x}$$

where
$$\underset{(n\times n)}{\hat{A}} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{bmatrix}, \quad \underset{(q\times n)}{\hat{C}} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 & \cdots & \hat{C}_l \end{bmatrix}$$

$$\underset{(\sigma_i\times\sigma_i)}{J_i} = \begin{bmatrix} J_{i1} & & & \\ & J_{i2} & & \\ & & \ddots & \\ & & & J_{i\alpha_i} \end{bmatrix}, \quad \underset{(q\times\sigma_i)}{\hat{C}_i} = \begin{bmatrix} \hat{C}_{i1} & \hat{C}_{i2} & \cdots & \hat{C}_{i\alpha_i} \end{bmatrix}$$

$$\underset{(r_{ik}\times r_{ik})}{J_{ik}} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}, \quad \underset{(q\times r_{ik})}{\hat{C}_{ik}} = \begin{bmatrix} \hat{c}_{1ik} & \hat{c}_{2ik} & \cdots & \hat{c}_{rik} \end{bmatrix}$$

For $i = 1, 2, \cdots, l$, the first columns of $\hat{C}_{i1}, \hat{C}_{i2}, \cdots, \hat{C}_{i\alpha_i}$ are linearly independent, that is,

$$rank\begin{bmatrix} \hat{c}_{1i1} & \hat{c}_{1i2} & \cdots & \hat{c}_{1i\alpha_i} \end{bmatrix} = \alpha_i, \qquad \forall i = 1, 2, \cdots, l$$

5.3.4 Observable Examples

Example 5.8 Is the state equation

$$\dot{x} = \begin{bmatrix} -1 & -4 & -2 \\ 0 & 6 & -1 \\ 1 & 7 & -1 \end{bmatrix} x, \quad n=3$$

$$y = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} x$$

observable?

Solution

$$rankQ_o = rank \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 19 & -3 \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} = 3 = n$$

Obviously, we can know that the matrix Q_o has full rank from the first three rows. Therefore, the system is completely observable from **Rank Criteria** .

Example 5.9 Is the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x, \quad n=4$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} x$$

observable?

Solution

Firstly , we compute the eigenvalues of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & -5 & \lambda \end{vmatrix} = 0$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \sqrt{5}, \quad \lambda_3 = -\sqrt{5}$$

According to **Observable PBH Rank Criteria**, we compute

$$rank\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}_{\lambda=0} = rank\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} = 4 = n$$

$$rank\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}_{\lambda=\sqrt{5}} = rank\begin{bmatrix} \sqrt{5} & -1 & 0 & 0 \\ 0 & \sqrt{5} & 1 & 0 \\ 0 & 0 & \sqrt{5} & -1 \\ 0 & 0 & -5 & \sqrt{5} \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} = 4 = n$$

$$rank\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}_{\lambda=-\sqrt{5}} = rank\begin{bmatrix} -\sqrt{5} & -1 & 0 & 0 \\ 0 & -\sqrt{5} & 1 & 0 \\ 0 & 0 & -\sqrt{5} & -1 \\ 0 & 0 & -5 & -\sqrt{5} \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} = 4 = n$$

which satisfy **Observable PBH Rank Criteria.** Therefore, the system is completely observable.

Example 5.10 Consider a system with pair-wise different eigenvalues, suppose its Jordan-canonical form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} x$$

Examine the observability of the system.

Solution

We know the matrix $\bar{C}$ does not consist of a column whose elements are all zero. According to **Observable Jordan-canonical Form Criteria,** we can know the system is observable.

Example 5.11 Consider the LTI system with duplicate eigenvalues, suppose the Jordan-canonical form is as follows:

$$\dot{\hat{x}} = \begin{bmatrix} -2 & 1 & & & & & \\ 0 & -2 & & & & & \\ & & -2 & & & & \\ & & & -2 & & & \\ & & & & 3 & 1 & \\ & & & & 0 & 3 & \\ & & & & & & 3 \end{bmatrix} \hat{x}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 7 & 0 & 0 & 1 \end{bmatrix} \hat{x}$$

Solution

Consider the first column of two Jordan blocks for $\lambda = -2$ and $\lambda = 3$, find the corresponding columns from the matrix $\hat{C}$ and construct the following two matrices:

$$\begin{bmatrix} \hat{c}_{111} & \hat{c}_{112} & \hat{c}_{113} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} \hat{c}_{121} & \hat{c}_{122} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$$

We can know the two matrices both are column linearly independent. According to **Observable Jordan-canonical Form Criteria,** we can know the system is observable.

5.4 Duality System

5.4.1 Definition

There are two systems. The system $\sum_1$ is shown as follows:

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$y_1 = C_1 x_1$$

The other system $\sum_2$

$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_2 = C_2 x_2$$

If the following conditions are satisfied, the system $\sum_1$ and $\sum_2$ are duality systems.

$$A_2 = A_1^T, \quad B_2 = C_1^T, \quad C_2 = B_1^T \tag{5-46}$$

where x_1, x_2 are n-dimensional state vectors, u_1, u_2 are r-dimensional and m-dimensional control vectors respectively; y_1, y_2 are m-dimensional and r-dimensional output vectors respectively; A_1, A_2 are $n \times n$ system matrices; A_1, A_2 are $n \times r$ and $n \times m$ control matrices respectively, and C_1, C_2 are $m \times n$ and $r \times n$ output matrices respectively.

Obviously, the system $\sum_1$ is an n-order system with r inputs and m outputs, while its duality systems $\sum_2$ is a n-order system with m inputs and r outputs. The configurations of the duality systems $\sum_1$ and $\sum_2$ are shown in Fig. 5.4 .

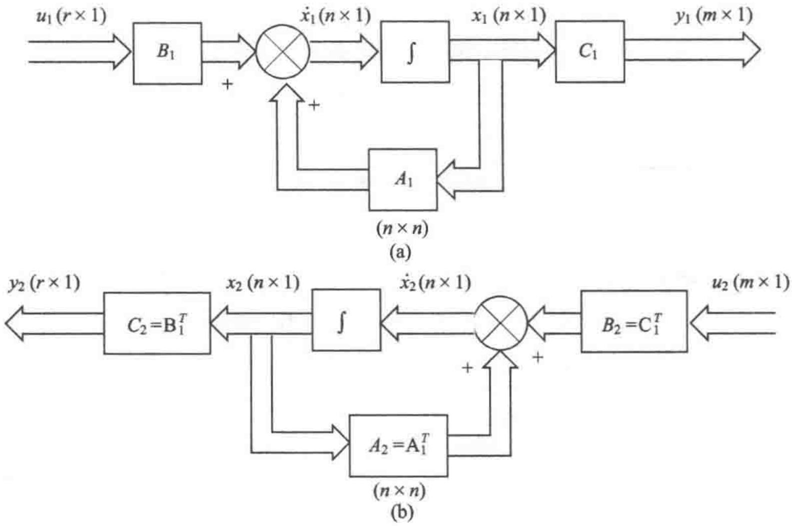


Fig.5.4 Configuration diagrams of duality systems

5.4.2 Properties of Duality Systems

Conclusion 5.1

No matter continuous-time system or discrete-time system, if the system $\sum_1$ is linear, its duality system $\sum_2$ is also linear, and if the system $\sum_1$ is time-variable or time-invariable, its duality system $\sum_2$ is also time-variable or time-invariable.

Conclusion 5.2

The transfer function matrices of the duality systems are mutually inverted.

Proof As shown in Fig.5.4(a), the transfer function matrix $W_1(s)$ of the system $\sum_1$ is a $m \times r$ matrix as follows.

$$W_1(s) = C_1 (sI - A_1)^{-1} B_1$$

As shown in Fig.5.4(b), the transfer function matrix $W_2(s)$ of the system $\sum_2$ is a $r \times m$ matrix as follows.

$$W_2(s) = C_2(sI - A_2)^{-1} B_2$$
$$= B_1^T (sI - A_1^T)^{-1} C_1^T$$
$$= B_1^T \left[(sI - A_1)^{-1}\right]^T C_1^T$$

Obviously, $$\left[W_2(s)\right]^T = C_1(sI - A_1)^{-1} B_1 = W_1(s)$$

The same way, we can know the input-state transfer function matrix $(sI - A_1)^{-1} B_1$ of the system $\sum_1$ and state - input transfer function matrix $C_2(sI - A_2)^{-1}$ of the system $\sum_2$ are mutually inverted, and the state - input transfer function matrix $C_1(sI - A_1)^{-1}$ of the system $\sum_1$ and input-state transfer function matrix $(sI - A_2)^{-1} B_2$ of the system $\sum_2$ are mutually inverted.

In addition, the characteristic equations of duality systems are the same, that is to say
$$|sI - A_2| = |sI - A_1^T| = |sI - A_1|$$

Conclusion 5.3

The system $\sum_1 = (A_1, B_1, C_1)$ and the system $\sum_2 = (A_2, B_2, C_2)$ are duality systems. The controllability of system $\sum_1$ is equivalent to the observability of system $\sum_2$, and the observability of system $\sum_1$ is equivalent to the controllability of system $\sum_2$. In other words, if the system $\sum_1$ is controllable or observable, the system $\sum_2$ is observable or controllable.

Proof The controllability matrix of system $\sum_2$ is
$$Q_{2c} = \begin{bmatrix} B_2 & A_2 B_2 & \cdots & A_2^{n-1} B_2 \end{bmatrix}$$
and its rank equals n. Therefore, the system $\sum_2$ is controllable.

The integration of the equation (5-46) in above equation results with the following equation:
$$Q_{2c} = \begin{bmatrix} C_1^T & A_1^T C_1^T & \cdots & (A_1^T) C_1^T \end{bmatrix} = Q_{1o}^T$$
where Q_{1o} is the observability matrix of system $\sum_1$.

This indicates the rank of the observability matrix of system $\sum_1$ is n, therefore, system $\sum_1$ is observable.

The same way, we can know
$$Q_{2o}^T = \begin{bmatrix} C_2^T & A_2^T C_2^T & \cdots & (A_2^T)^{n-1} C_2^T \end{bmatrix}$$
$$= \begin{bmatrix} B_1 & A_1 B_1 & \cdots & A_1^{n-1} B_1 \end{bmatrix} = Q_{1c}$$
If Q_{2o} has full rank, the system $\sum_2$ is observable, and Q_{1c} also has full rank, therefore, the system $\sum_1$ is controllable.

5.5 Canonical Form

This section discusses the controllability canonical forms and observability canonical forms of state equations.

5.5.1 Controllability Canonical Form of Single-Input Systems

Consider the n-dimensional time-invariant system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

if the system is controllable, we have

$$rank\begin{bmatrix} B & \vdots & AB & \cdots & A^{n-1}B \end{bmatrix} = n$$

hence there are at least n linearly independent n-dimensional column vectors in the controllability matrix, we select n linearly independent vectors from the nr column vectors and make some linear transformation, then we can get a certain controllability canonical whose columns are still linear independent. For single-input single-output systems, there are only one group of linearly independent vectors, so the controllability canonical form is unique. While for multi-input multi-output systems, there are multiple choices of n linearly independent vectors, so the form of controllability canonical is not unique. Obviously, the canonical forms exist if and only if the system is controllable.

- **Controllability Canonical Form** I

If the linear time-invariant single-input system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned} \tag{5-47}$$

is controllable, then there exists a linear nonsingular transformation

$$x = T_{c1}\overline{x}$$

$$T_{c1} = \begin{bmatrix} A^{n-1}b & A^{n-2}b & \cdots & b \end{bmatrix} \begin{bmatrix} 1 & & & & 0 \\ \alpha_{n-1} & 1 & & & \\ \vdots & & \ddots & & \\ \alpha_2 & \alpha_3 & & \ddots & \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \end{bmatrix} \tag{5-48}$$

which can transfer the state-space equation into

$$\begin{aligned}\dot{\overline{x}} &= \overline{A}\overline{x} + \overline{b}u \\ y &= \overline{C}\overline{x}\end{aligned} \tag{5-49}$$

where

$$\overline{A} = T_{c1}^{-1}AT_{c1} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}, \quad \overline{b} = T_{c1}^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{5-50}$$

$$\overline{C} = CT_{c1} = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} \tag{5-51}$$

Equation (5-49) is called the Controllability Canonical Form I, where $\alpha_i (i = 0,1,\cdots,n-1)$ are the coefficients of the following polynomial:

$$|\lambda I - A| = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$$

$\beta_i (i = 0,1,\cdots,n-1)$ are the results of CT_{c1}

$$\left.\begin{aligned} \beta_0 &= C(A^{n-1}b + \alpha_{n-1}A^{n-2}b + \cdots + \alpha_1 b) \\ &\vdots \\ \beta_{n-2} &= C(Ab + \alpha_{n-1}b) \\ \beta_{n-1} &= Cb \end{aligned}\right\} \tag{5-52}$$

Proof Suppose the system is controllable, then the $n \times 1$ vectors $b, Ab, \cdots, A^{n-1}b$ are linearly independent, the new vectors $e_1, e_2, \cdots, e_n$ in the following combination are also linearly independent:

$$\left.\begin{aligned} e_1 &= A^{n-1}b + \alpha_{n-1}A^{n-2}b + \alpha_{n-2}A^{n-3}b + \cdots + \alpha_1 b \\ e_2 &= A^{n-2}b + \alpha_{n-1}A^{n-3}b + \cdots + \alpha_2 b \\ &\vdots \\ e_{n-1} &= Ab + \alpha_{n-1}b \\ e_n &= b \end{aligned}\right\} \tag{5-53}$$

where $\alpha_i (i = 0,1,\cdots,n-1)$ are the coefficients of polynomial.

Thus the transformation matrix T_{c1} are composed of $e_1, e_2, \cdots, e_n$

$$T_{c1} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \tag{5-54}$$

As $\overline{A} = T_{c1}^{-1}AT_{c1}$, we have

$$T_{c1}\overline{A} = AT_{c1} = A\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} \tag{5-55}$$

Substituting equation (5-53) into the above equation yields

$$\begin{aligned} Ae_1 &= A(A^{n-1}b + \alpha_{n-1}A^{n-2}b + \cdots + \alpha_1 b) \\ &= (A^n b + \alpha_{n-1}A^{n-1}b + \cdots + \alpha_1 Ab + \alpha_0 b) - \alpha_0 b \\ &= -\alpha_0 b = -\alpha_0 e_n \end{aligned}$$

$$
\begin{aligned}
Ae_2 &= A(A^{n-2}b+\alpha_{n-1}A^{n-3}b+\cdots+\alpha_2 b)\\
&=(A^{n-1}b+\alpha_{n-1}A^{n-2}b+\cdots+\alpha_2 Ab+\alpha_1 b)-\alpha_1 b\\
&=e_1-\alpha_1 e_n\\
&\ \vdots\\
Ae_{n-1} &= A(Ab+\alpha_{n-1}b)\\
&=(A^2b+\alpha_{n-1}Ab+\alpha_{n-2}b)-\alpha_{n-2}b\\
&=e_{n-2}-\alpha_{n-2}e_n\\
Ae_n &= Ab=(Ab+\alpha_{n-1}b)-\alpha_{n-1}b=e_{n-1}-\alpha_{n-1}e_n
\end{aligned}
$$

then we substitute $Ae_1, Ae_2, \cdots, Ae_n$ into (5-55)

$$
\begin{aligned}
T_{c1}\overline{A} &= \begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} = \begin{bmatrix} -\alpha_0 e_n & (e_1-\alpha_1 e_n) & \cdots & (e_{n-1}-\alpha_{n-1}e_n) \end{bmatrix}\\
&= \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}\begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}
\end{aligned}
$$

Further we deduce $\overline{b}$. From $\overline{b}=T_{c1}{}^{-1}b$, we have $T_{c1}\overline{b}=b$. Substituting $b=e_n$ yields

$$
T_{c1}\overline{b}=e_n=\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}\begin{bmatrix} 0\\ 0\\ \vdots\\ 1 \end{bmatrix}
$$

Therefore,
$$
\overline{b}=\begin{bmatrix} 0\\ 0\\ \vdots\\ 1 \end{bmatrix}
$$

Finally, we deduce $\overline{C}$. From $\overline{C}=CT_{c1}$, we have

$$
\overline{C}=CT_{c1}=C\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}
$$

Substituting (5-53) into the above equation yields

$$
\begin{aligned}
\overline{C} &= C\begin{bmatrix} A^{n-1}b+\alpha_{n-1}A^{n-2}b+\alpha_{n-2}A^{n-3}b+\cdots+\alpha_1 b & \cdots & Ab+\alpha_{n-1}b & b \end{bmatrix}\\
&= \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0 &= C(A^{n-1}b+\alpha_{n-1}A^{n-2}b+\alpha_{n-2}A^{n-3}b+\cdots+\alpha_1 b)\\
&\ \vdots\\
\beta_{n-2} &= C(Ab+\alpha_{n-1}b)\\
\beta_{n-1} &= Cb
\end{aligned}
$$

We can derive the transfer function easily from the Controllability Canonical Form Ⅰ:

$$W(s)=\overline{C}(sI-\overline{A})^{-1}\overline{b}=\frac{\beta_{n-1}s^{n-1}+\beta_{n-2}s^{n-2}+\cdots+\beta_1 s+\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+\alpha_0} \tag{5-56}$$

From (5-56), we can see that the coefficients of denominator polynomial are the negative value of the elements of the last row of $\overline{A}$, and that the coefficients of numerator polynomial are the elements of $\overline{C}$. Hence, we can write out $\overline{A},\overline{b},\overline{C}$ directly from the coefficients of the denominator polynomial and numerator polynomial of the system transfer function.

- **Controllability Canonical Form Ⅱ**

If the linear time-invariant single-input system

$$\begin{aligned}\dot{x}&=Ax+bu\\ y&=Cx\end{aligned} \tag{5-57}$$

is controllable, then there exists a linear nonsingular transformation

$$x=T_{c2}\overline{x}=\begin{bmatrix}b & Ab & \cdots & A^{n-1}b\end{bmatrix}\overline{x} \tag{5-58}$$

which will transfer the state-space equation into

$$\begin{aligned}\dot{\overline{x}}&=\overline{A}\overline{x}+\overline{b}u\\ y&=\overline{C}\overline{x}\end{aligned} \tag{5-59}$$

where

$$\overline{A}=T_{c2}^{-1}AT_{c2}=\begin{bmatrix}0 & 0 & \cdots & 0 & -\alpha_0\\ 1 & 0 & \cdots & 0 & -\alpha_1\\ 0 & 1 & \cdots & 0 & -\alpha_2\\ \vdots & \vdots & & 0 & \vdots\\ 0 & 0 & \cdots & 1 & -\alpha_{n-1}\end{bmatrix},\quad \overline{b}=T_{c2}^{-1}b=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix} \tag{5-60}$$

$$\overline{C}=CT_{c2}=\begin{bmatrix}\beta_0 & \beta_1 & \cdots & \beta_{n-1}\end{bmatrix} \tag{5-61}$$

Equation (5-59) is called the Controllability Canonical Form Ⅱ, where α_0, $\alpha_1,\cdots$, α_{n-1} are the coefficients of the following polynomial:

$$\left|\lambda I-A\right|=\lambda^n+\alpha_{n-1}\lambda^{n-1}+\cdots+\alpha_1\lambda+\alpha_0$$

β_0, $\beta_1,\cdots$, β_{n-1} are the results of CT_{c2}

$$\left.\begin{aligned}\beta_0&=Cb\\ \beta_1&=CAb\\ &\vdots\\ \beta_{n-1}&=CA^{n-1}b\end{aligned}\right\} \tag{5-62}$$

Proof As the system is controllable, the controllability matrix

$$Q_c = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

is nonsingular. Let:

$$x = T_{c2}\overline{x}$$

where $T_{c2} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$, thus the sate equation and output equation after transformation are:

$$\dot{\overline{x}} = \overline{A}\overline{x} + \overline{b}u = T_{c2}^{-1}AT_{c2}\overline{x} + T_{c2}^{-1}bu$$
$$y = \overline{C}\overline{x} = CT_{c2}\overline{x}$$

First, we deduce $\overline{A}$.

$$AT_{c2} = A\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} Ab & A^2b & \cdots & A^n b \end{bmatrix} \tag{5-63}$$

According to Cayley-Hamilton Theorem, we have

$$A^n = -\alpha_{n-1}A^{n-1} - \alpha_{n-2}A^{n-2} - \cdots - \alpha_1 A - \alpha_0$$

Substituting the above equation into (5-63) yields

$$AT_{c2} = \begin{bmatrix} Ab & A^2b & \cdots & (-\alpha_{n-1}A^{n-1} - \alpha_{n-2}A^{n-2} - \cdots - \alpha_0)b \end{bmatrix} \tag{5-64}$$

Rewrite (5-64) into matrix form:

$$AT_{c2} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

thus

$$AT_{c2} = T_{c2} \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

multiply the above equation by T_{c2}^{-1}, then

$$\overline{A} = T_{c2}^{-1}AT_{c2} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix} \tag{5-65}$$

As $\overline{b} = T_{c2}^{-1}b$, equally $b = T_{c2}\overline{b} = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$. Obviously, in order to guarantee

(5-65), $\bar{b}$ should satisfy

$$\bar{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\bar{C} = CT_{c2} = \begin{bmatrix} Cb & CAb & \cdots & CA^{n-1}b \end{bmatrix}$$

that is,

$$\bar{C} = CT_{c2} = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}$$

5.5.2 Observability Canonical Form of Single-Output Systems

Similar to the condition for Controllability Canonical Form, the system has observability canonical form only when it is observable, that is,

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

The state-space equation have two types of observability canonical forms: Observability Canonical Form Ⅰ and Observability Canonical Form Π. They are respectively dual to the Controllability Canonical Form Ⅰ and Controllability Canonical Form Π.

- **Observability Canonical Form Ⅰ**

If the linear time-invariant single-output system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned} \tag{5-66}$$

is controllable, then there exists a linear nonsingular transformation

$$x = T_{o1}\tilde{x} \tag{5-67}$$

which will transfer the state-space equation (5-65) into

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{b}u \\ y &= \tilde{C}\tilde{x} \end{aligned} \tag{5-68}$$

where

$$\tilde{A} = T_{o1}^{-1}AT_{01} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}, \quad \tilde{b} = T_{o1}^{-1}b = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \tag{5-69}$$

$$\tilde{C} = CT_{o1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{5-70}$$

Equation (5-70) is called the Observability Canonical Form Ⅰ, where $\alpha_i (i = 0,1,\cdots,n-1)$ are the coefficients of the polynomial A. Define the inverse of the transformation matrix T_{o1}:

$$T_{o1}^{-1} = N = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5-71}$$

Proof We prove it by the Theory of Duality.

First we construct the duality system of $\sum = (A,b,C)$:

$$\begin{aligned} A^* &= A^T \\ b^* &= C^T \\ C^* &= b^T \end{aligned}$$

then we can have the Controllability Canonical Form Ⅱ of $\sum{}^* = (A^*,b^*,C^*)$. And the Observability Canonical Form Ⅰ of $\sum$ is just the Controllability Canonical Form Ⅱ of $\sum{}^*$, e.g.,

$$\begin{aligned} \tilde{A} &= A^* = \bar{A}^T \\ \tilde{b} &= b^* = \bar{C}^T \\ \tilde{C} &= C^* = \bar{b}^T \end{aligned}$$

where

$\bar{A},\bar{b},\bar{C}$ —the coefficient matrices of the Controllability Canonical Form Ⅱ of $\sum$.

$\tilde{A},\tilde{b},\tilde{C}$ —the coefficient matrices of the Observability Canonical Form Ⅰ of $\sum$.

A^*,b^*,C^* —the coefficient matrices of the Controllability Canonical Form Ⅱ of $\sum{}^*$.

- **Observability Canonical Form Ⅱ**

If the linear time-invariant single-output system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned} \tag{5-72}$$

is controllable, then there exists a linear nonsingular transformation

$$x = T_{o2}\tilde{x} \tag{5-73}$$

where

$$T_{o2} = \begin{bmatrix} 1 & \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1 \\ 0 & 1 & \cdots & \alpha_3 & \alpha_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} CA^{n-1} \\ CA^{n-2} \\ \vdots \\ CA \\ C \end{bmatrix} \tag{5-74}$$

The sate-space equation after transformation is:

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{b}u \\ y &= \tilde{C}\tilde{x} \end{aligned} \tag{5-75}$$

where

$$\tilde{A} = T_{o2}^{-1} A T_{02} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \vdots & -\alpha_2 \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}, \quad \tilde{b} = T_{o2}^{-1} b = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \tag{5-76}$$

$$\tilde{C} = C T_{o1} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \tag{5-77}$$

Equation (5-75) is called the Observability Canonical Form Ⅱ, where $\alpha_i (i = 0,1,\cdots,n-1)$ are the coefficients of the polynomial A, $\beta_i (i = 0,1,\cdots,n-1)$ are the results of $T_{o2}^{-1} b$ and is shown in (5-51).

The transformation above can be deduced directly by the Theory of Duality, the process is similar to that of the Observability Canonical Form Ⅰ. Besides, we can also derive the transfer function directly from the Controllability Canonical Form Ⅱ:

$$W(s) = \frac{\beta_{n-1} s^{n-1} + \beta_{n-2} s^{n-2} + \cdots + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \alpha_{n-2} s^{n-2} \cdots + \alpha_0} \tag{5-78}$$

where the coefficients of denominator polynomial are the negative value of the elements of the last column of $\tilde{A}$, and the coefficients of numerator polynomial are the elements of $\tilde{b}$.

5.5.3 Example

Example 5.12 Try to transform the following state-space equation into controllability canonical form Ⅰ

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

Solution

First we judge the controllability of the system,

$$Q_c = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 2 & 4 & 16 \\ 1 & 6 & 8 \\ 1 & 2 & 12 \end{bmatrix}$$

$rankQ_c = 3$, so the system is controllable.

Then we compute the characteristic polynomial

$$|\lambda I - A| = \lambda^3 - 9\lambda + 2$$

thus $\alpha_2 = 0, \alpha_1 = -9, \alpha_0 = 2$. From equations (5-49) and (5-50), we have

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 9 & 0 \end{bmatrix}$$

$$\bar{C} = C\begin{bmatrix} A^2b & Ab & b \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & 0 \\ \alpha_1 & \alpha_2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 16 & 4 & 2 \\ 8 & 6 & 1 \\ 12 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

so the controllability canonical form I of the system is

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 9 & 0 \end{bmatrix}\bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\bar{x}$$

From equation (5-56) we can write out the transfer function of the system:

$$W(s) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} = \frac{s^2 + 2s + 3}{s^3 - 9s + 2}$$

Example 5.13 Try to transform the state-space equation of Example 5.11 into controllability canonical form II.

Solution

As shown in Example 5.11, we know $\alpha_2 = 0, \alpha_1 = -9, \alpha_0 = 2$. From equations (5-60)~(5-61), we have

$$\bar{A} = \begin{bmatrix} 0 & 0 & -\alpha_0 \\ 1 & 0 & -\alpha_1 \\ 0 & 1 & -\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 9 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\bar{b}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\bar{C}=\begin{bmatrix}Cb & CAb & CA^2b\end{bmatrix}=\begin{bmatrix}1 & 2 & 12\end{bmatrix}$$

thus the controllability canonical form Ⅱ of the system is

$$\dot{\bar{x}}=\begin{bmatrix}0 & 0 & -2\\1 & 0 & 9\\0 & 1 & 0\end{bmatrix}\bar{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 2 & 12\end{bmatrix}\bar{x}$$

Example 5.14 Try to transform the state-space equation of Example 5.11 into observability canonical forms.

Solution

(1) Find the observability canonical form Ⅰ of the system

First we compute the observability matrix Q_o

$$Q_o=\begin{bmatrix}C\\CA\\CA^2\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\0 & 2 & 0\\6 & -2 & 2\end{bmatrix}$$

$rankQ_o=3$, so the system can be transformed into observability canonical forms. From equations (5-69) and (5-70), we have

$$\tilde{A}=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-2 & 9 & 0\end{bmatrix},\quad \tilde{b}=\begin{bmatrix}1\\2\\12\end{bmatrix},\quad \tilde{C}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

thus the observability canonical form Ⅰ of the system is

$$\dot{\tilde{x}}=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-2 & 9 & 0\end{bmatrix}\tilde{x}+\begin{bmatrix}1\\2\\12\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\tilde{x}$$

(2) Find the observability canonical form Ⅱ of the system

From equations (5-76) and (5-77), we have

$$\tilde{A}=\begin{bmatrix}0 & 0 & -2\\1 & 0 & 9\\0 & 1 & 0\end{bmatrix},\quad \tilde{b}=\begin{bmatrix}3\\2\\1\end{bmatrix},\quad \tilde{C}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

thus the observability canonical form Ⅱ of the system is

$$\dot{\tilde{x}}=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & 9\\ 0 & 1 & 0\end{bmatrix}\tilde{x}+\begin{bmatrix}3\\ 2\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\tilde{x}$$

5.5.4 Observability and Controllability Canonical Form of Multi-Input Multi-Output Systems

Consider the following transfer function

$$G(s)=\frac{B_{n-1}s^{n-1}+\cdots+B_1s+B_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0}+B_n \tag{5-79}$$

where B_i is $m\times r$ dimensional matrix.

The **controllable canonical form** is

$$\dot{x}=\begin{bmatrix}O_r & I_r & & & \\ O_r & & \ddots & & \\ \vdots & & & \ddots & \\ O_r & & & & I_r\\ -a_0I_r & & & & -a_{n-1}I_r\end{bmatrix}x+\begin{bmatrix}O_r\\ O_r\\ \vdots\\ O_r\\ I_r\end{bmatrix}u \tag{5-80}$$

$$y=\begin{bmatrix}B_0 & B_1 & \cdots & B_{n-1}\end{bmatrix}x+B_nu$$

where O_r and I_r are $r\times r$ dimensional zero matrix and unit matrix; r stands for dimension of input vector, n is the order of denominator polynomial.

Example 5.15 The system transfer function matrix is given below, try to give a controllable canonical form of the system

$$W(s)=\begin{bmatrix}\dfrac{s+2}{s+1} & \dfrac{1}{s+3}\\ \dfrac{s}{s+1} & \dfrac{s+1}{s+2}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{1}{s+3}\\ -\dfrac{1}{s+1} & -\dfrac{1}{s+2}\end{bmatrix}+\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

$$=\frac{\begin{bmatrix}s^2+5s+6 & s^2+3s+2\\ -(s^2+5s+6) & -(s^2+4s+3)\end{bmatrix}}{(s+1)(s+2)(s+3)}+\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

$$=\frac{\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}s^2+\begin{bmatrix}5 & 3\\ -5 & -4\end{bmatrix}s+\begin{bmatrix}6 & 2\\ -6 & -3\end{bmatrix}}{s^3+6s^2+11s+6}+\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Solution

For the system shown with the above model, we have n=3, m=r=2 and

$$a_0 = 6, \quad a_1 = 11, \quad a_2 = 6$$

$$D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 6 & 2 \\ -6 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5 & 3 \\ -5 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

So we get the matrix of the controllable canonical form as follows

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -6 & 0 & -11 & 0 & -6 & 0 \\ 0 & -6 & 0 & -11 & 0 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 2 & 5 & 3 & 1 & 1 \\ -6 & -3 & -5 & -4 & -1 & -1 \end{bmatrix}$$

The **observable canonical form** is

$$\dot{x} = \begin{bmatrix} 0_m & 0_m & \cdots & 0_m & 0_m & -\alpha_0 I_m \\ I_m & 0_m & \cdots & 0_m & 0_m & -\alpha_1 I_m \\ 0_m & I_m & \cdots & 0_m & 0_m & -\alpha_2 I_m \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0_m & 0_m & \cdots & 0_m & I_m & -\alpha_{n-1} I_m \end{bmatrix} x + \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n-1} \end{bmatrix} u \tag{5-81}$$

$$y = \begin{bmatrix} 0_m & 0_m & \cdots & I_m \end{bmatrix} x + B_n u$$

where 0_m and I_m are $m \times m$ dimensional zero matrix and unit matrix, m stands for dimension of output vector, n is the order of denominator polynomial.

5.6 System Decomposition

5.6.1 Controllability Decomposition

Suppose the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5-82}$$

is partly controllable. The controllability matrix is

$$Q_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

and

$$rankQ_c = n_1 < n$$

Therefore, there exists a nonsingular transformation

$$x = R_c\hat{x} \tag{5-83}$$

which can transfer the state equation into the following form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u \\ y &= \hat{C}\hat{x} \end{aligned} \tag{5-84}$$

where

$$\hat{x} = \begin{array}{c|l} \left[\begin{array}{c} \hat{x}_1 \\ \hline \hat{x}_2 \end{array}\right] & \begin{array}{l} n_1 \\ n - n_1 \end{array} \end{array}$$

$$\hat{A} = R_c^{-1}AR_c = \begin{array}{cl} \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline 0 & \hat{A}_{22} \end{array}\right] & \begin{array}{l} n_1 \\ n - n_1 \end{array} \\ \begin{array}{cc} n_1 & n - n_1 \end{array} & \end{array} \tag{5-85}$$

$$\hat{B} = R_c^{-1}B = \begin{array}{cl} \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] & \begin{array}{l} n_1 \\ n - n_1 \end{array} \end{array} \tag{5-86}$$

$$\hat{C} = CR_c\begin{array}{c} \left[\begin{array}{c|c} \hat{C}_1 & \hat{C}_2 \end{array}\right] \\ \begin{array}{cc} n_1 & n - n_1 \end{array} \end{array} \tag{5-87}$$

From the above equations, we can know that after the system is transformed into equation (5-84), the state-space description of the system is decomposed into controllable and uncontrollable parts. The n_1 -dimensional subspace $\dot{\hat{x}}_1 = \hat{A}_{11}\hat{x}_1 + \hat{B}_1u + \hat{A}_{12}\hat{x}_2$ is controllable, while the $(n - n_1)$-dimensional subspace $\dot{\hat{x}}_2 = \hat{A}_{22}\hat{x}_2$ is uncontrollable. The state decomposition is shown in Fig.5.5. Because $\hat{x}_2$ can not be detected from

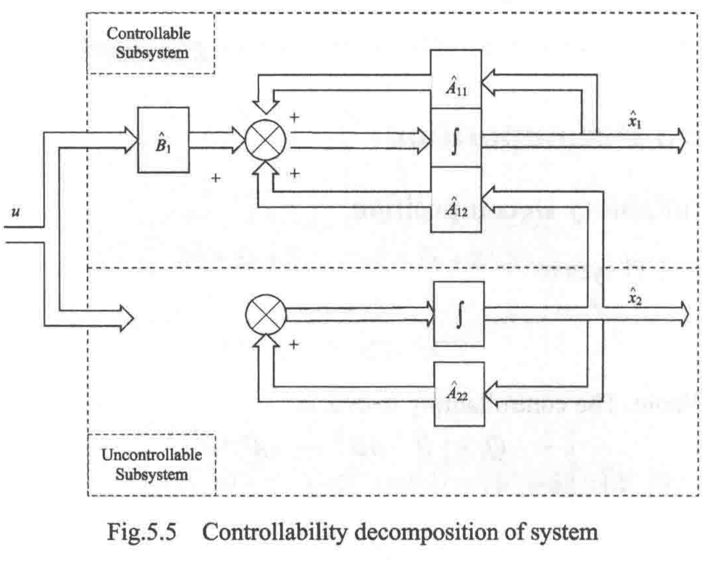


Fig.5.5 Controllability decomposition of system

u, $\hat{x}_2$ only has uncontrollable free response. Obviously, if we neglect the $(n-n_1)$-dimensional subsystem, we can obtain a controllable system with less dimension.

We form the nonsingular transfer matrix

$$R_c = \begin{bmatrix} R_1 & R_2 & \cdots & R_{n_1} & \cdots & R_n \end{bmatrix} \tag{5-88}$$

where the first n_1 columns are any linearly independent columns of controllability matrix Q_c, and the remaining columns can be chosen as long as R_c is nonsingular.

Example 5.16 Is the state equations

$$\dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x$$

controllable? If not, please write the controllability decomposition of the system.

Solution

The controllability matrix is

$$Q_c = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

and $rankQ_c = 2 < n$, therefore, the system is partly controllable.

We form the nonsingular transfer matrix as equation (5-88).

$$R_1 = b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad R_2 = Ab = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad R_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

thus,

$$R_c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

where R_3 is chosen arbitrarily as long as R_c is nonsingular.

After the transformation, the new state equation is

$$\dot{\hat{x}} = R_c^{-1} A R_c \hat{x} + R_c^{-1} b u$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$
= \left[\begin{array}{cc|c} 0 & -1 & -1 \\ 1 & -2 & -2 \\ \hline 0 & 0 & -1 \end{array}\right] \hat{x} + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u
$$

$$
y = CR_c\hat{x} = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \hat{x}
$$

If we choose

$$
R_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}
$$

Thus,

$$
\dot{\hat{x}} = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & -2 & -2 \\ \hline 0 & 0 & -1 \end{array}\right] \hat{x} + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u
$$

$$
y = CR_c\hat{x} = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \hat{x}
$$

5.6.2 Observability Decomposition

Suppose the LTI system

$$
\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5-89}
$$

is partly observable. The observability matrix is

$$
Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
$$

and $\qquad rankQ_o = n_1 < n$

Therefore, there exists a nonsingular transformation

$$
x = R_o\tilde{x} \tag{5-90}
$$

which can transfer the state equation into the following form

$$
\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}u \\ y &= \tilde{C}\tilde{x} \end{aligned} \tag{5-91}
$$

where

$$
\tilde{x} = \left[\begin{array}{c} \tilde{x}_1 \\ \hline \tilde{x}_2 \end{array}\right] \begin{array}{l} n_1 \\ n - n_1 \end{array}
$$

$$\tilde{A} = R_o^{-1} A R_o = \begin{array}{c} \left[\begin{array}{c:c} \tilde{A}_{11} & 0 \\ \hdashline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \begin{array}{l} n_1 \\ n-n_1 \end{array} \\ \begin{array}{cc} n_1 & n-n_1 \end{array} \end{array} \tag{5-92}$$

$$\tilde{B} = R_o^{-1} B = \left[\begin{array}{c} \tilde{B}_1 \\ \hdashline \tilde{B}_2 \end{array}\right] \begin{array}{l} n_1 \\ n-n_1 \end{array} \tag{5-93}$$

$$\tilde{C} = C R_o \begin{array}{c} \left[\begin{array}{c:c} \tilde{C}_1 & 0 \end{array}\right] \\ \begin{array}{cc} n_1 & n-n_1 \end{array} \end{array} \tag{5-94}$$

From the above equations, we can know when the system is transferred into equation (5-91), the state-space description of the system is decomposed into observable and unobservable parts. The n_1-dimensional subspace

$$\dot{\tilde{x}}_1 = \tilde{A}_{11}\tilde{x}_1 + \tilde{B}_1 u$$
$$y = \tilde{C}_1 \tilde{x}_1$$

is observable, while the $(n-n_1)$-dimensional subspace

$$\dot{\tilde{x}}_2 = \tilde{A}_{21}\tilde{x}_1 + \tilde{A}_{22}\tilde{x}_2 + \tilde{B}_2 u$$

is unobservable. The state decomposition is shown in Fig.5.6. Obviously, if we do not consider the $(n-n_1)$-dimensional unobservable subsystem, we can obtain a n_1-dimensional observable system.

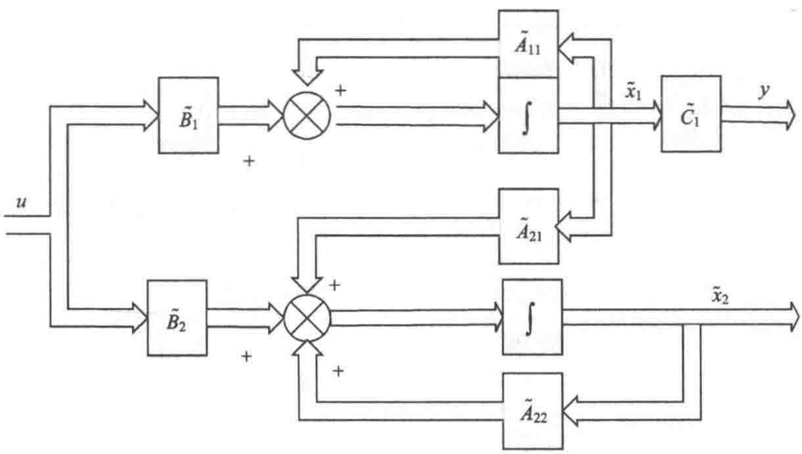


Fig.5.6 Observability decomposition of a system

We form the nonsingular transfer matrix

$$R_o^{-1} = \begin{bmatrix} R_1' \\ R_2' \\ \vdots \\ R_{n_1}' \\ \vdots \\ R_n' \end{bmatrix} \tag{5-95}$$

where the first n_1 rows are any linearly independent rows of observability matrix Q_o, and the remaining rows can be chosen as long as R_o^{-1} is nonsingular.

Example 5.17 Is the state equations

$$\dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x$$

observable? If not, please write the observability decomposition of the system.

Solution

The observability matrix is

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix}$$

and $rankQ_o = 2 < n$, therefore, the system is partly observable.

We form the nonsingular transfer matrix as equation (5-95).

$$R_1' = C = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \quad R_2' = CA = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \quad R_3' = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

thus, $R_o^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $R_o = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u$

where R_3 is chosen arbitrarily as long as R_o^{-1} is nonsingular.

After the transformation, the new state equation is

$$\dot{\tilde{x}} = R_o^{-1} A R_o \tilde{x} + R_o^{-1} b u$$

$$= \left[\begin{array}{cc:c} 0 & -1 & 0 \\ -1 & -2 & 0 \\ \hdashline 1 & 0 & -1 \end{array}\right] \tilde{x} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u$$

$$y = CR_o \tilde{x} = \left[\begin{array}{cc:c} 1 & 0 & 0 \end{array}\right] \tilde{x}$$

5.6.3 Controllability and Observability Decomposition

1) Suppose the LTI system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned} \tag{5-96}$$

is partly controllable and observable. Therefore, there exists a nonsingular transformation

$$x = R\overline{x} \tag{5-97}$$

which can transfer the state equation into the following form

$$\begin{aligned}\dot{\overline{x}} &= \overline{A}\overline{x} + \overline{B}v \\ y &= \overline{C}\overline{x}\end{aligned} \tag{5-98}$$

where

$$\overline{A} = R^{-1}AR = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \tag{5-99}$$

$$\overline{B} = R^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \tag{5-100}$$

$$\overline{C} = CR = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} \tag{5-101}$$

From the configuration of $\overline{A}, \overline{B}, \overline{C}$, we can know the n-dimensional state space is divided into four subspaces according to the controllability and observability of the system. The equation (5-98) can be rewritten as follows:

$$\begin{aligned}\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\overline{o}} \\ \dot{x}_{\overline{c}o} \\ \dot{x}_{\overline{c}\overline{o}} \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\overline{o}} \\ x_{\overline{c}o} \\ x_{\overline{c}\overline{o}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\overline{o}} \\ x_{\overline{c}o} \\ x_{\overline{c}\overline{o}} \end{bmatrix}\end{aligned} \tag{5-102}$$

and the subsystem (A_{11}, B_1, C_1) is controllable and observable.

The block diagram of equation (5-98) is shown in Fig.5.7.

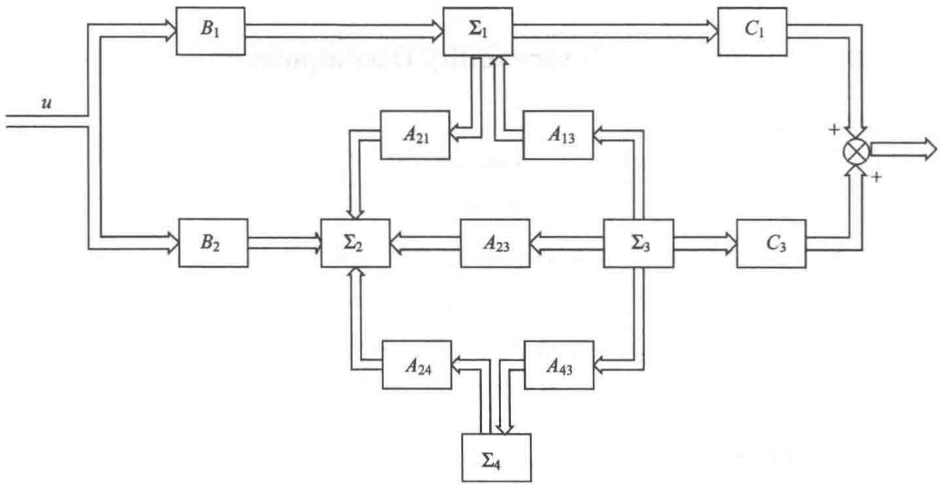


Fig.5.7 The block diagram of equation (5-98)

From the Fig.5.7, we can know the condition for the transfer of the information between the four subsystems.

2) The method of determining matrix R

Step 1: The transfer matrix

$$x = R_c \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} \tag{5-103}$$

will transform system $\sum = (A, B, C)$ into

$$\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} &= R_c^{-1} A R_c \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + R_c^{-1} B u \\ &= \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_4 \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix} u \\ y &= C R_c \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} \end{aligned} \tag{5-104}$$

where x_c is controllable state; $x_{\bar{c}}$ is uncontrollable state; and R_c is formed according to equation (5-88).

Step 2: The transfer matrix $x_{\bar{c}} = R_{o2} \begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$ will transform the uncontrollable subsystem $\sum_{\bar{c}} = \left(\bar{A}_4, 0, \bar{C}_2 \right)$ into

$$\begin{bmatrix} \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = R_{o2}^{-1} \bar{A}_4 R_{o2} \begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{33} & 0 \\ A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

$$y_2 = \overline{C}_2 R_{o2}\begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} C_3 & 0 \end{bmatrix}\begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

where $x_{\bar{c}o}$ is uncontrollable but observable state; $x_{\bar{c}\bar{o}}$ is uncontrollable and unobservable state;and R_{o2} is formed according to equation (5-95) for system $\sum_{\bar{c}} = \left(\overline{A}_4, 0, \overline{C}_2\right)$.

Step 3: The transformation $x_c = R_{o1}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$ will transform the controllable subsystem $\sum_c = \left(\overline{A}_1, B, \overline{C}_1\right)$ according to observability.

According to equation (5-104), we can obtain

$$\dot{x}_c = \overline{A}_1 x_c + \overline{A}_2 x_{\bar{c}} + Bu$$

Substitute the state transfer equations into the above equation:

$$R_{o1}\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = \overline{A}_1 R_{o1}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} + \overline{A}_2 R_{o2}\begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Bu$$

Multiply above equation by R_{o1}^{-1}, we will have

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = R_{o1}^{-1}\overline{A}_1 R_{o1}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} + R_{o1}^{-1}\overline{A}_2 R_{o2}\begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + R_{o1}^{-1}Bu$$

$$= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} + \begin{bmatrix} A_{13} & 0 \\ A_{23} & A_{24} \end{bmatrix}\begin{bmatrix} x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u$$

$$y = \overline{C}R_{o1}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

where x_{co} is controllable and observable state; $x_{c\bar{o}}$ is controllable but observable state; and R_{o1} is formed according to equation (5-95) for system $\sum_c = \left(\overline{A}_1, B, \overline{C}_1\right)$.

After the above three transformations, we can have the decomposition description as follows according to controllability and observability.

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix}\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

Example 5.18 The LTI system

$$\dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -3 \end{bmatrix} x$$

is partly controllable and observable. Give the controllability and observability decomposition.

Solution

$$R_c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

After transformation, we have

$$\begin{bmatrix} \dot{x}_c \\ \\ \dot{x}_{\bar{c}} \end{bmatrix} = \left[\begin{array}{cc:c} 0 & -1 & -1 \\ 1 & -2 & -2 \\ \hdashline 0 & 0 & -1 \end{array}\right] \begin{bmatrix} x_c \\ \\ x_{\bar{c}} \end{bmatrix} + \left[\begin{array}{c} 1 \\ 0 \\ \hdashline 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc:c} 1 & -1 & -2 \end{array}\right] \left[\begin{array}{c} x_c \\ \hdashline x_{\bar{c}} \end{array}\right]$$

From the above, we can know the uncontrollable subspace $x_{\bar{c}}$ is one-dimensional. Obviously, the subspace is observable, therefore, there is no need to decompose the subspace.

The controllable subsystem $\sum_c$ is

$$\dot{x}_c = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} x_c + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_{\bar{c}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} x_c$$

Then, we decompose the subsystem $\sum_c$ according to observability.

We form the nonsingular matrix according to the equation (5-95):

$$R_o^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

which transfer the system $\sum_c$ into

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

$$+\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\-2\end{bmatrix}x_{\bar{c}}+\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}u$$

equivalently,

$$\begin{bmatrix}\dot{x}_{co}\\\dot{x}_{c\bar{o}}\end{bmatrix}=\begin{bmatrix}-1 & 0\\1 & -1\end{bmatrix}\begin{bmatrix}x_{co}\\x_{c\bar{o}}\end{bmatrix}+\begin{bmatrix}1\\-2\end{bmatrix}x_{\bar{c}}+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y_1=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}x_{co}\\x_{c\bar{o}}\end{bmatrix}=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_{co}\\x_{c\bar{o}}\end{bmatrix}$$

With the above two transformations, we have the decomposition description:

$$\begin{bmatrix}\dot{x}_{co}\\\dot{x}_{c\bar{o}}\\\dot{x}_{\bar{c}o}\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1\\1 & -1 & -2\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_{co}\\x_{c\bar{o}}\\x_{\bar{c}o}\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & -2\end{bmatrix}\begin{bmatrix}x_{co}\\x_{c\bar{o}}\\x_{\bar{c}o}\end{bmatrix}$$

3) Another method of decomposition

First, we can transfer the system into Jordan-canonical form, then examine the controllability and observability of all state variables according to controllability and observability criteria. Finally, we form the corresponding subsystems with those state variables.

For example, the given Jordan-canonical form of system $\Sigma=(A,B,C)$ is

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\\\dot{x}_5\\\dot{x}_6\end{bmatrix}=\begin{bmatrix}-4 & 1 & & & & \\0 & -4 & & & & \\ & & 3 & 1 & & \\ & & 0 & 3 & & \\ & & & & -1 & 1\\ & & & & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}+\begin{bmatrix}1 & 3\\5 & 7\\4 & 3\\0 & 0\\1 & 6\\0 & 0\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}3 & 1 & 0 & 5 & 0 & 0\\1 & 4 & 0 & 2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}$$

According to controllability and observability criteria of Jordan-canonical form , we know that x_1, x_2 are controllable and observable variables, x_3, x_5 are controllable but unobservable variables , x_4 are uncontrollable and observable variables and x_6 are uncontrollable and unobservable variables and.

Thus , $x_{co} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_{c\bar{o}} = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$ $x_{\bar{c}o} = x_4$ $x_{\bar{c}\bar{o}} = x_6$

Rearrange according to this order, and you can get

$$\begin{bmatrix} \dot{x}_{co} \\ \\ \dot{x}_{c\bar{o}} \\ \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \left[\begin{array}{cc|cc|c|c} -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} x_{co} \\ \\ x_{c\bar{o}} \\ \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \left[\begin{array}{cc} 1 & 3 \\ 5 & 7 \\ \hline 4 & 3 \\ 1 & 6 \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\tilde{A}_m = P^{-1} A_m P$$

5.6.4 Minimum Realization

- **Definition of Realization**

Given a transfer matrix $W(s)$, if there exists a state-space equation $\sum$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

that has $W(s)$ as its transfer matrix, e.g.,

$$C(sI - A)^{-1} B + D = W(s)$$

then $W(s)$ is realizable and $\sum$ is one of the realizations of $W(s)$.

It is noticeable that not every transfer matrix $W(s)$ is realizable, $W(s)$ is physically realizable if it qualifies the following items:

(1) All the coefficients of the numerator and denominator polynomials of each element $W_{ik}(s)(i = 1, 2, \cdots, m; k = 1, 2, \cdots, r)$ are real constants.

(2) $W_{ik}(s)$ is real rational fraction of s, i.e, the order of the numerator polynomial is no more than that of the denominator polynomial. When all the elements of $W(s)$ are strictly real rational fractions, the realization of $W(s)$ has the form of (A, B, C). Apart from this, the realization has the form of (A, B, C, D) and $D = \lim\limits_{s \to 0} W(s)$.

- **Minimum Realization**

If a transfer function is realizable, then it has infinitely many realizations, not necessarily of the same dimension. From the engineering point of view, it is significant to find the class of minimal-dimensional realizations of the system.

(1) Definition of Minimum Realization

Consider one realization of the transfer function $W(s)$, $\sum$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

if any other realization

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$y = \tilde{C}\tilde{x}$$

has more dimension than $\sum$,then $\sum$ is the Minimum Realization of the system.

Because the transfer matrix can only reflect the dynamic behaviors of the controllable and observable subsystem, removing the uncontrollable or unobservable states will not change the transfer matrix of the system. Thus, the state-space expression with uncontrollable or unobservable states can not be the minimum realization. As stated above, we have the following methods to verify the minimum realization.

(2) Steps to find minimum realization

Theorem 5.12 The realization of transfer matrix $W(s)$, $\sum$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is the minimum realization if and only if $\sum(A, B, C)$ is controllable and observable.

According to this theorem, we can find the minimum realization of any transfer matrix $W(s)$ whose elements are all strictly real rational fractions. Usually, we can obtain the minimum realization as follows:

1) For a given transfer matrix $W(s)$, first we select one realization $\sum(A, B, C)$. More often we choose the controllability canonical or observability canonical for the sake of convenience.

2) For the $\sum(A, B, C)$ chosen above, we find its controllable and observable part $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1)$. So this part is just the minimum realization of the system.

Example 5.19 Try to find the minimum realization of the transfer matrix

$$W(s) = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{(s+2)(s+3)} \end{bmatrix}$$

Solution

$W(s)$ is a strictly real rational fraction of s, rewrite it in the descending order of s as the following form:

$$W(s) = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)(s+3)} & \dfrac{s+1}{(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$=\frac{1}{(s+1)(s+2)(s+3)}\left[(s+3) \quad (s+1)\right]$$

$$=\frac{1}{s^3+6s^2+11s+6}\{[1 \quad 1]s+[3 \quad 1]\}$$

From equation (5-56), we can know that

$$\alpha_0=6,\alpha_1=11,\alpha_2=6$$

$$\beta_0=[3 \quad 1],\beta_1=[1 \quad 1],\beta_2=[0 \quad 0]$$

the dimension of the output vector is $m=1$, and the dimension of the input vector is $r=2$: First we adopt the controllability canonical form realization.

$$A_0=\begin{bmatrix} 0_m & 0_m & -\alpha_0 I_m \\ I_m & 0_m & -\alpha_1 I_m \\ 0_m & 0_m & -\alpha_2 I_m \end{bmatrix}=\begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$B_0=\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C_0=\begin{bmatrix} 0_m & 0_m & I_m \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

then we check whether the realization $\sum=(A_0,B_0,C_0)$ is controllable or not.

$$Q_c=[B_0 \quad A_0B_0 \quad A_0^2B_0]=\begin{bmatrix} 3 & 1 & 0 & 0 & -6 & -6 \\ 1 & 1 & 3 & 1 & -11 & -11 \\ 0 & 0 & 1 & 1 & -3 & -5 \end{bmatrix}$$

$$rankQ_c=3=n$$

therefore, $\sum=(A_0,B_0,C_0)$ is controllable and observable, so it is the minimum realization.

Example 5.20 Try to find the minimum realization of the transfer matrix

$$W(s)=\begin{bmatrix} \dfrac{s+2}{s+1} & \dfrac{1}{s+1} \\ \dfrac{s}{s+1} & \dfrac{s+1}{s+2} \end{bmatrix}$$

Solution

First, we simplify $W(s)$ into the form of strictly real rational function and write its controllability canonical form (or observability canonical form). After computing, the controllability canonical form of the system is:

$$A=\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\-6&0&-11&0&-6&0\\-6&-3&-5&-4&-1&-1\end{bmatrix};\quad B=\begin{bmatrix}0&0\\0&0\\0&0\\0&0\\1&0\\0&1\end{bmatrix}$$

$$C=\begin{bmatrix}6&2&5&3&1&1\\-6&-3&-5&-4&-1&-1\end{bmatrix};\quad D=\begin{bmatrix}1&0\\1&1\end{bmatrix}$$

Step 2: Examine whether the states realized by the controllability canonical form are observable or not

$$Q_o=\begin{bmatrix}C\\CA\\CA^2\end{bmatrix}=\begin{bmatrix}6&2&5&3&1&1\\-6&-3&-5&-4&-1&-1\\-6&-6&-5&-9&-1&-3\\6&6&5&8&1&2\\6&18&5&27&1&9\\-6&-12&-5&-16&-1&-4\end{bmatrix}$$

as $rankQ_o=3<n=6$, the controllability canonical form is not the minimum realization. Thus we decompose the structure according to the observability.

Step 3: Construct the transfer matrix R_0^{-1} and decompose the system according to the observability.

$$R_0^{-1}=\left[\begin{array}{ccc|ccc}6&2&5&3&1&1\\-6&-3&-5&-4&-1&-1\\-6&-6&-5&-9&-1&-3\\\hline 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{array}\right]$$

thus,

$$R_0=\left[\begin{array}{ccc|ccc}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\-1&-1&0&0&-1&0\\\hline \frac{3}{2}&0&\frac{1}{2}&-6&0&-5\\\frac{5}{2}&3&-\frac{1}{2}&0&1&0\end{array}\right]$$

so

$$\hat{A}=R_0^{-1}AR_0=\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{3}{2} & -2 & -\frac{1}{2} & 0 & 0 & 0 \\ -3 & 0 & -4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ \frac{3}{2} & 0 & -2 & -6 & 0 & -5 \end{array}\right]=\begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$$

$$\hat{B}=R_0^{-1}B=\left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \\ -1 & -3 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]=\begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

$$\hat{C}=CR_0=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]=\begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}$$

After examination, $\sum=(\hat{A}_{11},\hat{B}_1,\hat{C}_1)$ is a controllable and observable subsystem, thus, the minimum realization of $W(s)$ is

$$A_m=\hat{A}_{11}=\begin{bmatrix} 0 & 0 & 1 \\ -\frac{3}{2} & -2 & -\frac{1}{2} \\ -3 & 0 & -4 \end{bmatrix};\quad B_m=\hat{B}_1=\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ -1 & -3 \end{bmatrix};$$

$$C_m=\hat{C}_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};\quad D=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Compute the transfer function according to the above A_m, B_m, C_m, D, we can check the result.

$$C_m(sI-A_m)^{-1}B_m+D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} s & 0 & -1 \\ \frac{3}{2} & s+2 & \frac{1}{2} \\ 3 & 0 & s+4 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ -1 & -3 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+3} \\ \frac{s}{s+1} & \frac{s+1}{s+2} \end{bmatrix}$$

We can also write out the realization of controllability canonical form $\sum=(A_0,B_0,C_0)$

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 \\ 1 & 0 & 0 & 0 & -11 & 0 \\ 0 & 1 & 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -6 \end{bmatrix}; \quad B_0 = \begin{bmatrix} 6 & 2 \\ -6 & -3 \\ 5 & 3 \\ -5 & -4 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then we decompose $\sum = (A_0, B_0, C_0)$ by controllability, chose the transform matrix R_c according to equation (5-88)

$$R_c = \begin{bmatrix} 6 & 2 & -6 & 1 & 0 & 0 \\ -6 & -3 & 6 & 0 & 1 & 0 \\ 5 & 3 & -9 & 0 & 0 & 1 \\ -5 & -4 & 8 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

and we have

$$R_c^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 & -7 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 4 & -2 & -16 \\ 0 & 1 & 0 & -3 & 0 & 9 \\ 0 & 0 & 1 & 2 & -3 & 8 \end{bmatrix}$$

thus

$$\tilde{A} = R_c^{-1} A_0 R_c = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -6 & 0 & 1 & -2 \\ 0 & 1 & -5 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & -3 \end{array}\right]$$

$$\tilde{B} = R_c^{-1} B_0 = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{C} = C_0 R_c = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} = \left[\begin{array}{ccc:ccc} 1 & 1 & -3 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \end{array}\right]$$

$\sum = (\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is a controllable and observable subsystem, so the minimum realization of $W(s)$ is

$$\tilde{A}_m = \tilde{A}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & -5 \end{bmatrix}; \quad \tilde{B}_m = \tilde{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{C}_m = \tilde{C}_1 = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 2 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

From the above calculation, we can see that if a transfer function is realizable, then it has infinitely many realizations, not necessarily of the same dimension. But we can prove that if $\sum(A_m, B_m, C_m)$ and $\sum(\tilde{A}_m, \tilde{B}_m, \tilde{C}_m)$ are the minimum realizations of the same transfer matrix $W(s)$, then there must exist a state transformation $x = P\tilde{x}$ such that

$$\tilde{A}_m = P^{-1} A_m P \quad \tilde{B}_m = P^{-1} B_m \quad \tilde{C}_m = C_m P$$

We can see that the minimum realizations of the same transfer matrix are equivalent in algebra.

Example 5.21 Try to find the minimum realization of the transfer matrix using MATLAB

$$W(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

Solution the controllability canonical form of the system is:

$$A = \begin{bmatrix} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{bmatrix}; \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

This six-dimensional realization is clearly not minimal realizations. It can be reduced to minimal realizations by calling the MATLAB function **minreal**. We type:

```
a=[-4.5 0 -6 0 -2 0;0 -4.5 0 -6 0 -2;1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0;0 0 0 1 0 0];
b=[1 0;0 1;0 0;0 0;0 0;0 0];
c=[-6 3 -24 7.5 -24 3;0 1 0.5 1.5 1 0.5];
d=[2 0;0 0];;
[am,bm,cm,dm]=minreal(a,b,c,d)
Yield
am =
-1.3387      0.2185      -1.6003
2.5335      -1.1599      4.8338
-0.0007     -0.0002      -2.0014
bm =
-0.2666        0.2026
 0.2513       -0.6119
-0.0001      0.3483
cm =
   32.7210      10.8346         8.6137
   -0.8143      -0.8632         1.8281
dm =
     2    0
     0    0
```

Thus, the minimum realization is expressed as:

$$\dot{x}(t)=\begin{bmatrix}-1.3387 & 0.2185 & -1.6003\\ 2.5335 & -1.1599 & 4.8338\\ -0.0007 & -0.0002 & -2.0014\end{bmatrix}x+\begin{bmatrix}-0.2666 & 0.2026\\ 0.2513 & -0.6119\\ -0.0001 & 0.3483\end{bmatrix}u$$

$$y(t)=\begin{bmatrix}32.7210 & 10.8346 & 8.6137\\ -0.8143 & -0.8632 & 1.8281\end{bmatrix}x+\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}u$$

5.7 Summary

The controllability and observability are both important properties of a system, and the definitions and criteria are described separately. The duality system is an important conception; the duality systems have a series of interesting properties. The decomposition of a system is analyzed based on the controllability and observablity, and the minimum realization can be obtained accordingly.

Exercise

5.1 Check the controllability of the following systems

(1) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

(2) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

(3) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

(4) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$

(5) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} u$

5.2 Check the observability of the following systems

(1) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(2) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(3) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(4) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(5) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

5.3 Is it possible to find a set of p and q such that the state equation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 12\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}p\\ -1\end{bmatrix}u$$

$$y=\begin{bmatrix}q & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

is not controllable? Observable?

5.4 Try to prove that the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}20 & -1 & 0\\ 4 & 16 & 0\\ 12 & 0 & 18\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}a\\ b\\ c\end{bmatrix}u$$

is not controllable no matter what a,b and c are.

5.5 Try to transform the following state-space equation into controllability canonical form Ⅰ.

$$\dot{x}=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}x+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

5.6 Try to transform the following state-space equation into controllability canonical form Ⅰ.

$$\dot{x}=\begin{bmatrix}-1 & -2 & -2\\ 0 & -1 & 1\\ 1 & 0 & 1\end{bmatrix}x+\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}u$$

5.7 Try to transform the following state-space equation into observability canonical form Ⅱ.

$$\dot{x}=\begin{bmatrix}-1 & -2 & -2\\ 0 & -1 & 1\\ 1 & 0 & 1\end{bmatrix}x+\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 1 & 0\end{bmatrix}x$$

5.8 Try to transform the following state-space equation into observability canonical form Ⅱ.

$$\dot{x}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}x+\begin{bmatrix}2\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}-1 & 1\end{bmatrix}x$$

5.9 Is the state equations

$$\dot{x}=\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}x+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

controllable? If not, please give the controllability decomposition of the system.

5.10 Is the state equations

$$\dot{x}(t)=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 3\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$
$$y(t)=\begin{bmatrix}1 & -1 & 1\end{bmatrix}x(t)$$

controllable? If not, please give the controllability decomposition of the system.

5.11 Is the state equations

$$\dot{x}(t)=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 3\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$
$$y(t)=\begin{bmatrix}1 & -1 & 1\end{bmatrix}x(t)$$

observable? If not, please give the observability decomposition of the system.

5.12 Try to find the minimum realization of the transfer matrix

$$W(s)=\begin{bmatrix}\dfrac{s+1}{s+2}\\ \dfrac{s+3}{(s+2)(s+4)}\end{bmatrix}$$

5.13 The state equation of the inverted pendulum was developed in Example 1.10. Suppose for a given pendulum, the equation becomes

$$\dot{x}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 5 & 0\end{bmatrix}x+\begin{bmatrix}0\\1\\0\\-2\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}x$$

if $x_3=\theta$ deviates from zero slightly, can we find a control u to push it back to zero? Why?

Chapter 6
State Feedback and Observer

6.1 Introduction

Generally, the control theory can be divided into two parts, system analysis and system synthesis. In previous chapters, the solution of state-space equation, the stability analysis and the controllability and observability of a control system are introduced. In this chapter, the system synthesis will be discussed. The controller is designed through feedback and the performance of a system is improved by pole assignment. The state estimator or state observer is designed to generate an estimation of the state.

6.2 Linear Feedback

Feedback is the most popular way to improve the performance of a system. Three kinds of linear feedback are discussed in this chapter, e.g., the state feedback, the output feedback and feedback from output y to $\dot{x}$. The algorithms are described in detail.

6.2.1 State Feedback

Consider an n-dimensional linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{6-1}$$

where $x \in R^{n\times 1}$; $u \in R^{p\times 1}$; $y \in R^{m\times 1}$, $A \in R^{n\times n}, B \in R^{n\times p}, C \in R^{m\times n}, D \in R^{m\times p}$.

Assume $D = 0$ to simplify the discussion. In state feedback, the input u is given by

$$u = Kx + v = v + [k_1 \quad k_2 \quad \cdots \quad k_n]x = v + \sum_{i=1}^{n} k_i x_i \tag{6-2}$$

where $v \in R^{p\times 1}$ is the reference input; $K \in R^{p\times n}$ is the feedback gain matrix.

As shown in Fig.6.1, each feedback gain k_i is a real constant. This is called the constant gain negative state feedback, or state feedback in simplicity.

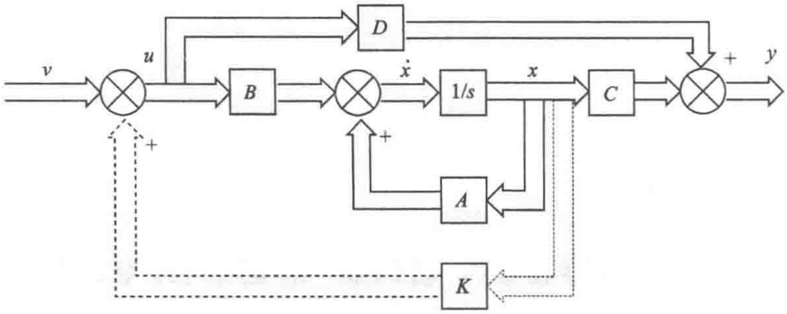


Fig.6.1 State feedback

Substituting (6-2) into (6-1) yields the closed-loop state-space equation

$$\begin{aligned} \dot{x} &= (A+BK)x + Bv \\ y &= Cx \end{aligned} \tag{6-3}$$

the closed-loop transfer function is

$$W_k(s) = C[sI-(A+BK)]^{-1}B \tag{6-4}$$

6.2.2 Output Feedback

Output feedback is another linear feedback law using the output vector y, as shown in Fig.6.2.

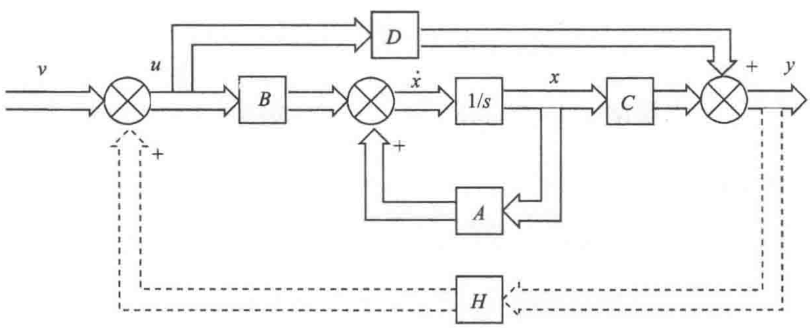


Fig.6.2 Output feedback

The control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{6-5}$$

The input u is given by

$$u = Hy + v \tag{6-6}$$

where $H \in R^{p\times m}$ is output feedback gain matrix.

If $D=0$, the closed-loop state-space equation is

$$\begin{aligned} \dot{x} &= (A+BHC)x+Bv \\ y &= Cx \end{aligned} \tag{6-7}$$

The closed-loop transfer function is

$$W_H(s)=C[sI-(A+BHC)]^{-1}B \tag{6-8}$$

From equation (6-4), HC in output feedback is comparative to K in state feedback. As $m<n$, the optional degree of H is much smaller than that of K. Only when $C=I$, $HC=K$, output feedback is absolutely equivalent to state feedback. Therefore, the effect of output feedback is obviously not as good as that of state feedback without compensator. But output feedback shows its unique advantage in the facility in technique implementation.

6.2.3 Feedback from output to $\dot{x}$

This linear feedback from system output y to state vector $\dot{x}$ is widely used in state estimator. This kind of feedback configuration is shown in Fig.6.3.

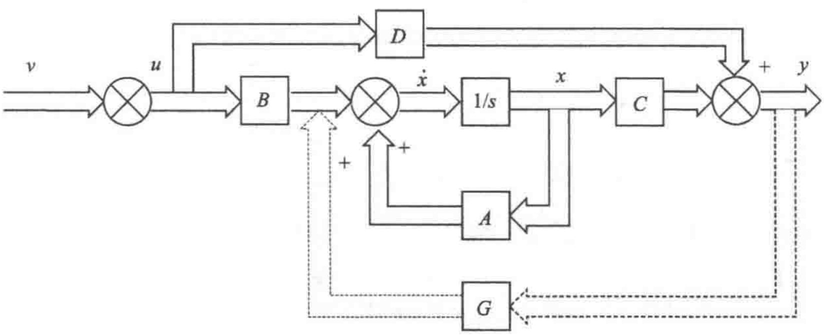


Fig.6.3 Feedback from output y to $\dot{x}$

The control system is

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx+Du \end{aligned} \tag{6-9}$$

Considering the feedback gain matrix $G, G\in R^{n\times m}$ from the output y to the derivative of the state vector $\dot{x}$, the closed-loop system is

$$\begin{aligned} \dot{x} &= Ax+Gy+Bu \\ y &= Cx+Du \end{aligned} \tag{6-10}$$

Substituting y into $\dot{x}$

$$\dot{x} = (A+GC)x + (B+GD)u$$
$$y = Cx + Du \tag{6-11}$$

If $D=0$, then

$$\dot{x} = (A+GC)x + Bu$$
$$y = Cx \tag{6-12}$$

The closed-loop transfer function

$$W_G(s) = C[sI-(A+GC)]^{-1}B \tag{6-13}$$

As equation (6-13) shows, the change of matrix G will affect the eigenvalue of the closed-loop system, thus affect the characteristic of the system.

6.3 Pole Assignment

6.3.1 Sufficient and Necessary Condition for Arbitrary Pole Assignment

Theorem 6.1

There exists a state feedback matrix K which can assign eigenvalues of the matrix $A+BK$ of the closed-loop system to arbitrary place of the state space, if and only if the state vector of the open-loop system is controllable, i.e, if and only if

$$rank Q_c = n$$

where

$$Q_c = \begin{bmatrix} B \vdots & AB \vdots & A^2B \vdots & \cdots & \vdots A^{n-1}B \end{bmatrix} \tag{6-14}$$

Proof Only for sufficiency. If the state vector of the open-loop system is absolutely controllable, the following equation can be obtained with state feedback.

$$\det\left[\lambda I-(A+BK)\right] = f^*(\lambda) \tag{6-15}$$

where $f^*(\lambda)$ is the desired characteristic polynomial.

$$W_o(s) = C(sI-A)^{-1}B$$

$$f^*(\lambda) = \prod_{i=1}^{n}(\lambda-\lambda_i^*) = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^* \tag{6-16}$$

where λ_i^* $(\lambda = 1,2,\cdots,n)$ are desired closed-loop poles.

1) If $\sum_o = (A,B,C)$ is absolutely controllable, the following nonsingular transform exists,

$$x = T_{c\mathrm{I}}\bar{x}$$

where $T_{c\mathrm{I}}$ is transfer matrix of controllable canonical form Ⅰ.

Transform $\sum_o$ into the controllable canonical form Ⅰ.

$$\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u \tag{6-17}$$

$$y = \overline{C}\overline{x}$$

where $\overline{A} = T_{cI}^{-1} A T_{cI} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$

$$\overline{B} = T_{cI}^{-1} B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\overline{C} = C T_{cI} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}$$

The transfer function of the controllable system $\sum_o$

$$W_o(s) = C(sI - A)^{-1} B = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{6-18}$$

2) Consider the following state feedback gain matrix

$$\overline{K} = \begin{bmatrix} \overline{k}_0 & \overline{k}_1 & \cdots & \overline{k}_{n-1} \end{bmatrix} \tag{6-19}$$

Then we can obtain the closed-loop state-space description to $\overline{x}$

$$\dot{\overline{x}} = (\overline{A} + \overline{B}\overline{K})\overline{x} + \overline{B}u \tag{6-20}$$

$$y = \overline{C}\overline{x}$$

Where $\overline{A} + \overline{B}\overline{K} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_0 - \overline{k}_0) & -(a_1 - \overline{k}_1) & \cdots & \cdots & -(a_{n-1} - \overline{k}_{n-1}) \end{bmatrix}$

Closed-loop characteristic polynomial,

$$\begin{aligned} f(\lambda) &= \left| \lambda I - (\overline{A} + \overline{B}\overline{K}) \right| \\ &= \lambda^n + (a_{n-1} - \overline{k}_{n-1})\lambda^{n-1} + \cdots + (a_1 - \overline{k}_1)\lambda + (a_0 - \overline{k}_0) \end{aligned} \tag{6-21}$$

Closed-loop transfer function,

$$\begin{aligned} W_k(s) &= \overline{C}\left[sI - (\overline{A} + \overline{B}\overline{K}) \right]^{-1} \overline{B} \\ &= \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + (a_{n-1} - \overline{k}_{n-1})s^{n-1} + \cdots + (a_1 - \overline{k}_1)s + (a_0 - \overline{k}_0)} \end{aligned} \tag{6-22}$$

3) To accord with the desired poles, the following equation has to be satisfied.

$$f(\lambda) = f^*(\lambda)$$

The coefficients of the feedback matrix can be obtained by equaling the coefficients of λ with same order in both sided of the above equation.

$$\bar{k}_i = a_i - a_i^* \tag{6-23}$$

Thus $\bar{K} = \begin{bmatrix} a_0 - a_0^* & a_1 - a_1^* & \cdots & a_{n-1} - a_{n-1}^* \end{bmatrix}$.

4) Transform $\bar{K}$ corresponding with $\bar{x}$ into K corresponding with x, by using the following equation.

$$K = \bar{K} T_{c\mathrm{I}}^{-1} \tag{6-24}$$

This is due to $u = v + \bar{K}\bar{x} = v + KT_{c\mathrm{I}}^{-1}x$.

Theorem 6.2

For the case of pole assignment via output feedback wherein $u = Hy + v$, a theorem similar to the Theorem 1 has not yet been proven. The determination of the output feedback matrix H is, in general, a very difficult task. A method for determining the matrix H , which is closely related to the method of determining the matrix K presented earlier, is based on the equation

$$K = HC \tag{6-25}$$

This method starts with the determination of the matrix K and in the sequel the matrix H is determined by using equation(6-25).It is fairly easy to determine the matrix H from equation (6-25) since this equation is linear in H. Note that equation (6-25) is only a sufficient condition. That is, if equation (6-25) does not have a solution for H, it does not follow that pole assignment by output feedback is impossible.

Theorem 6.3

Even for the absolutely controllable SISO system $\Sigma_o = (A,b,c)$, arbitrary pole assignment via output feedback can not be guaranteed.

Proof The closed-loop transfer function of SISO feedback system $\Sigma_h = \left[(A+bhc),b,c\right]$ is

$$W_h(s) = c\left[sI - \left(A + bhc\right)\right]^{-1} b = \frac{W_o(s)}{1 + hW_o(s)} \tag{6-26}$$

Where $W_o(s) = c\left(sI - A\right)^{-1} b$ is the transfer function of the controllable system.

From the closed-loop characteristic polynomial, we can obtain the closed-loop root locus equation

$$hW_o(s) = -1 \tag{6-27}$$

When $W_o(s)$ is fixed on beforehand, we can obtain a series of root locus with the reference variable h varying from 0 to ∞. Obviously, no matter what h is, the desired pole which is not consisted in the root locus can not appear in root locus.

Output linear feedback has an important drawback: **Arbitrary pole assignment is not realizable.**

To overcome the drawback, we always introduce additional regulatory network to affect the root locus by increasing open-loop poles and zeros. Therefore, the regulated root locus consists of the desired pole.

Theorem 6.4

For an absolutely controllable SISO system $\Sigma_o = (A, b, c)$, the following is sufficient and necessary condition of arbitrary pole assignment by the output feedback with dynamic compensator.

1) Σ_o is absolutely observable.

2) The order of dynamic compensator is n=1.

6.3.2 Methods to Assign the Poles of a System

(1) Pole Assignment via State Feedback

Consider a linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{6-28}$$

Where we assume that all states are accessible and known. To this system we apply a linear state feedback control law of the following form

$$u(t) = -Kx(t) \tag{6-29}$$

Then the closed-loop system is given by the homogeneous equation

$$\dot{x}(t) = (A - BK)x(t) \tag{6-30}$$

It is remarked that the feedback law $u(t)=-Kx(t)$ is used rather than the feedback law $u(t)=Kx(t)$. This different chosen sign is utilized to facilitate the observer design problem.

Here, the design problem is to find the appropriate controller matrix K so as to improve the performance of the closed-loop system (6-30). One method to improve the performance of (6-30) is pole assignment. The pole-assignment method consists in finding a particular matrix K, such that the poles of the closed-loop system (6-30) are set to the desirable pre-assigned values. Using this method, the behavior of the open-loop system may be improved significantly. For example, the method can stabilize an unstable system, increase or decrease the speed of response, widen or narrow the system's bandwidth, increase or decrease the steady-state error, etc. For these reasons, improving the system performance via the pole-assignment method is widely used in practice.

The pole assignment or eigenvalue assignment problem can be defined as follows:

let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the matrix A of the open-loop system (6-30) and $\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*$ be the desired eigenvalues of matrix $A-BK$ of the closed-loop system (6-30), where all complex eigenvalues appear in complex conjugate pairs. Denote $f(\lambda)$ and $f^*(\lambda)$ to be the characteristic polynomial and the desired characteristic polynomial, i.e., to find a matrix K so that equation (6-32) is satisfied.

$$f(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i)=|\lambda I-A|=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0 \tag{6-31}$$

$$f^*(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i^*)=|\lambda I-A+BK|=\lambda^n+a_{n-1}^*\lambda^{n-1}+\cdots+a_1^*\lambda+a_0^* \tag{6-32}$$

The pole-assignment problem has attracted considerable attention for many years. The first significant results were established by Wonham in the late 1960s and are given by the Theorem 1 in the Section 6.3.1.

According to Theorem 1, in cases that the open-loop system (6-30) is not controllable, at least one eigenvalue of the matrix A remains invariant under the state feedback law (6-31). In such cases, in order to assign all eigenvalues, one must search for an appropriate dynamic controller wherein the feedback law (6-31) may involve, not only proportion, but also derivative, integral and other terms. Dynamic controllers have the disadvantage in that they increase the order of the system.

Now, consider the case that the system (A,B) is controllable, a fact which guarantees that there exists a K which satisfies the pole-assignment problem. Next, we will deal with the problem of determining such a feedback matrix K. For simplicity, we will first study the case of single-input systems, in which the matrix B reduces to a vector b and the matrix K reduces to a row vector k. Equation (6-32) then becomes

$$f^*(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i^*)=|\lambda I-A+bk|=\lambda^n+a_{n-1}^*\lambda^{n-1}+\cdots+a_1^*\lambda+a_0^* \tag{6-33}$$

It is remarked that the solution of equation (6-33) for k is unique.

Several methods have been proposed for determining k. We present three well-known methods.

Method 1 The Base-Gura Formula. One of the most popular pole-assignment methods gives the following simple solution:

$$k=-\bar{K}T_{\mathrm{cI}}^{-1} \tag{6-34}$$

Where $\bar{K}$ is defined in equation (6-23) and

$$T_{cI} = \left[A^{n-1}b, \quad \cdots \quad Ab, \quad b \right] \begin{bmatrix} 1 & & & \\ a_{n-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_1 & \cdots & a_{n-1} & 1 \end{bmatrix} \tag{6-35}$$

Method 2 The Phase Canonical Formula. Consider the special case that the system under control is described in its phase-variable canonical form, i.e., A and b have the special forms A^* and b^*, where

$$A^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\hat{a}_0 & -\hat{a}_1 & -\hat{a}_2 & \cdots & -\hat{a}_{n-1} \end{bmatrix}, \quad b^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{6-36}$$

One of the most popular pole-assignment methods gives the following simple solution:

$$k^T = \left[W^T Q_c^T \right]^{-1} (a^* - a) \tag{6-37}$$

Where Q_c is defined in equation (6-14) and

$$W = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_1 \\ 0 & 1 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad a^* = \begin{bmatrix} a_{n-1}^* \\ a_{n-2}^* \\ \vdots \\ a_0^* \end{bmatrix}, \quad a = \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{bmatrix} \tag{6-38}$$

Then, it can be easily shown that

$$Q_c^* = \left[b^* \vdots \;\; A^* b^* \vdots \;\; A^{*2} b^* \vdots \;\; \cdots \;\; \vdots A^{*n-1} b^* \right]$$

The product $W^T Q_c^{*T}$ reduces to the simple form

$$W^T Q_c^{*T} = \tilde{I} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{6-39}$$

In this case, the vector k^{*T} in expression (6-37) reduces to $k^{*T} = \tilde{I}(a^* - a)$, i.e, it reduces to the following form:

$$k^{*T} = \tilde{I}(a^* - a) = \begin{bmatrix} a_0^* - a_0 \\ a_1^* - a_1 \\ \vdots \\ a_{n-1}^* - a_{n-1} \end{bmatrix} \tag{6-40}$$

It is evident that expression (6-40) is extremely simple to apply, provided that the matrix

A and the vector b of the system under control are in the phase-variable canonical form (6-38).

Method 3 The Ackermann's Formula. Another approach for computing k has been proposed by Ackermann, which leads to the following expression:

$$k = e^T Q_c^{-1} f^*(A) \tag{6-41}$$

The matrix Q_c is given in equation (6-14), wherein the variable s is replaced by the matrix A, i.e.,

$$f^*(A) = A^n + a_{n-1}^* A^{n-1} + \cdots + a_1^* A + a_0^* I \tag{6-42}$$

In the general case of multi-input system, the determination of the matrix K is somewhat complicated. A simple approach to the problem is to assume that K has the following form:

$$K = qp^T \tag{6-43}$$

Where q and p are n-dimensional vectors. Then, the matrix $A - BK$ becomes

$$A - BK = A - Bqp^T = A - \beta p^T, \quad \text{where} \quad \beta = Bq \tag{6-44}$$

Therefore, assuming that K has the form of equation (6-43), then the multi-input system is reduced to single-input system studied previously. In other words, the solution for the vector p is equation (6-37) or equation (6-41) and differs only in that the matrix Q_c is now the matrix $\tilde{Q}_c$, which takes the following form

$$\tilde{Q}_c = \left[\beta \vdots \ A\beta \vdots \ A^2\beta \vdots \ \cdots \ \vdots A^{n-1}\beta\right], \quad K = HC \tag{6-45}$$

The vector $\beta = Bq$ involves arbitrary parameters, which are the elements of the arbitrary vector q. These arbitrary parameters can have any value, provided that $rank\tilde{Q}_c = n$.

(2) Pole Assignment via Output Feedback

Consider a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{6-46}$$

Where we assume that all states are accessible and known. We apply the following linear state feedback control law to the above system

$$u(t) = -Hy(t) + v \tag{6-47}$$

Then the closed-loop system is given by the homogeneous equation

$$\dot{x}(t) = (A - BHC)x(t) \tag{6-48}$$

The pole assignment or eigenvalue assignment problem can be defined as follows: denote $\lambda_1, \lambda_2, \cdots, \lambda_n$ as the eigenvalues of the matrix A of the open-loop system (6-46) and $\lambda_1^*, \lambda_2^*, \ldots. \lambda_n^*$ as the desired eigenvalues of matrix $A-BHC$ of the closed-loop system (5.3-35), where all complex eigenvalues appear in complex conjugate pairs.

In the case of pole assignment via output feedback wherein $u=-Hy+v$, Theorem 6.2 has been proven. According to Theorem 6.2, we can obtain the matrix H by $K=HC$.

This method starts with the determination of the matrix K, and in the following content, the matrix H is determined by using equation (6-25). It is fairly easy to determine the matrix H from equation (6-25) since this equation is linear in H. Note that equation (6-25) is only a sufficient condition, i.e., if equation (6-25) does not have a solution for H, it does not follow that pole assignment by output feedback is impossible.

6.3.3 Examples

Example 6.1 Consider a system in the form (6-28), where

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find a vector k to make the closed-loop system eigenvalues be $\lambda_1^*=-1$ and $\lambda_2^*=-1.5$.

Solution

We have

$$f(\lambda)=|\lambda I-A|=\lambda^2+1 \quad \text{and} \quad f^*(\lambda)=(\lambda-\lambda_1^*)(\lambda-\lambda_2^*)=\lambda^2+2.5\lambda+1.5$$

Method 1 Here we use equation (6-35) and (6-23)

$$T_{cI}=\begin{bmatrix} Ab & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\bar{K}=\begin{bmatrix} a_0-a_0^* & a_1-a_1^* \end{bmatrix}=\begin{bmatrix} 1-1.5 & 0-2.5 \end{bmatrix}=\begin{bmatrix} -0.5 & -2.5 \end{bmatrix}$$

Therefore

$$T_{cI}^{-1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence

$$k=-\bar{K}T_{cI}^{-1}=-\begin{bmatrix} -0.5 & -2.5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0.5 & 2.5 \end{bmatrix}$$

Method 2 Since the system is in phase-variable canonical form, the vector k can readily be determined by equation (6-40), as follows:

$$k^T=k^{*T}=\begin{bmatrix} a_0^*-a_0 \\ a_1^*-a_1 \end{bmatrix}=\begin{bmatrix} 1.5-1 \\ 2.5-0 \end{bmatrix}=\begin{bmatrix} 0.5 \\ 2.5 \end{bmatrix}$$

Method 3 Here we apply equation (6-41). We have

$$f^*(A) = A^2 + a_1^* A + a_0^* I = A^2 + 2.5A + 1.5I$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 + 2.5\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 1.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2.5 \\ -2.5 & 0 \end{bmatrix} + \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 2.5 \\ -2.5 & 0.5 \end{bmatrix}$$

$$Q_c^{-1} = \begin{bmatrix} b & \vdots Ab \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore

$$k = e^T S^{-1} f^*(A) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 2.5 \\ -2.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 2.5 \end{bmatrix}$$

Clearly, the resulting three controller vectors derived by the three methods are identical. This is due to the fact that for single-input system, k is unique.

Example 6.2 Consider a system in the form (6-28), where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find a vector k to make the eigenvalues of the closed-loop system are $\lambda_1^* = -1$, $\lambda_2^* = -2$, and $\lambda_3^* = -2$.

Solution

We have

$$f(\lambda) = |\lambda I - A| = \lambda^3 - 1$$

and

$$f^*(\lambda) = (\lambda - \lambda_1^*)(\lambda - \lambda_2^*)(\lambda - \lambda_3^*) = \lambda^3 + 5\lambda^2 + 8\lambda + 4$$

Method 1 Here we use equation (6-37) and (6-23)

$$T_{cI} = \begin{bmatrix} A^2 b & Ab & b \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\bar{K} = \begin{bmatrix} a_0 - a_0^* & a_1 - a_1^* & a_2 - a_2^* \end{bmatrix} = \begin{bmatrix} -1-4 & 0-8 & 0-5 \end{bmatrix} = \begin{bmatrix} -5 & -8 & -5 \end{bmatrix}$$

Therefore

$$T_{cI}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence

$$k = -\bar{K}T_{cI}^{-1} = -\begin{bmatrix} -5 & -8 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 5 \end{bmatrix}$$

Method 2 Since the system is in phase-variable canonical form, the vector k can readily be determined by equation (6-40), as follows:

$$k^T = k^{*T} = \begin{bmatrix} a_0^* - a_0 \\ a_1^* - a_1 \\ a_2^* - a_2 \end{bmatrix} = \begin{bmatrix} 4-(-1) \\ 8-0 \\ 5-0 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}$$

Method 3 Here we apply equation (6-41). We have

$$f^*(A) = A^3 + a_2^* A^2 + a_1^* A + a_0^* I = A^3 + 5A^2 + 8A + 4I$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^3 + 5\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2 + 8\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 8 & 0 \\ 0 & 0 & 8 \\ 8 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 5 \\ 5 & 5 & 8 \\ 8 & 5 & 5 \end{bmatrix}$$

Therefore

$$k = e^T S^{-1} f^*(A) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 8 & 5 \\ 5 & 5 & 8 \\ 8 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 5 \end{bmatrix}$$

Example 6.3 Consider the following system with transfer function as

$$W(s) = \frac{10}{s(s+1)(s+2)}$$

Try to find a state feedback controller to make the closed loop poles be -2 and $-1\pm j1$.

Solution

Since the system is controllable and observable, the poles can be assigned arbitrarily. Choose the following controllable canonical form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} x$$

With state feedback , the closed-loop characteristic polynomial is:

$$f(\lambda) = \det[\lambda I - (A + bK)] = \lambda^3 + (3 - k_2)\lambda^2 + (2 - k_1)\lambda - k_0$$

The desired closed-loop characteristic polynomial is

$$f^*(\lambda) = (\lambda + 2)(\lambda + 1 - j)(\lambda + 1 + j) = \lambda^3 + 4\lambda^2 + 6\lambda + 4$$

Compare relative parameters in the above two functions, we have

$$k_0 = -4, \quad k_1 = -4, \quad k_2 = -1$$

Thus:

$$K = \begin{bmatrix} -4 & -4 & -1 \end{bmatrix}$$

The closed-loop transfer function is

$$G(s) = \frac{10}{s^3 + 4s^2 + 6s + 4}$$

The block diagram of the closed-loop system is shown in Fig. 6.4.

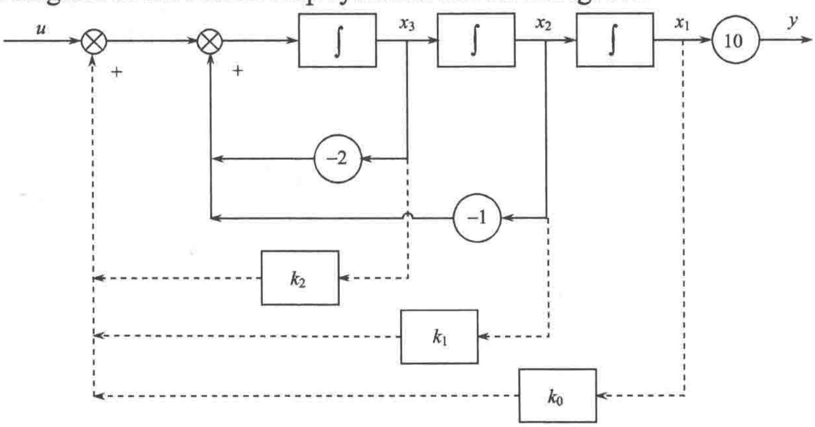


Fig.6.4 Block diagram of the closed-loop system

Example 6.4 The state space model of a system is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Try to find a state feedback controller to make the closed loop poles be -2 and $-1 \pm j1$.

Solution

To determine the controllability of the system

$$M=\begin{bmatrix} b & Ab & A^2b \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -2 & -4 \end{bmatrix}$$

$|M| \neq 0$, so the system is controllable, the closed-loop poles of the system can be can be arbitrary assigned.

Transform the above state space model into the controllable Canonical form. The characteristic function is

$$|sI-A|=s^3+3s^2+2s$$

So we choose

$$I=\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then

$$T_{c1}=MI=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad T_{c1}^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Let $\hat{k}=\begin{bmatrix} \hat{k}_0 & \hat{k}_1 & \hat{k}_2 \end{bmatrix}$ the closed-loop characteristic polynomials can be expressed as

$$f(\lambda)=\left|\lambda I-(\hat{A}+\hat{b}\hat{k})\right|=\left|\lambda I-(T^{-1}AT+T^{-1}b\hat{k})\right|$$

And

$$f^*(\lambda)=(\lambda+2)(\lambda+1-j)(\lambda+1+j)=\lambda^3+4\lambda^2+6\lambda+4$$

To achieve the desired closed-loop ploes, we have $f^*(\lambda)=f(\lambda)$

And $k=\hat{k}T_{c1}^{-1}$, so $k=\begin{bmatrix}-4 & -4 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}=\begin{bmatrix}-4 & -3 & -1\end{bmatrix}$

The block diagram of the closed-loop system is shown in Fig. 6.5.

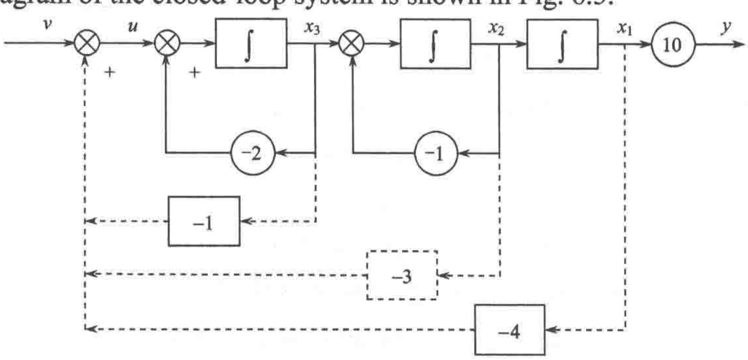


Fig.6.5 Block diagram of the closed-loop system

Example 6.5 Consider a plant described by

$$\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

Let us introduce state feedback $u=r-[k_1 \quad k_2 \quad k_3]x$ to place the three eigenvalues at -2, $-1\pm j$, figure out how to realize it using MATLAB.

The MATLAB function place computes state feedback gains for eigenvalue placement or assignment. For the example, we type

```
a=[0 1 0;0 0 1;0 -2 -3];b=[0;0;1];
p=[-2,-1+j,-1-j];
k=place(a,b,p)
yield
k =
       4.0000    4.0000    1.0000
```

This is the matrix $[\mathrm{k}_1 \quad \mathrm{k}_2 \quad \mathrm{k}_3]=\begin{bmatrix}4 & 4 & 1\end{bmatrix}$.

6.4 State Estimator

6.4.1 Introduction

In the preceding sections, we introduce state feedback under the implicit assumption that all state variables are available for feedback. This assumption may not hold in practice, either because the state variables are not accessible for direct connection or because sensors or transducers are not available. In this case, in order to apply state feedback, we must design a device, called **a state estimator or state observer**, so that the output of the device will generate an estimation of the state.

6.4.2 State Estimator

1. Full-Dimensional State Estimator

Consider the linear time-invariant system $\sum_0=(A,B,C)$

$$\begin{aligned}\dot{x}&=Ax+Bu\\ y&=Cx\end{aligned}\tag{6-49}$$

Where A, B, and C are given and the input $u(t)$ and the output $y(t)$ are available to us. The problem is to estimate x from u and y with the knowledge of A, B, and C. If we know A and B, we can duplicate the original system as

$$\dot{\hat{x}}=A\hat{x}+Bu\tag{6-50}$$

which is shown in Fig.6.6. The duplication will be called an open-loop estimator. Now if (6-49) and (6-50) have the same initial state, then for any input, we have $\hat{x}(t) = x(t)$ for all $t \geqslant 0$. Therefore the remaining question is how to find the initial state of (6-49) and then set the initial state of (6-50) to that state. If (6-49) is observable, its initial state $x(0)$ can be computed from u and y over any time interval, say, $[0, t_1]$. We can then compute the state at t_2 and set $\hat{x}(t_2) = x(t_2)$. Then we have $\hat{x}(t) = x(t)$ for all $t \geqslant t_2$. Thus if (6-49) is observable, an open-loop estimator can be used to generate the state vector.

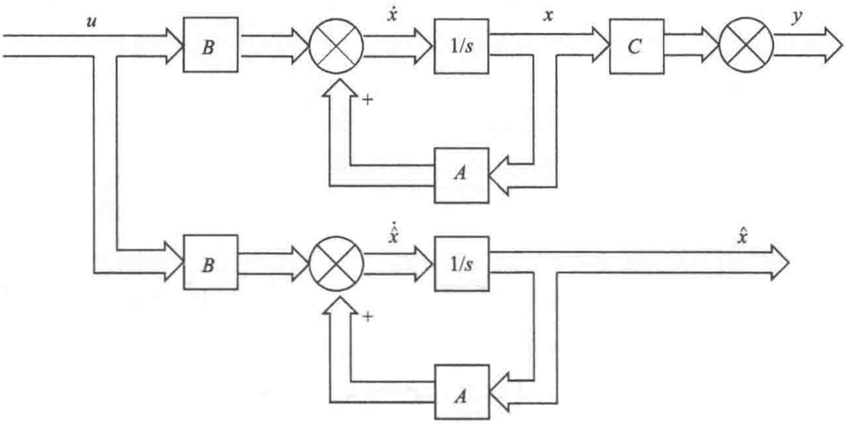


Fig.6.6 Block diagram of open-loop state estimator

There are, however, two disadvantages, in using an open-loop estimator. First, the initial state must be computed and set each time we use the estimator. This is very inconvenient. Second, and more seriously, if the matrix A has eigenvalues with positive real part, then even for a very small difference between $x(t_0)$ and $\hat{x}(t_0)$ for some t_0, which may be caused by disturbance or imperfect estimation of the initial state, the difference between $x(t)$ and $\hat{x}(t)$ will grow with time. Therefore the open-loop estimator is, in general, not satisfactory.

We see from Fig.6.6 that even though the input and output of (6-49) are available, we use only the input to drive the estimator. Now we shall modify the estimator in Fig.6.6 to the one in Fig.6.7, in which the output $y(t) = Cx(t)$ of (6-49) is compared with $C\hat{x}(t)$. Their difference, passing through an $n \times 1$ constant gain vector G, is used as a correcting term. If the difference is zero, no correction is needed. If the difference is nonzero and if the gain G is properly designed, the difference will drive the estimated state to the actual state. Such an estimator is called a closed-loop or an asymptotic estimator or, simply, an estimator.

The open-loop estimator is now modified as,

$$\dot{\hat{x}} = A\hat{x} + Bu + G(y - \hat{y}) = A\hat{x} + Bu + Gy - GC\hat{x} \tag{6-51}$$

which is shown in Fig.6.7. (6-51) can be written as

$$\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy \tag{6-52}$$

and is shown in Fig.6.8. It has two inputs u and y and its output yields an estimated state $\hat{x}$.

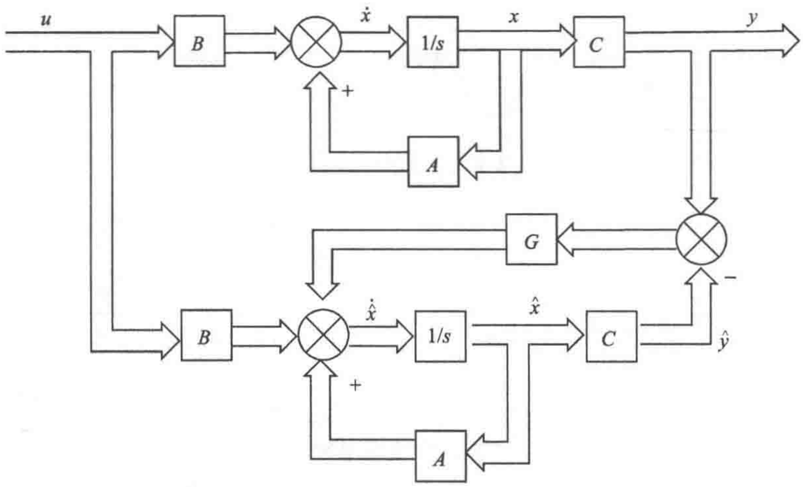


Fig.6.7 Block diagram of closed-loop state estimator Ⅰ

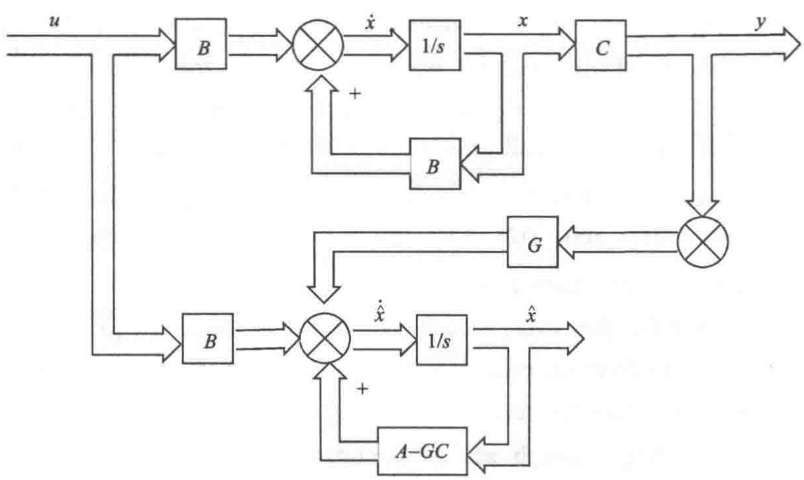


Fig.6.8 Block diagram of closed-loop state estimator Π

Let us define

$$e = x - \hat{x}$$

It is the error between the actual state and estimated state. Differentiating e and then substituting (6-49) and (6-51) into it, we obtain

$$\begin{aligned}\dot{e} = \dot{x} - \dot{\hat{x}} &= Ax + Bu - (A - GC)\hat{x} - G(Cx) - Bu \\ &= Ax - (A - GC)\hat{x} - GCx = (A - GC)(x - \hat{x})\end{aligned}$$

or

$$\dot{e} = (A - GC)e \tag{6-53}$$

This equation governs the estimation error. If all eigenvalues of $(A - GC)$ can be assigned arbitrarily, then we can control the rate for e to approach zero or, equivalently, for the estimated state to approach the actual state. For example, if all eigenvalues of $A - GC$ have negative real parts smaller than $-\sigma$, then all entire of e will approach zero at rates faster than $e^{-\sigma t}$. Therefore, even if there is a large error between $\hat{x}(t_0)$ and $x(t_0)$ at the initial time t_0, the estimated state will approach the actual state rapidly. Thus there is no need to compute the initial state of the original state equation. In conclusion, if all eigenvalues of $(A - GC)$ are properly assigned, a closed-loop estimator is much more desirable than an open-loop estimator.

Theorem 6.5

Consider the pair (A,C). All eigenvalues of $(A - GC)$ can be assigned arbitrarily by selecting a real constant vector G if and only if (A,C) is observable.

Example 6.6 Consider the state equation

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} x$$

Try to find the state estimator, so that the desired closed loop eigenvalues can be –10,–10.

Solution

(1) Examine the system's observability. We have

$$N = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

Since $rankN=2$, there exists a full-dimensional state estimator.

(2) Transfer the system to observability criterion form Π.

The characteristic polynomial of the system is

$$\det[\lambda I - A] = \det\begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{bmatrix} = \lambda^2 - \lambda$$

We have $a_1 = -1, a_0 = 0, L = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

and $T^{-1} = LN = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

and thus $$\dot{\overline{x}} = T^{-1}AT\overline{x} + T^{-1}bu = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\overline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = CT\overline{x}$$

(3) Introducing feedback matrix $\overline{G} = \begin{bmatrix} \overline{g}_1 \\ \overline{g}_2 \end{bmatrix}$, the characteristic polynomial of estimator yields

$$f(\lambda) = \left|\lambda I - (\overline{A} - \overline{G}\overline{C})\right| = \begin{vmatrix} \lambda & \overline{g}_1 \\ -1 & \lambda - (1 - \overline{g}_2) \end{vmatrix} = \lambda^2 - (1 - \overline{g}_2)\lambda + \overline{g}_1$$

(4) The desired characteristic polynomial is

$$f^*(\lambda) = (\lambda + 10)^2 = \lambda^2 + 20\lambda + 100$$

(5) Comparing the corresponding coefficient of $f(\lambda)$ and $f^*(\lambda)$, we have

$$\overline{g}_1 = 100, \quad \overline{g}_2 = 21, \quad \text{and } \overline{G} = \begin{bmatrix} 100 \\ 21 \end{bmatrix}$$

(6) Transforming to the state of x

$$G = T\overline{G} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 100 \\ 21 \end{bmatrix} = \begin{bmatrix} 60.5 \\ 100 \end{bmatrix}$$

(7) The proposed estimator is

$$\dot{\hat{x}} = (A - Gc)\hat{x} + bu + Gy$$
$$= \begin{bmatrix} -120 & 60.5 \\ -200 & 100 \end{bmatrix}\hat{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u + \begin{bmatrix} 60.5 \\ 100 \end{bmatrix}y$$

or $$\dot{\hat{x}} = A\hat{x} + bu + G(y - \hat{y}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\hat{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u + \begin{bmatrix} 60.5 \\ 100 \end{bmatrix}(y - \hat{y})$$

Example 6.7 Consider the transfer function of a controlled system

$$W_0(s) = \frac{1}{s(s+6)}$$

Find a vector k such that the closed-loop system has eigenvalues $\lambda_1^* = -1$ and $\lambda_2^* = -1.5$ by state feedback. And design a full-dimensional state estimator that can realize the above feedback.

Solution

(1) From the transfer function above, we can know that the system is controllable and observable. Therefore, the state feedback matrix and estimator can be designed independently due to the separation principle.

(2) Design the state feedback matrix K.

For convenient of estimator design, we use the observable canonical form Π of the system directly.

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 1 & -6 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

(3) We have

$$f(\lambda) = |\lambda I - A| = \lambda^2 + 6\lambda$$

and $$f^*(\lambda) = (\lambda - \lambda_1^*)(\lambda - \lambda_2^*) = \lambda^2 + 8\lambda + 52$$

Here we use equation (6-37) and (6-23)

$$T_{cI} = \begin{bmatrix} Ab & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$$

and $$\bar{K} = \begin{bmatrix} a_0 - a_0^* & a_1 - a_1^* \end{bmatrix} = \begin{bmatrix} 0 - 52 & 6 - 8 \end{bmatrix} = \begin{bmatrix} -52 & -2 \end{bmatrix}$$

Therefore

$$T_{cI}^{-1} = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}$$

Hence

$$k = -\bar{K} T_{cI}^{-1} = -\begin{bmatrix} -52 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 40 \end{bmatrix}$$

(4) Design the full-dimensional estimator

Suppose $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, then

$$A - GC = \begin{pmatrix} 0 & 0 \\ 1 & -6 \end{pmatrix} - \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -g_1 \\ 1 & -6 - g_2 \end{pmatrix}$$

and $$\det\left[\lambda I - (A - GC)\right] = \det \begin{pmatrix} \lambda & g_1 \\ -1 & \lambda + 6 + g_2 \end{pmatrix} = \lambda^2 + (6 + g_2)\lambda + g_1$$

Comparing with $$f^*(\lambda) = (\lambda - \lambda_1^*)(\lambda - \lambda_2^*) = \lambda^2 + 20\lambda + 100$$

we can obtain $$G = \begin{pmatrix} 100 \\ 4 \end{pmatrix}$$

The full-dimensional estimator equation:

$$\dot{\hat{x}} = (A - GC)\hat{x} + Gy + bu$$
$$= \begin{pmatrix} 0 & -100 \\ 1 & -20 \end{pmatrix}\hat{x} + \begin{pmatrix} 100 \\ 4 \end{pmatrix}y + \begin{pmatrix} 1 \\ 0 \end{pmatrix}u$$

Example 6.8 the state space model of a system is

$$\begin{cases} \dot{X} = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}X + \begin{bmatrix} 2 \\ 4 \end{bmatrix}u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix}X \end{cases}$$

Try to construct a two dimensional state estimator with poles to be –4 and –6. Give out the model of the stator estimator and plot the diagram of the system with state estimator.

Solution

The desired characteristic equation is:

$$(s+4)(s+6) = s^2 + 10s + 24 = 0, \quad |N| = \begin{vmatrix} C \\ CA \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} = -3 \neq 0, \quad rank\ N = 2,$$

the system is observable and the state estimator can be constructed

$$|sI - A + GC| = s^2 + (g_2 - 4)s + 3g_1 - 3 = 0, \quad g_1 = 9, \quad g_2 = 14;$$

The state estimator $\dot{X}_g = (A - GC)X_g + Bu + Gy = \begin{bmatrix} 0 & -8 \\ 3 & -10 \end{bmatrix}X_g + \begin{bmatrix} 2 \\ 4 \end{bmatrix}u + \begin{bmatrix} 9 \\ 14 \end{bmatrix}y$.

The diagram of the system is shown in Fig. 6.9.

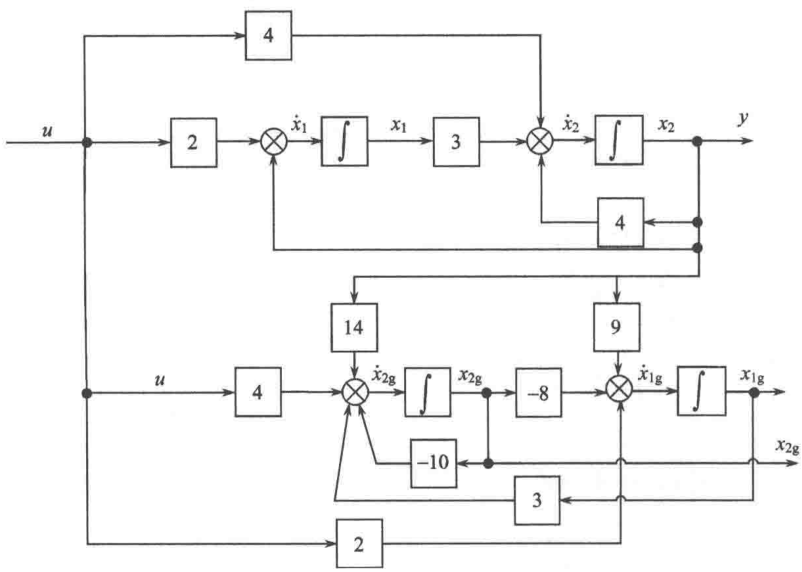


Fig.6.9 Block diagram of the system

Example 6.9 Consider a plant described by the following state equation

$$\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1.24 & 0.3965 & -3.145\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 1.244\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}x$$

Try to design a state estimator to place the eigenvalues at $-5\pm j5\sqrt{3}$, -10 by MATLAB.

For the example, we type

```
a=[0 1 0;0 0 1;1.244 0.3965 -3.145];
b=[0;0;1.244];
c=[1 0 0];
v=[-5+j*5*sqrt(3) -5-j*5*sqrt(3) -10];
l=(acker(a',c',v))'
yield
l =
     16.8550
     147.3875
     544.3932
```

Then we can get

$$L=\begin{bmatrix}16.855\\ 147.3875\\ 544.3932\end{bmatrix}$$

So the state estimator is

$$\dot{x}=(\Lambda-LC)x+Bu+Ly$$

$$=\left\{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1.244 & 0.3965 & -3.145\end{bmatrix}-\begin{bmatrix}16.855\\ 147.3875\\ 544.3932\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\right\}x+\begin{bmatrix}0\\ 0\\ 1.244\end{bmatrix}u+\begin{bmatrix}16.855\\ 147.3875\\ 544.3932\end{bmatrix}y$$

$$=\begin{bmatrix}-16.855 & 1 & 0\\ -147.3875 & 0 & 1\\ -544.3932 & 0.3965 & -3.145\end{bmatrix}x+\begin{bmatrix}0\\ 0\\ 1.244\end{bmatrix}u+\begin{bmatrix}16.855\\ 147.3875\\ 544.3932\end{bmatrix}y$$

2. Reduced-Dimensional State Estimator

The estimator presented in Section 6.4.2.1, usually called a full-dimensional estimator, has the same dimension with the controlled system. Actually the output vector y is always measurable. We can derive a part of state variables directly from y, thus reducing the dimension of the estimator.

Consider a observable system, assume the rank of the output matrix C is m, then m dimension state variables can be acquired by the output y, the other $n-m$ dimension state variables can be acquired by an $(n-m)$-dimensional state estimator. This estimator with the output equation can then be used to estimate all n state variables. This estimator has a lesser dimension than the system (6-49) and is called a reduced-dimensional estimator.

The controllable system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6-54}$$

With $rankC = m$, the pair (A,C) is observable. The design consists of two steps.

(1) decompose the state to $\overline{x}_1$ and $\overline{x}_2$, m dimension $\overline{x}_2$ can be derived from y while $n-m$ dimension $\overline{x}_1$ are to be observed.

(2) construct the $(n-m)$-dimensional state estimator.

Let $x = T\overline{x}$

$$\overline{A} = T^{-1}AT = \begin{array}{c} \left[\begin{array}{c:c} \overline{A}_{11} & \overline{A}_{12} \\ \hdashline \overline{A}_{21} & \overline{A}_{22} \end{array}\right] \begin{array}{l} n-m \\ m \end{array} \\ \begin{array}{cc} n-m & m \end{array} \end{array}$$

$$\overline{B} = T^{-1}B = \left[\begin{array}{c} \overline{B}_1 \\ \hdashline \overline{B}_2 \end{array}\right] \begin{array}{l} n-m \\ m \end{array}$$

$$\overline{C} = CT = \begin{array}{c} \left[\begin{array}{c:c} 0 & I \end{array}\right] m \\ \begin{array}{c:c} n-m & m \end{array} \end{array}$$

$$\overline{A} = T^{-1}AT = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \quad \overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} \quad \overline{C} = CT = \begin{bmatrix} 0 & I \end{bmatrix}$$

The transform matrix T

$$T^{-1} = \left[\begin{array}{c} C_0 \\ \hdashline C \end{array}\right] \begin{array}{l} n-m \\ m \end{array} \quad T = \begin{bmatrix} C_0 \\ C \end{bmatrix}^{-1}$$

where C_0 is a $(n-m)\times n$ matrix to guarantee that T is nonsingular.

$$CT = C\begin{bmatrix} C_0 \\ C \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \end{bmatrix}$$

The state-space equation can be written as

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u$$
$$\bar{y} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \bar{x}_2 \tag{6-55}$$

As the system (6-54) is observable, (6-55) is also observable.

From (6-55), we can see $\bar{x}_2$ can be directly detected from $\bar{y}$, $\bar{x}_1$ can be obtained from the estimator. The decomposed system structure is shown in Fig.6.10.

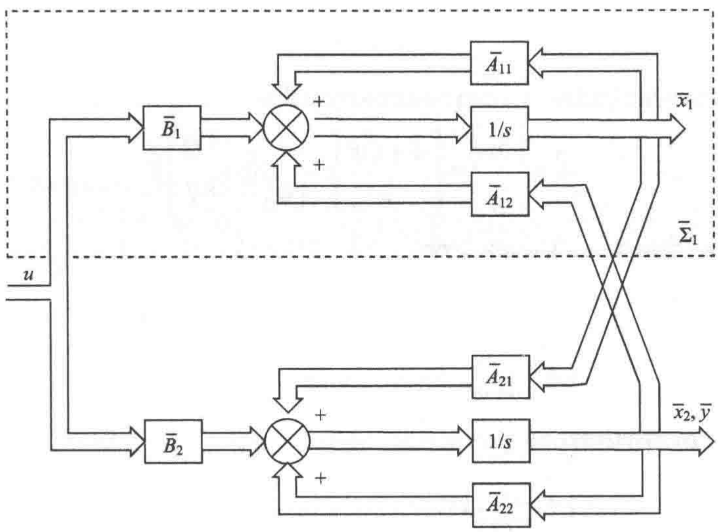


Fig.6.10 Structure of decomposed system

The subsystem $\bar{\Sigma}_1 = (\bar{A}_{11}, \bar{A}_{12}, \bar{B}_1, 0)$ is to be reconfigured. Following the strategy of full-dimension state estimator, we can duplicate $\bar{\Sigma}_1$ from (6-55) as

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u = \bar{A}_{11}\bar{x}_1 + M \tag{6-56}$$

where

$$M = \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u \tag{6-57}$$

Let $Z = \bar{A}_{22}\bar{x}_2$, then $Z = \dot{\bar{x}}_2 - \bar{A}_{22}\bar{x}_2 - \bar{B}_2 u$.

Consider M and Z as the input and output of $\bar{\Sigma}_1$, then $\bar{A}_{21}$ is the output matrix. Because of the observable pair $(\bar{A}, \bar{C})$, the pair $(\bar{A}_{11}, \bar{A}_{21})$ for $\bar{\Sigma}_1$ is also observable, thus the subsystem $\bar{\Sigma}_1$ can be estimated. Consulting equation (6-52), the estimator can be written as

$$\dot{\hat{\bar{x}}}_1 = (\bar{A}_{11} - \bar{G}\bar{A}_{21})\hat{\bar{x}}_1 + M + \bar{G}Z \tag{6-58}$$

Similarly, the eigenvalues of $(\bar{A}_{11} - \bar{G}\bar{A}_{21})$ can be assigned at desired positions by choosing a $(n-m)\times m$ dimensional matrix $\bar{G}$.

Substituting (6-57) into (6-58) yields

$$\dot{\hat{\bar{x}}}_1=(\bar{A}_{11}-\bar{G}\bar{A}_{21})\hat{\bar{x}}_1+(\bar{A}_{12}-\bar{G}\bar{A}_{22})\bar{y}+(\bar{B}_1-\bar{G}\bar{B}_2)u+\bar{G}\dot{\bar{y}}$$

Considering the difficulty of implementation of $\dot{\bar{y}}$, we introduce a new variable

$$\hat{\bar{w}}=\hat{\bar{x}}_1-\bar{G}\bar{y}$$

So the estimator equation can be described as

$$\dot{\hat{\bar{w}}}=(\bar{A}_{11}-\bar{G}\bar{A}_{21})\hat{\bar{x}}_1+(\bar{A}_{12}-\bar{G}\bar{A}_{22})\bar{y}+(\bar{B}_1-\bar{G}\bar{B}_2)u$$

$$\hat{\bar{x}}_1=\hat{\bar{w}}+\bar{G}\bar{y} \tag{6-59}$$

Hence all n state variables $\hat{\bar{x}}$ can be constructed as

$$\hat{\bar{x}}=\begin{bmatrix}\hat{\bar{x}}_1\\ \hat{\bar{x}}_2\end{bmatrix}=\begin{bmatrix}\hat{\bar{w}}+\bar{G}\bar{y}\\ \bar{y}\end{bmatrix}=\begin{bmatrix}I\\ 0\end{bmatrix}\hat{\bar{w}}+\begin{bmatrix}\bar{G}\\ I\end{bmatrix}\bar{y}$$

Then transform $\hat{\bar{x}}$ to $\hat{x}$, we have $\bar{x}=T\hat{\bar{x}}$.

The whole structure of the estimator is shown in Fig.6.11.

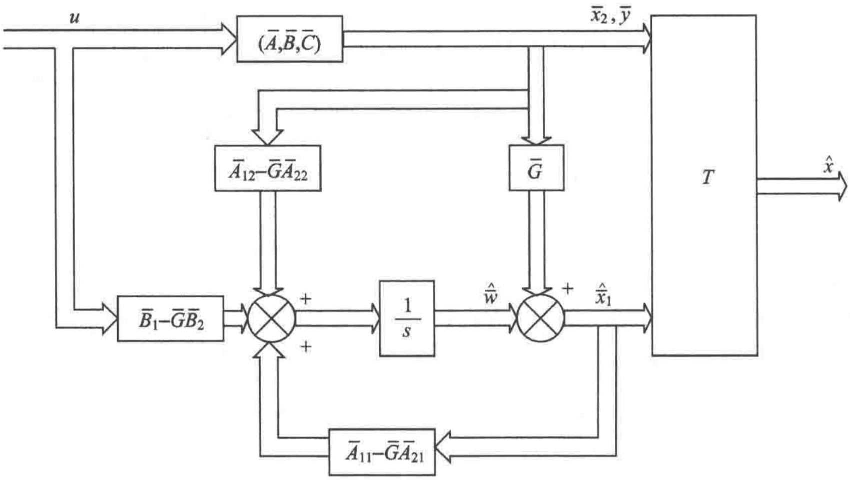


Fig.6.11 Block diagram of reduced-dimensional estimator

From equation (6-57), we can see that $\bar{x}_2=\bar{y}$, so there are no estimated errors of these m dimension state variables. Subtracting (6-59) from (6-56), we can obtain the estimated error equation

$$\dot{e}_1=\dot{\bar{x}}_1-\dot{\hat{\bar{x}}}_1=\bar{A}_{11}\bar{x}_1+\bar{A}_{12}y+\bar{B}_1u-(\bar{A}_{11}-\bar{G}\bar{A}_{21})\bar{x}_1-(\bar{A}_{12}-\bar{G}\bar{A}_{22})\bar{y}-(\bar{B}_1-\bar{G}\bar{B}_2)u-\bar{G}\dot{\bar{y}}$$

Considering $\bar{A}_{21}\bar{x}_1=\dot{\bar{y}}-\bar{A}_{22}\bar{y}-\bar{B}_2u$, the above equation can be simplified as

$$\dot{e}_1=(\bar{A}_{11}-\bar{G}\bar{A}_{21})(\bar{x}_1-\hat{\bar{x}}_1)=(\bar{A}_{11}-\bar{G}\bar{A}_{21})e_1 \tag{6-60}$$

where e_1 is the error between $\bar{x}$ and $\hat{\bar{x}}_1$. As the subsystem $\bar{\sum}_1$ is observable, the eigenvalues of $(\bar{A}_{11}-\bar{G}\bar{A}_{21})$ can be assigned at desired positions by choosing $\bar{G}$, thus guarantee that the error e_1 can approach zero at the desired rate.

Example 6.10 Considering the system

$$\dot{x}=\begin{bmatrix}4 & 4 & 4\\ -11 & -12 & -12\\ 13 & 14 & 13\end{bmatrix}x+\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 1 & 1\end{bmatrix}x$$

Find a reduced-dimensional state estimator with the poles $-3,-4$.

Solution

Examine the system's observability, there exists a reduced-dimensional state estimator. $rankc=1$.

Construct the transform matrix T

$$T^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\qquad T=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}$$

Let

$$\bar{A}=T^{-1}AT=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}4 & 4 & 4\\ -11 & -12 & -12\\ 13 & 14 & 13\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 4\\ 1 & 0 & -12\\ 1 & 1 & 5\end{bmatrix}$$

we have

$$\bar{b}=T^{-1}b=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$$

$$\bar{c}=cT=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

Since $\bar{x}_3$ can be provided directly by $\bar{y}$, a second-dimensional state estimator is needed.

(1) Introducing feedback matrix $\bar{G}=\begin{bmatrix}\bar{g}_1\\ \bar{g}_2\end{bmatrix}$, the characteristic polynomial of estimator yields

$$f(\lambda)=\left|\lambda I-(\bar{A}_{11}-\bar{G}\bar{A}_{21})\right|=\left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}-\begin{bmatrix}\bar{g}_1\\ \bar{g}_2\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\right|$$
$$=\begin{vmatrix}\lambda+\bar{g}_1 & \bar{g}_1\\ -1+\bar{g}_2 & \lambda+\bar{g}_2\end{vmatrix}=\lambda^2+(\bar{g}_1+\bar{g}_2)\lambda+\bar{g}_1$$

(2) The desired characteristic polynomial is

$$f^*(\lambda)=(\lambda+3)(\lambda+4)=\lambda^2+7\lambda+12$$

(3) Comparing the corresponding coefficient of $f(\lambda)$ and $f^*(\lambda)$, we have

$$\bar{g}_1=12,\quad \bar{g}_2=-5,\quad \text{and}\quad \bar{G}=\begin{bmatrix}12\\-5\end{bmatrix}$$

(4) From equation (6-59), we obtain the estimator equation

$$\dot{\hat{\bar{w}}}=\begin{bmatrix}-12 & -12\\ 6 & 5\end{bmatrix}\hat{\bar{x}}_1+\begin{bmatrix}-56\\13\end{bmatrix}\bar{y}+\begin{bmatrix}1\\-1\end{bmatrix}u$$

$$\hat{\bar{x}}_1=\hat{\bar{w}}+\begin{bmatrix}12\\-5\end{bmatrix}\bar{y}$$

The estimation of the state after linear transformation is

$$\hat{\bar{x}}=\begin{bmatrix}\hat{\bar{x}}_1\\ \hat{\bar{x}}_3\end{bmatrix}=\begin{bmatrix}\hat{\bar{w}}+\bar{G}\bar{y}\\ \bar{y}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar{w}_1\\ \bar{w}_2\end{bmatrix}+\begin{bmatrix}12\\-5\\1\end{bmatrix}\bar{y}=\begin{bmatrix}\bar{w}_1+12\bar{y}\\ \bar{w}_2-5\bar{y}\\ \bar{y}\end{bmatrix}$$

(5) To get the state estimation of the original system, transform $\hat{\bar{x}}$ as follows

$$\hat{x}=T\hat{\bar{x}}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}\bar{w}_1+12\bar{y}\\ \bar{w}_2-5\bar{y}\\ \bar{y}\end{bmatrix}=\begin{bmatrix}\bar{w}_1+12\bar{y}\\ \bar{w}_2-5\bar{y}\\ -\bar{w}_1-\bar{w}_2-6\bar{y}\end{bmatrix}$$

Example 6.11 the state space model of a system is $\begin{cases}\dot{X}=\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}X+\begin{bmatrix}1\\3\end{bmatrix}u\\ y=\begin{bmatrix}0 & 1\end{bmatrix}X\end{cases}$, try to construct a one dimensional state estimator with pole to be −5, and plot the diagram of the system.

Solution the system model is an observable canonical, so the system is observable, the state estimator can be constructed and the pole can be assigned arbitrary $y=x_2$, only x_1 need to be constructed

$$\left|sI-\hat{A}_{11}+g\hat{A}_{21}\right|=s-0+g=s+5=0,\quad g{=}5;$$

$$\dot{W}=\dot{w}=(A_{11}-GA_{21})W+[(A_{12}-GA_{22})+(A_{11}-GA_{21})G]Y+(B_1-GB_2)U=-5w-38y-14u$$

$$x_{1g}=\hat{x}_{1g}=W+GY=w+5y,\quad x_{2g}=\hat{x}_{2g}=y$$

The state estimator is

$$\begin{cases}\dot{w}=-5w-38y-14u\\ x_{1g}=w+5y\\ x_{2g}=y\end{cases}$$

The diagram of the system with state estimator is shown in Fig. 6.12.

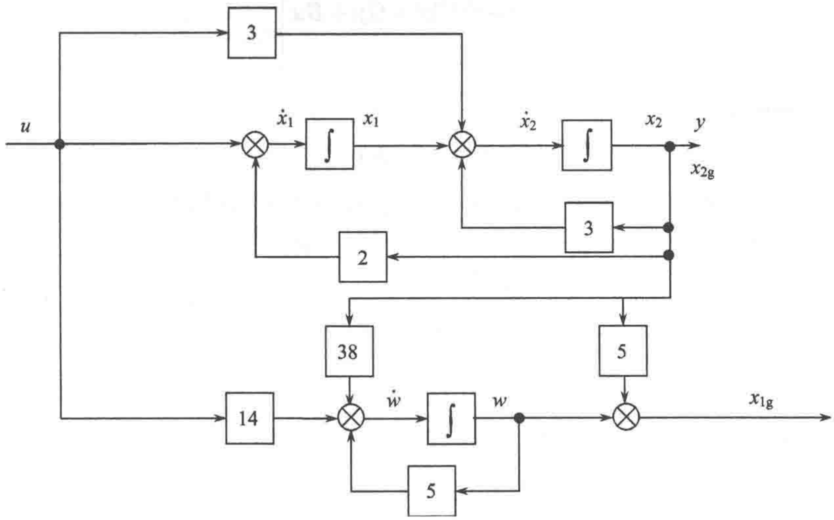


Fig.6.12 Diagram of the system

6.5 State Feedback Based on State Estimator

Fig. 6.13 is a state feedback system based on full-dimensional state estimator.

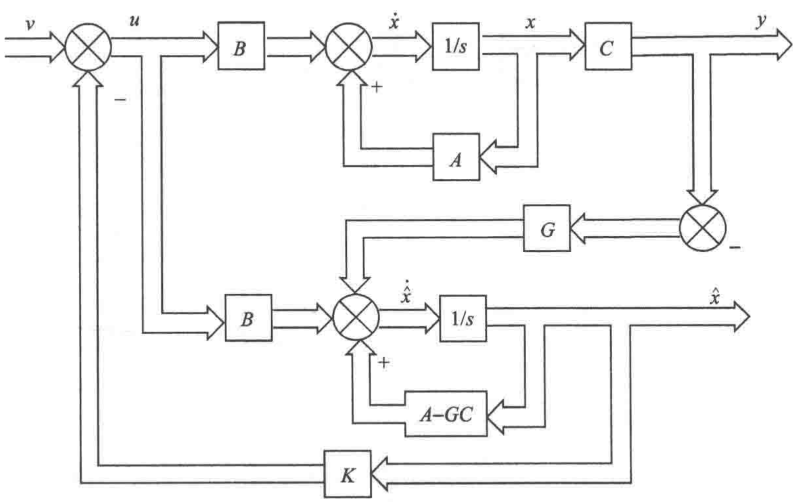


Fig.6.13 State feedback system based on full-dimensional state estimator

Consider the controllable and observable controlled system $\sum_0 = (A, B, C)$:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{6-61}$$

The state estimator $\sum_G$:

$$\left.\begin{aligned}\dot{\hat{x}} &= (A - GC)\hat{x} + Gy + Bu \\ \hat{y} &= C\hat{x}\end{aligned}\right\} \tag{6-62}$$

The state feedback law:

$$u = -K\hat{x} + v \tag{6-63}$$

Substituting equation(6-63) into equation (6-61) and equation (6-62), and you can obtain the state-space description of the total closed-loop system.

$$\left.\begin{aligned}\dot{x} &= Ax - BK\hat{x} + Bv \\ \dot{\hat{x}} &= GCx + (A - GC - BK)\hat{x} + Bv \\ y &= Cx\end{aligned}\right\} \tag{6-64}$$

equation (6-64) can be written in the following matrix form.

$$\left.\begin{aligned}\begin{pmatrix}\dot{x} \\ \dot{\hat{x}}\end{pmatrix} &= \begin{pmatrix}A & -BK \\ GC & A - GC - BK\end{pmatrix}\begin{pmatrix}x \\ \hat{x}\end{pmatrix} + \begin{pmatrix}B \\ B\end{pmatrix}v \\ y &= \begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}x \\ \hat{x}\end{pmatrix}\end{aligned}\right\} \tag{6-65}$$

This is a closed-loop system with dimension of $2n$.

Define the state error as $\tilde{x} = x - \hat{x}$. And introduce the following equivalent transformation:

$$\begin{pmatrix}x \\ \tilde{x}\end{pmatrix} = \begin{pmatrix}I & 0 \\ I & -I\end{pmatrix}\begin{pmatrix}x \\ \hat{x}\end{pmatrix} = \begin{pmatrix}x \\ x - \hat{x}\end{pmatrix} \tag{6-66}$$

Let transfer matrix

$$T = \begin{pmatrix}I & 0 \\ I & -I\end{pmatrix} \tag{6-67}$$

Then,

$$T^{-1} = \begin{pmatrix}I & 0 \\ I & -I\end{pmatrix}^{-1} = \begin{pmatrix}I & 0 \\ I & -I\end{pmatrix} = T \tag{6-68}$$

With linear transformation, the system turns to $(\bar{A}_1,\ \bar{B}_1,\ \bar{C}_1)$:

$$\begin{aligned}\bar{A}_1 &= T^{-1}A_1T = \begin{pmatrix}I & 0 \\ I & -I\end{pmatrix}\begin{pmatrix}A & -BK \\ GC & A - GC - BK\end{pmatrix}\begin{pmatrix}I & 0 \\ I & -I\end{pmatrix} = \begin{pmatrix}A - BK & BK \\ 0 & A - GC\end{pmatrix} \\ \bar{B}_1 &= T^{-1}B_1 = \begin{pmatrix}I & 0 \\ I & -I\end{pmatrix}\begin{pmatrix}B \\ B\end{pmatrix} = \begin{pmatrix}B \\ 0\end{pmatrix} \\ \bar{C}_1 &= C_1T = \begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}I & 0 \\ I & -I\end{pmatrix} = \begin{pmatrix}C & 0\end{pmatrix}\end{aligned} \tag{6-69}$$

Linear transformation does not change the poles of system, therefore

$$\det\left[\lambda I-\bar{A}_1\right]=\det\begin{pmatrix}\lambda I-(A-BK) & -BK \\ 0 & \lambda I-(A-GC)\end{pmatrix}$$
$$=\det\left[\lambda I-(A-BK)\right]\det\left[\lambda I-(A-GC)\right]$$
$$=\det\left[\lambda I-(A-BK)\right]\det\left[\lambda I-(A-GC)\right] \tag{6-70}$$

The results are very interesting, because they illustrate the fact that the characteristic polynomial of the closed-loop state feedback system based on state estimator equals the product of the characteristic polynomial of the matrix $(A-BK)$ and that of the matrix $(A-GC)$. The poles of the closed-loop system equals the sum of the poles of direct state feedback $(A-BK)$ and that of state estimator $(A-GC)$. Indeed, if system (A,B) is controllable, then the matrix K of the state feedback law (6-63) can be chosen so that the poles of the closed-loop system $\sum_0=(A,B,C)$ have any desired arbitrary values. The same applies to equation (6-62), where, if the system (A,C) is observable, the matrix G of estimator can be chosen so as to force the error to go rapidly to zero. This property, where the two design problems (the estimator and the matrix K of the closed-loop system) can be handled independently, is called the *separation principle.* This principle is clearly a very important design feature, since it reduces a very difficult design task to two separate simpler design problem.

Consider the pole assignment and the estimator design problem. The pole-assignment problem is called the *control problem* and it is rather a simple control design tool for improving the closed-loop system performance. The estimator design problem is called *estimator problem*, since it produces a good estimate of $x(t)$ in case where $x(t)$ is not measurable. The solution of the estimator design problem reduces to that of solving a pole-assignment problem. In case where an estimate of $x(t)$ is used in the control problem, one faces the problem of simultaneous solving the estimation and the control problem. At first sight this appears to be a formidable task. However, thanks to the separation theorem, the solution of the combined problem of estimation and control breaks down to separately solving the estimation and the control problem. Since the solution of the combined problem of estimation and control requires twice of the pole assignment problem.

6.6 Summary

Three types of feedback are introduced in this chapter. They can be used to improve the performance of a system. The precondition and algorithm of every feedback are discussed in detail. The pole assignment can be realized with some feedback. The desired poles come from the request for the performance of a system. The state estimator can be designed when a system is observable to realize the state estimation so as to fulfill the state feedback to optimize the system performance.

Appendix: State Feedback and Observer for Main Steam-temperature Control in Power Plant Steam-boiler Generation System

The superheater is an important part of the steam generation process in the boiler-turbine system, where steam is superheated before entering the turbine that drives the generator. The objective is to control the superheated steam temperature by controlling the flow of spray water using the spray water valves. From Fig. 6.14, a two stage water sprayers is used to control the superheated temperature. The steam generated from the boiler drum passes through the low-temperature superheater before it enters the radiant-type platen superheater. Water is sprayed onto the steam to control the superheated steam temperature in both the low and high temperature superheaters. Proper control of the superheated steam temperature is extremely important to ensure the overall efficiency and safety of the power plant. Therefore, the superheated steam temperature is to be controlled by adjusting the flow of spray water.

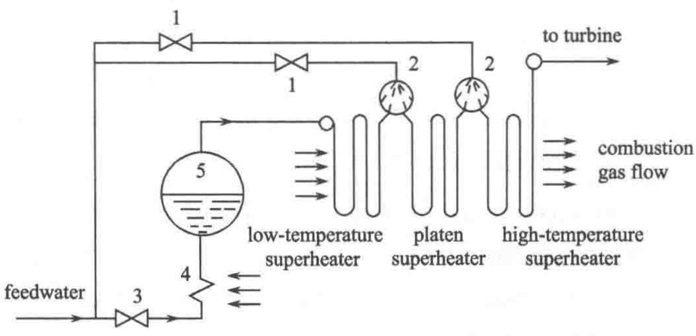


Fig.6.14 Boiler and superheater steam generation process

The typical mathematic model of the superheated steam temperature control process is a sixth-order transfer function as follow:

$$G_0(s)=\frac{\theta(s)}{W(s)}=\frac{1.589\times 2.45}{(1+14s)^2(1+15.8s)^4}$$

where θ and W represent the superheater steam temperature and the water flow rate of spray superheating, respectively.

Then the transfer function can be transformed to the controllable canonical form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where $A=\begin{bmatrix} -0.396 & -0.0653 & -0.0057 & -2.8355e-6 & -7.4664e-6 & -8.1868e-8 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$$B=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.3187e-6 \end{bmatrix}$$

Here we need to find a state feedback controller $u=r-\begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{bmatrix}x$ to make the closed loop poles at $\begin{bmatrix} -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 \end{bmatrix}$.

The block diagram of the system is shown in Fig.6.15.

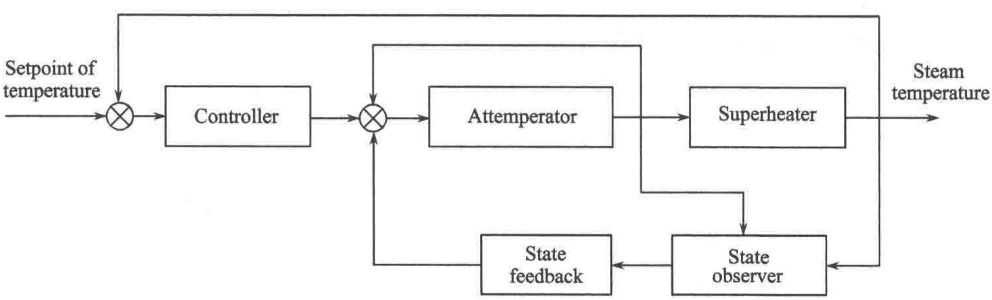


Fig.6.15 The steam temperature control system with state feedback and state observer

The system is controllable according to the matrix A. With state feedback , the closed-loop characteristic polynomial is:

$$\begin{aligned} f(\lambda) &= \det[\lambda I - (A - BK)] \\ &= \lambda^6 + (0.396 + k_1)\lambda^5 + (0.0653 + k_2)\lambda^4 + (0.0057 + k_3)\lambda^3 + (2.8355e-6 + k_4)\lambda^2 \\ &\quad + (7.4664e-6 + k_5)\lambda + 8.1868e-8 + k_6 \end{aligned}$$

The desired closed-loop characteristic polynomial is

$$f^*(\lambda) = (\lambda + 0.1)^6 = \lambda^6 + 0.6\lambda^5 + 0.15\lambda^4 + 0.02\lambda^3 + 0.0015\lambda^2 + 0.00006\lambda + 0.000001$$

Compare relative parameters in the above two functions, we have

$$k_1 = 0.204, \quad k_2 = 0.0847, \quad k_3 = 0.0143,$$
$$k_4 = 1.4972e-3, \quad k_5 = 5.2534e-5, \quad k_6 = 9.1813e-7$$

Thus: $K = \begin{bmatrix} 0.204 & 0.0847 & 0.0143 & 1.4972e-3 & 5.2534e-5 & 9.1813e-7 \end{bmatrix}$

The observable canonical form of the system is :

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$$
$$y(t) = \bar{C}x(t) + \bar{D}u(t)$$

where
$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -8.1868e-8 \\ 1 & 0 & 0 & 0 & 0 & -7.4664e-6 \\ 0 & 1 & 0 & 0 & 0 & -2.8355e-6 \\ 0 & 0 & 1 & 0 & 0 & -0.0057 \\ 0 & 0 & 0 & 1 & 0 & -0.0653 \\ 0 & 0 & 0 & 0 & 1 & -0.396 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0.3187e-6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Next we could design a full-dimensional state estimator with poles to be $\begin{bmatrix} -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & -0.25 \end{bmatrix}$.

Design the full-dimensional estimator.

Suppose $G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{bmatrix}$, then

$$\bar{A} - G\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -8.1868e-8 \\ 1 & 0 & 0 & 0 & 0 & -7.4664e-6 \\ 0 & 1 & 0 & 0 & 0 & -2.8355e-6 \\ 0 & 0 & 1 & 0 & 0 & -0.0057 \\ 0 & 0 & 0 & 1 & 0 & -0.0653 \\ 0 & 0 & 0 & 0 & 1 & -0.396 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -8.1868e-8-g_1 \\ 1 & 0 & 0 & 0 & 0 & -7.4664e-6-g_2 \\ 0 & 1 & 0 & 0 & 0 & -2.8355e-6-g_3 \\ 0 & 0 & 1 & 0 & 0 & -0.0057-g_4 \\ 0 & 0 & 0 & 1 & 0 & -0.0653-g_5 \\ 0 & 0 & 0 & 0 & 1 & -0.396-g_6 \end{bmatrix}$$

and

$$\det\left[\lambda I-\left(\bar{A}-G\bar{C}\right)\right]=\lambda^6+(0.396+g_6)\lambda^5+(0.0653+g_5)\lambda^4+(0.0057+g_4)\lambda^3$$
$$+(2.8355e-6+g_3)\lambda^2+(7.4664e-6+g_2)\lambda+8.1868e-8+g_1$$

Comparingwith

$$f^*(\lambda)=(\lambda+0.25)^6=\lambda^6+1.5\lambda^5+0.9375\lambda^4+0.3125\lambda^3+0.0586\lambda^2+0.00586\lambda+0.000244$$

we can obtain $G=\begin{bmatrix} 0.0002 \\ 0.0059 \\ 0.0583 \\ 0.3068 \\ 0.8722 \\ 1.104 \end{bmatrix}$

The full-dimensional estimator equation:

$$\dot{\hat{x}}=\left(\bar{A}-G\bar{C}\right)\hat{x}+Gy+\bar{B}u$$
$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -0.000244 \\ 1 & 0 & 0 & 0 & 0 & -0.00586 \\ 0 & 1 & 0 & 0 & 0 & -0.0586 \\ 0 & 0 & 1 & 0 & 0 & -0.3125 \\ 0 & 0 & 0 & 1 & 0 & -0.9375 \\ 0 & 0 & 0 & 0 & 1 & -1.5 \end{bmatrix}\hat{x}+\begin{bmatrix} 0.0002 \\ 0.0059 \\ 0.0583 \\ 0.3068 \\ 0.8722 \\ 1.104 \end{bmatrix}y+\begin{bmatrix} 0.3187e-6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}u$$

Exercises

6.1 Determine whether the following systems can realize arbitrary pole assignment with state feedback:

(1) $\dot{x}=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}x+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u$

(2) $\dot{x}=\begin{bmatrix} 4 & 2 \\ 0 & -2 \end{bmatrix}x+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u$

(3) $\dot{x}=\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & 1\\ 0 & 0 & -2\end{bmatrix}x+\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}u$

(4) $\dot{x}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2 & -4 & -3 & -5\end{bmatrix}x+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}u$

6.2 Consider a single-input continuous-time linear time-invariant system

$$\dot{x}=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}x+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

Try to find a state feedback matrix k which makes the closed-loop eigenvalues $\lambda_1^*=-2+j$, $\lambda_2^*=-2-j$.

6.3 Given the transfer function of a SISO continuous-time linear time-invariant system

$$G(s)=\frac{1}{s(s+4)(s+8)}$$

try to find a state feedback matrix k which makes the closed-loop eigenvalues $\lambda_1^*=-2$, $\lambda_2^*=-4$, $\lambda_3^*=-7$.

6.4 Given a single-input linear time-invariant system

$$\dot{x}=\begin{bmatrix}0 & 0 & 0\\ 1 & -6 & 0\\ 0 & 1 & -12\end{bmatrix}x+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u$$

try to find a state feedback matrix $u=-Kx$ which makes the closed-loop eigenvalues $\lambda_1^*=-2$, $\lambda_2^*=-1+j$, $\lambda_3^*=-1-j$.

6.5 Consider a continuous-time linear time-invariant system

$$\dot{x}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}x+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}x$$

Try to find an output feedback matrix f which makes the closed-loop eigenvalues be $\lambda_1^*=-2$, $\lambda_2^*=-4$.

6.6 Consider the following fourth-order system

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

Determine state feedback matrics to place the closed-loop system poles at:

(1) $\lambda_1^* = -2$, $\lambda_2^* = -2$, $\lambda_3^* = -2$, $\lambda_3^* = -2$

(2) $\lambda_1^* = -3$, $\lambda_2^* = -3$, $\lambda_3^* = -3$, $\lambda_3^* = -2$

(3) $\lambda_1^* = -3$, $\lambda_2^* = -4$, $\lambda_3^* = -3$, $\lambda_3^* = -3$

6.7 Consider a continuous-time linear time-invariant system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} u$$

Try to find the state feedback matrix to place the closed-loop eigenvalues at $\lambda_1^* = -2$, $\lambda_2^* = -1 + j2$, $\lambda_3^* = -1 - j2$.

6.8 Given the transfer function of a SISO continuous-time linear time-invariant system

$$g_0(s) = \frac{(s+2)(s+3)}{(s+1)(s-2)(s+4)}$$

try to determine if there exists a state feedback matrix k which can make the closed-loop transfer function as

$$g(s) = \frac{s+3}{(s+2)(s+4)}$$

If does, find a state feedback matrix k.

6.9 Design a full-dimensional state estimator with eigenvalues to be $-r, -2r$ ($r > 0$) for the state equation below

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

6.10 Design a reduced-dimensional state estimator with eigenvalues to be –4 and –5for the state equation below

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

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